# Reciprocal Eigenvalue Properties of Graphs

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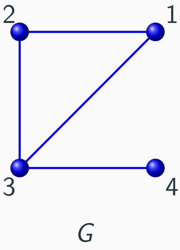
# Background

Let $G = (V, E)$ be a graph on $n$ vertices.

The **adjacency matrix** $A(G)$ is an $n \times n$ matrix where $A(i, j)$ equals the number of edges between $i$ and $j$.




$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

**Characteristic polynomial** of $A(G)$:

$$P(G; x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n.$$

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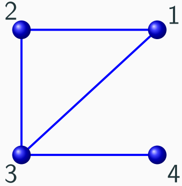
- The $ij$-th entry of $A(G)^k$$=$ the number of walks of length $k$ from vertex $i$ to vertex $j$.

**Spectrum:** $\sigma(G) = \big(\lambda_1, \lambda_2, \ldots, \lambda_n\big)$, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $A(G)$.

By **eigenvalues of** $G$, we mean the eigenvalues of $A(G)$.

We say $G$ is **nonsingular** if $A(G)$ is nonsingular.

**Spectral radius of $G$:** $\rho(\mathbf{G}) \longrightarrow$ the largest eigenvalue of $A(G)$




$G$

$$\sigma(G) = (-1.481, -1, 0.311, 2.170)$$

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- If $\lambda_{n-2} = -1$, then $G^c$ is isomorphic to the union of a complete bipartite graph and some isolated vertices.

If $T$ is a tree on $n$ vertices, then

$$\rho(P_n) \leq \rho(T) \leq \rho(S_n),$$

with left hand equality if and only if $T \cong P_n$ and right hand equality if and only if $T \cong S_n$.

**The spectral radius is closely related to the maximum degree of the graph and indicates the presence of a highly connected vertex.**

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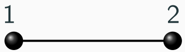
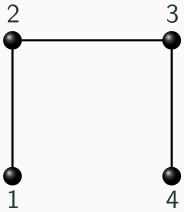
- If $G$ is a **nonsingular bipartite graph** on $n = 2m$ vertices, then

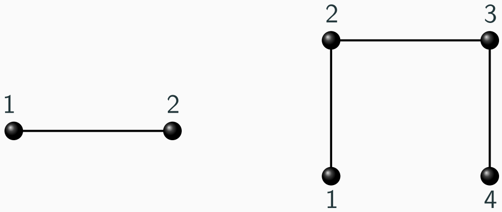
$$\sigma(G) = \Big(-\lambda_m, \dots, -\lambda_1, \lambda_1, \dots, \lambda_m\Big).$$




$P_2$




$P_4$




$P_2$ $P_4$

$$\sigma(P_2) = (-1, 1), \quad \sigma(P_4) = (\frac{-1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2})$$

## Question

Characterize the graphs that satisfy the property that the reciprocal of each eigenvalue is also an eigenvalue.

- A graph $G$ is said to have the **reciprocal eigenvalue property** (property (R)) if $\frac{1}{\lambda}$ is an eigenvalue of $G$ whenever $\lambda$ is an eigenvalue of $G$.

[1] D. M. Cvetković, I. Gutman and S. K. Simić, On self pseudo-inverse graphs, *Univ. Beograd. Publ. Elektrotehn. Fak.*, (1978).

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- Cvetković, Gutman and Simić [1]: Property (SR) was termed as **property** $C$.

- Godsil and Mckay [2]: Property (SR) was termed as **symmetry property**.

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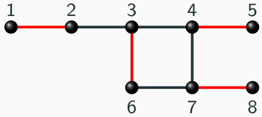
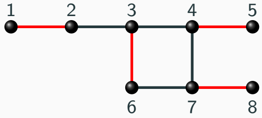
A **perfect matching** in a graph $G$ is a collection of vertex disjoint edges that covers every vertex.




$G$

- A graph in general can have more than one perfect matchings.
- If a tree has a perfect matching, it is unique and such trees are precisely the trees which are nonsingular.
- In general, any graph with a unique perfect matching is nonsingular.

