INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
MA5040 - Topology
Problem Sheet 2
Spring 2025

Let X and Y denote topological spaces.

Problem 1. Let Y be Hausdorff topological space. Let $f, g : X \to Y$ be continuous functions. Let $Z = \{x \in X : f(x) = g(x)\}$. Show that Z is closed in X.

Problem 2. Let $f : X \to Y$ be a continuous function and Y is Hausdorff. Show that the graph of the function $\{(x, f(x)) \in X \times Y : x \in X\}$ is closed.

Problem 3. Let $f : X \to Y$ be continuous. True or false: If x is a limit point of the subset A of X, then $f(x)$ is a limit point of $f(A)$.

Problem 4. Given $x_0 \in X$ and $y_0 \in Y$. Show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by $f(x) = x \times y_0$ and $g(y) = x_0 \times y$ are imbeddings.

Problem 5. Show that the subspace (a, b) is homeomorphic to $(0, 1)$, and the subspace $[0, 1]$ is homeomorphic to $[a, b]$.

Problem 6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is such that

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_l$ to $\mathbb{R}$.

Problem 7. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

1. Show that the set $\{x : f(x) \leq g(x)\}$ is closed in X.
2. Let $h : X \to Y$ be a function defined as follows:

$$h(x) = \min\{f(x), g(x)\}.$$

   Show that h is continuous.

Problem 8. Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \cup_\alpha A_\alpha$. Let $f : X \to Y$, and suppose that $f|A_\alpha$ is continuous for each α.

1. Show that if the collection $\{A_\alpha\}$ is finite, and each set A_α is closed, then f is continuous.

2. Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

3. An indexed family of sets $\{A_\alpha\}$ is said to be locally finite if each point x of X has neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Problem 9. Let X and Y be topological spaces. Let $\pi_1 : X \times Y \to X$ be the projection map onto the first component $\pi_1(x, y) = x$. Show that π_1 is open. Is π_1 a closed map?

Problem 10. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Define the function $f \times g : A \times C \to B \times D$ by

$$(f \times g)(a \times c) = f(a) \times g(c).$$

Show that $f \times g$ is a continuous function.

Problem 11. Let $F : X \times Y \to Z$. The function F is continuous in each variable separately if for each $y_0 \in Y$, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and the map $k : Y \to Z$ defined by $h(y) = F(x_0 \times y)$ is continuous.

Problem 12. Consider the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$F(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } x \times y \neq 0 \times 0 \\ 0 & \text{if } x \times y = 0 \times 0 \end{cases}$$

1. Show that F is continuous in each variable separately.

2. Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.

3. Show that the function F is not continuous.

Problem 13. Let $A \subset X$, ane let $f : A \to Y$ be continuous and Y be Hausdorff. Show that if f can be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Problem 14. Let X and Y be topological spaces and $f : X \to Y$ be a function. Show that f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$. Find an example to show that the inclusion can be strict.

Problem 15. Give an example to show that a continuous bijective map $f : X \to Y$ need not be a homeomorphism.

Problem 16. Given an example to show that the existence of imbeddings $f : X \to Y$ and $g : Y \to X$ does not imply that the topological spaces are homeomorphic.