INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
MA5040 - Topology
Problem Sheet 1
Spring 2025

Notation :

- **$\mathcal{T}_{cf}$ be the co-finite topology.**
- **$\mathcal{T}_{cc}$ be the co-countable topology.**
- **$\mathcal{T}_{disc}$ be the discrete topology.**
- **$\mathcal{T}_{indisc}$ be the indiscrete topology.**

Problem 1. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that $\cap\mathcal{T}_\alpha$ is a topology on X. Is $\cup\mathcal{T}_\alpha$ a topology on X?

Problem 2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

Problem 3. In $\mathbb{Z}_+$ define a topology as follows: A subset of U of $\mathbb{Z}_+$ is open if and only if whenever $n \in U$ all the divisors of n belong to U. Show that this is a topology on $\mathbb{Z}_+$ and it is not same as the discrete topology.

Problem 4. Show that the collection

$$\mathbb{B} = \{(a, b) : a < b, a \text{ and } \; b \; \text{ are rational}\}$$

is a basis for the standard topology on $\mathbb{R}$.

Problem 5. Let $\mathbb{N}$ denote the set of natural numbers. Let $A_n = \{m \in \mathbb{N} : m \geq n\}$. Let $\mathcal{T}_1 = \{\emptyset\} \cup \{A_n\}n \in \mathbb{N}$. Show that $\mathbb{T}_1$ is a topology on $\mathbb{N}$. Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_{cf}$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_f$.

Problem 6. Let X be a nonempty set.

(a) Show that $\mathcal{T}_{indisc} \subseteq \mathcal{T}_{cf} \subseteq \mathcal{T}_{cc} \subseteq \mathcal{T}_{disc}$.

(b) Let X be a finite set. Show that $\mathcal{T}_{indisc} \subsetneq \mathcal{T}_{cf} = \mathcal{T}_{cc} = \mathcal{T}_{disc}$.

(c) Let X be a countably infinite set. Show that $\mathcal{T}_{indisc} \subsetneq \mathcal{T}_{cf} \subsetneq \mathcal{T}_{cc} \subseteq \mathcal{T}_{disc}$.

(d) Let X be an uncountable set. Show that $\mathcal{T}_{indisc} \subsetneq \mathcal{T}_{cf} \subsetneq \mathcal{T}_{cc} \subsetneq \mathcal{T}_{disc}$.

Definition 0.1. A property P of a topological space X is *hereditary*, if every subspace of X has property P, whenever X has property P.

Problem 7. Show that the following properties are hereditary.

(a) Discreteness,

(b) Indiscreteness,

(c) Cofiniteness,

(d) Cocountableness.

Problem 8. If $\mathcal{T}$ and $\mathcal{T}^{'}$ are topologies on X, and $\mathcal{T}^{'}$ is strictly finer than $\mathcal{T}$. What can you say about the corresponding subspace topologies on the subset Y of X?

Problem 9. A map $f : X \to Y$ is said to be an open map if for every open set U of X, the set $f(U)$ is open in Y. Show that π_1 and π_2 are open maps.

Problem 10. Show that if A is closed in X and B is closed in Y , then $A \times B$ is closed in $X \times Y$.

Problem 11. Let X be an ordered set in the order topology. Show that $\overline{(a, b)} \subseteq [a, b]$. Under what conditions does equality hold?

Problem 12. Let A, B and $A\alpha$ denote the subsets of a topological space X. Show that

1. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
3. $\cup \overline{A_\alpha} \subseteq \overline{\cup A_\alpha}$. What about reverse inclusion?

Problem 13. Let A, B and $A\alpha$ denote the subsets of a topological space X. Show that

1. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
2. $\cap \overline{A_\alpha} = \overline{\cap A_\alpha}$.

Problem 14. Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

Problem 15. Determine the closure of the set $K = \{\frac{1}{n} : n \in \mathbb{Z}_+\}$ in $\mathbb{R}$ with respect to the following topologies:

(a) Standard topology,

(b) Lower-limit topology,

(c) Upper-limit topology,

(d) K-topology,

(e) Discrete topology,

(f) Indiscrete topology,

(g) Co-countable topology,

(h) Co-finite topology.

Problem 16. Which of the above topologies are Hausdorff?

Problem 17. Show that X is Hausdorff if and only if $\Delta = \{x \times x : x \in X\}$ is closed in $X \times X$.

Problem 18. For $A \subseteq X$, define the boundary of A as follows:

$$\operatorname{Bd} A = \overline{A} \cap \overline{X \setminus A}.$$

(a) Show that Int A and $\operatorname{Bd} A$ are disjoin, and $\overline{A} = Int(A) \cup \operatorname{Bd}(A)$.

(b) Show that $\operatorname{Bd}(A) = \emptyset$ if and only if A is both open and closed.

(c) Show that U is open if and only $\operatorname{Bd}(U) = \overline{U} \setminus U$.

(d) True or false : If U is open, then $U = Int(\overline{U})$. Justify your answer.