**Alternating path:** Let $G$ be a graph with a unique perfect matching. A path $P(i,j) = [i = i_1, i_2, \ldots, i_{2k} = j]$ is said to be an **alternating path** if the edges $\{i_1, i_2\}, \{i_3, i_4\}, \ldots, \{i_{2k-1}, i_{2k}\}$ are edges in the perfect matching.




$G$

The path $P = [1, 2, 3, 6]$ is an alternating path.

**Corona:** A corona of a graph $G$, denoted by $\hat{G}$, is the graph obtained from $G$ by adding a new pendant vertex at each vertex of $G$.

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$P_4$

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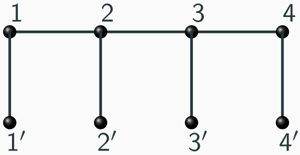


$\hat{P}_4$

**Lemma:** If $G = \hat{G}_1$, a corona of $G_1$, then $\lambda$ is an eigenvalue of $G$ if and only if $\frac{-1}{\lambda}$ is an eigenvalue of $G$. Furthermore, if $G_1$ is bipartite then $G$ has strong reciprocal eigenvalue property.

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$$A(\hat{G}_1) = \begin{bmatrix} A(G_1) & I \\ I & \mathbf{0} \end{bmatrix}$$

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- If $\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of $G_1$, then

$$\frac{\mu_1 + \sqrt{\mu_1^2 + 4}}{2}, \frac{\mu_1 - \sqrt{\mu_1^2 + 4}}{2}, \ldots, \frac{\mu_n + \sqrt{\mu_n^2 + 4}}{2}, \frac{\mu_n - \sqrt{\mu_n^2 + 4}}{2}$$

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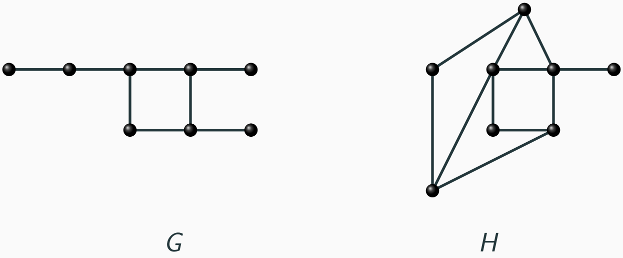


**Figure 1:** Non-corona graphs with property (SR)

$$P(G;x) = x^8 - 8x^6 + 15x^4 - 8x^2 + 1$$

$$P(H;x) = x^8 - 11x^6 - 2x^5 + 24x^4 - 2x^3 - 11x^2 + 1$$

# Trees with property (SR)/ (R)

**Definition:** A polynomial $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ of degree $n$ with real coefficients is called *self-reciprocal* or *palindromic* if $P(x) = x^n P(\frac{1}{x})$, that is, $a_i = a_{n-i}$, for $i = 0, 1, \ldots, n$. It is called *anti-palindromic* if $P(x) = -x^n P(\frac{1}{x})$.

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**Pless**[1]**:** If $\alpha$ is a root of a polynomial $P(x)$ that is either palindromic or anti-palindromic, then $\frac{1}{\alpha}$ is also a root of $P(x)$ and they both have the same multiplicity.

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**Lemma:** Let $G$ be a graph on $n$ vertices with property (SR) and

$$P(G;x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n.$$

Then $P(x) = -x^n P(\frac{1}{x})$ and hence, $|c_i| = |c_{n-i}|$ for $i = 1, 2, \ldots, n-1$ and $|c_n| = 1$.

- If $G$ satisfies property (SR), then $|\det A(G)| = 1$.

[1] S. Barik, S. Pati, and B. K. Sarma, The spectrum of the corona of two graphs, SIAM J. Discrete Math., (2007).

**Theorem:** Let $T$ be a tree on $n = 2k$ vertices. Then $T$ has property (SR) if and only if $T$ is a corona tree.

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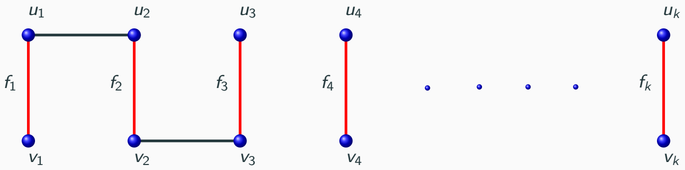


- $c_2 =$ number of edges $= 2k - 1$

- $c_{2k-2} =$ the number of pairwise disjoint edge subsets of size $k - 1 = k + k - 1 + 1 > c_2$.

