INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
MA5040 - Topology
Problem Sheet 3
Spring 2025

Problem 1. Let $\mathbf{x_1}, \mathbf{x_2}, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x_1}), \pi_\alpha(\mathbf{x_2}), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Problem 2. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are"eventually zero", that is, all sequences $(x_1, x_2, , \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answers.

Problem 3. Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots).$$

Show that, if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Problem 4.

1. In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

2. More generally, given $p \geq 1$, define,

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p \right]^{\frac{1}{p}}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $d^{'}$ is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

Problem 5. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Problem 6. Consider the product, box and uniform topologies on $\mathbb{R}^\omega$.

1. In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?
   (a) $f(t) = (t, 2t, 3t, \dots)$,
   (b) $g(t) = (t, t, t, \dots)$,
   (c) $h(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \dots)$.
2. In which topologies do the following sequences converge?
   (a) $\mathbf{w_1} = (1, 1, 1, 1, \dots), \mathbf{w_2} = (0, 2, 2, 2, \dots), \mathbf{w_3} = (0, 0, 3, 3, \dots), \dots$
   (b) $\mathbf{x_1} = (1, 1, 1, 1, \dots), \mathbf{x_2} = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots), \mathbf{x_3} = (0, 0, \frac{1}{3}, \frac{1}{3}, \dots), \dots$
   (c) $\mathbf{y_1} = (1, 0, 0, 0, \dots), \mathbf{y_2} = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots), \mathbf{y_3} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots)$,
   (d) $\mathbf{z_1} = (1, 1, 0, 0, \dots), \mathbf{z_2} = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots), \mathbf{z_3} = (\frac{1}{3}, \frac{1}{3}, 0, 0, \dots), \dots$

Problem 7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

Problem 8. Let $\overline{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \dots$$

1. Show that $U(\mathbf{x}, \epsilon)$ is not equal to the $\epsilon-$ball $B_{\overline{\rho}}(\mathbf{x}, \epsilon)$.
2. Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.
3. Show that

$$B_{\overline{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \epsilon).$$

Problem 9. Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined by the equation:

$$h((x_1, x_2, \dots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots).$$

Give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i and b_i is h continuous? a homeomorphism?

Problem 10. Show that if d is a metric for X, then

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a bounded metric that gives the topology of X. [Hint: If $f(x) = x/(1 + x)$ for $x > 0$, use the mean-value theorem to show that $f(a + b) - f(b) \le f(a)$.]

Problem 11. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called an *isometric imbedding* of X in Y.

Problem 12. Let X_n be a metric space with metric d_n, for $n \in \mathbb{Z}_+$.

1. Show that
$$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\}$$
is a metric for the product space $X_1 \times \cdots \times X_n$.

2. Let $\tilde{d}_i = \min\{d_i, 1\}$. Show that
$$D(x, y) = \sup \frac{\tilde{d}_i(x_i, y_i)}{i}$$
is a metric for the product space $\prod X_i$.

Problem 13. Show that R_l and the ordered square satisfy the first countability axiom.

Problem 14. Let X be a set and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\overline{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \overline{\rho})$.

Problem 15. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function.

1. Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

2. Show that f_n does not converge uniformly to f.

Problem 16. Using the closed set formulation of continuity, show that the following are closed subsets of $\mathbb{R}^2$:

1. $A = \{x \times y |\ xy = 1\}$

2. $S^1 = \{x \times y |\ x^2 + y^2 = 1\}$

3. $B^2 = \{x \times y |\ x^2 + y^2 \leq 1\}$.

The set B^2 is called the closed unit ball in $\mathbb{R}^2$.