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**Graph inverse:** Let $G$ be a nonsingular graph. If there is a signature matrix $S$ such that $SA(G)^{-1}S$ is nonnegative, then $G$ is said to be invertible and the weighted associated with the matrix $SA(G)^{-1}S$ is called the inverse graph of $G$ and is denoted by $G^{-1}$.

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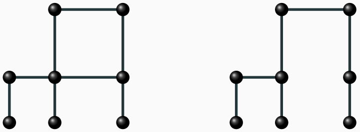

$G$ $\qquad$ $G^{-1}$

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**Problem (1985):** (Godsil) Characterize the graphs $G$ such that $G^{-1}$ is isomorphic to $G$.

**Theorem:** Let $G$ be a bipartite connected graph on $n$ vertices with a unique perfect matching $\mathcal{M}$ and $\mathcal{P}_G$ denote the collection of all alternating paths in $G$. Let $B = [b_{ij}]$, where

$$b_{ij} = \begin{cases} \displaystyle\sum_{P(i,j)\in\mathcal{P}_G} (-1)^{\frac{|P(i,j)|-1}{2}}, & \text{if at least one } P(i,j) \in \mathcal{P}_G, \\ 0, & \text{otherwise.} \end{cases}$$

Then $B = A(G)^{-1}$.

[1] C. D. Godsil, Inverses of trees, *Combinatorica*, (1985).

[2] S. Barik, M. Neumann, and S. Pati, On nonsingular trees and a reciprocal eigenvalue property, *Linear Multilinear Algebra*, (2006).

**Example:**

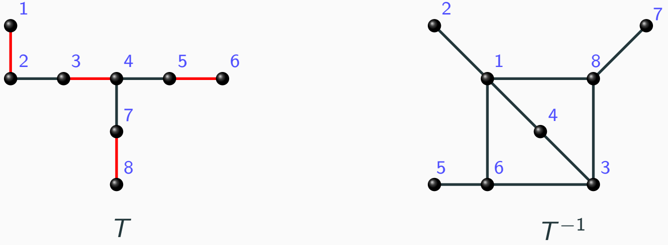


**Figure 2:** A nonsingular tree and its inverse

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**Theorem:** Let $T$ be a nonsingular tree on $n = 2k$ vertices. Then the following are equivalent.

(i) $T$ has property (SR).

(ii) $T$ has property (R).

(iii) $T$ is a corona tree.

(iv) $|\mathcal{P}_T| = 2k - 1$.

(v) $T^{-1}$ is a tree.

(vi) $T^{-1}$ is isomorphic to $T$.

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# Unicyclic graphs with property (SR)

If $G$ is a unicyclic graph with an even cycle and is also a simple corona, then $G$ satisfies property (SR).

Let us ask the converse. Suppose that we have a unicyclic graph with property (SR). Is it necessarily bipartite? Is it necessarily a corona?

**Lemma:** Let $G$ be a unicyclic graph with property (SR) with cycle length $g$. Then $g$ is even.

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[2] R.B. Bapat, S.K. Panda, and S. Pati, Self-inverse unicyclic graphs and strong reciprocal eigenvalue property, *Linear Algebra Appl.*, (2017).

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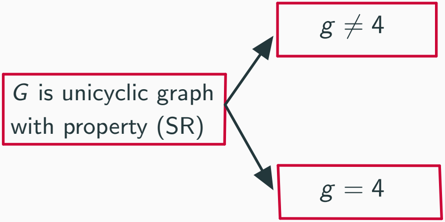


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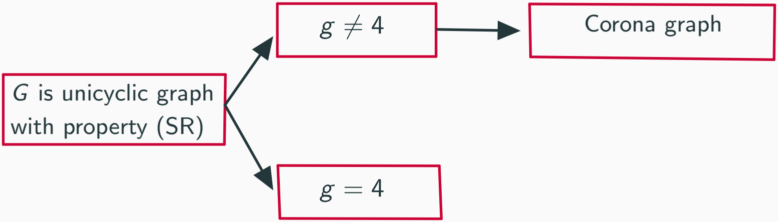


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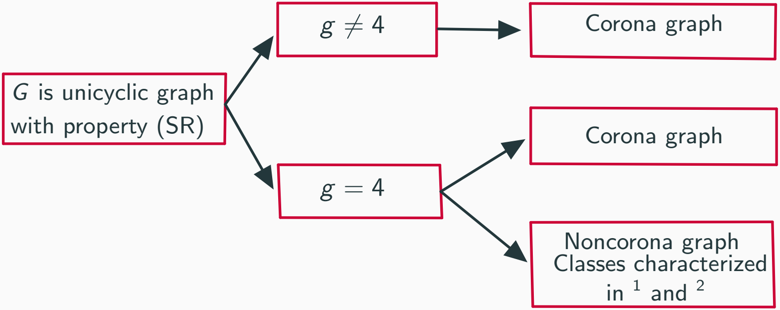


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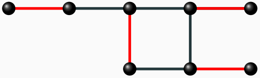
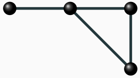
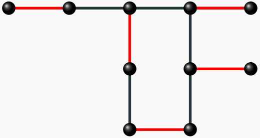
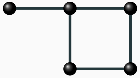
# Bipartite graphs with property (SR)

## Contraction of matching edges:









**Theorem (Panda and Pati, Simion and Cao):** Let $G$ be a connected bipartite graph with a unique perfect matching such that the graph obtained by contracting the matching edges is also bipartite. Then the following conditions are equivalent.

(i) The reciprocal of the largest eigenvalue is the smallest positive eigenvalue of $G$.

(ii) $G^{-1}$ exists and is isomorphic to $G$.

(iii) $G$ satisfies property (R).

(iv) $G$ satisfies property (SR).

(v) $G$ is a simple corona of some bipartite graph.

[1] S. Panda and S. Pati, On some graphs which satisfy reciprocal eigenvalue properties, *Linear Algebra Appl.*, (2017).

[2] R. Simion and D. S. Cao, Solution to a problem of C. D. Godsil regarding bipartite graphs with unique perfect matching, *Combinatorica*, (1989).

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- Whether this is true in general?

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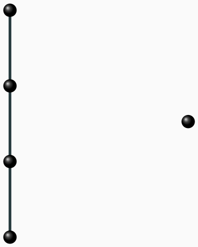
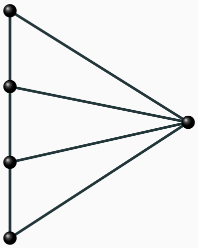
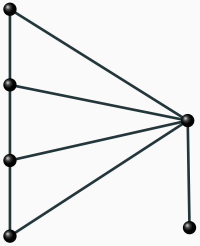
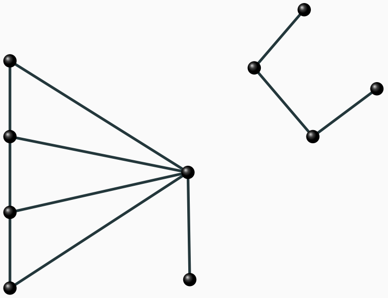
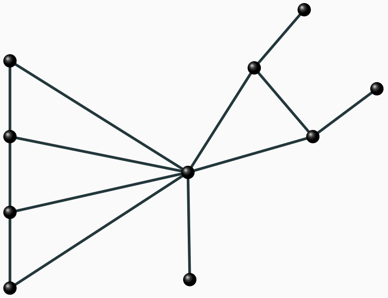
- Whether this is true in general?
- Probably in bipartite case.

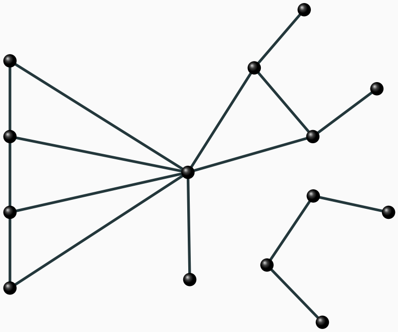












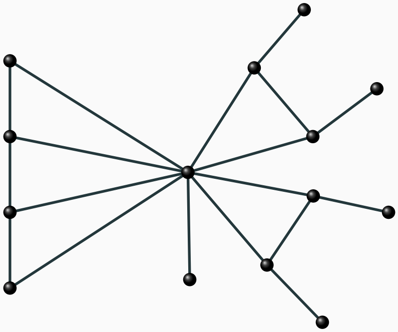
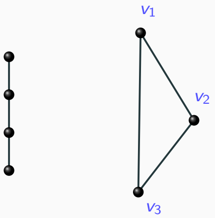
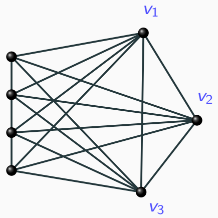


**Figure 3:** A graph satisfying property (R) but not (SR)

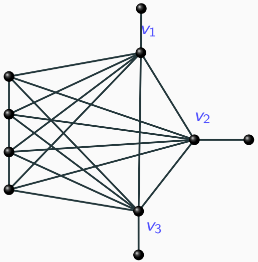
$$\begin{pmatrix} \frac{-3-\sqrt{5}}{2}, & \frac{-1-\sqrt{5}}{2}, & \frac{1-\sqrt{5}}{2}, & \frac{-3+\sqrt{5}}{2}, & 2-\sqrt{3}, & \frac{-1+\sqrt{5}}{2}, & \frac{1+\sqrt{5}}{2}, & 2+\sqrt{3} \\ \\ 1 & 3 & 2 & 1 & 1 & \mathbf{3} & \mathbf{2} & 1 \end{pmatrix}$$



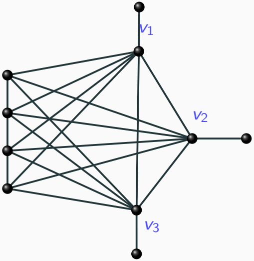
$v_1$
$v_2$
$v_3$

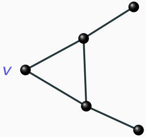


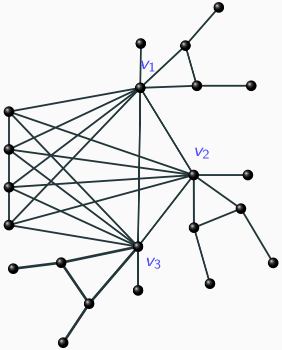
$v_1$
$v_2$
$v_3$



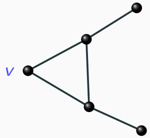
$v_1$
$v_2$
$v_3$


$v_1$
$v_2$
$v_3$


$v$




$G_3^3$




$H_v$

**Theorem:** The graph $\mathbf{G_n^n}$ satifies property (R) but not (SR).

In $P(\mathbf{G_n^n}; x)$, the power of $(x^2 + x - 1)$ is $n$ more than the power of $(x^2 - x - 1)$. Thus, $\mathbf{G_n^n}$ does not satisfy property (SR).

[1] S. Barik and S. Pati, Classes of nonbipartite graphs with reciprocal eigenvalue property, *Linear Algebra Appl.*, (2023).

# Singular graphs and the reciprocal eigenvalue property

**Can there exist a graph which has at least one zero eigenvalue and whose nonzero eigenvalues satisfy the reciprocal eigenvalue property?**

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**Graphs with the weak reciprocal eigenvalue property.**

Trivial example: $K_1$, $\sigma(K_1) = (0)$

Nontrivial graph?

Nontrivial tree?

**Direct Product:** Let $F$ and $H$ be two graphs with disjoint vertex sets $V(F) = \{u_1, \ldots, u_m\}$ and $V(H) = \{v_1 \ldots, v_n\}$, respectively.

The direct product of $F$ and $H$, denoted by $F \times H$, is the graph with the vertex set $V(F) \times V(H)$, and

$(u_i, v_j) \sim (u_r, v_s)$ in $F \times H$ if $u_i \sim u_r$ in $F$ and $v_j \sim v_s$ in $H$.

[1] P. M. Weichsel, The Kronecker product of graphs, *Proc. Am. Math. Soc.*, (1962).

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**Theorem (Weichsel's Theorem):** $F \times H$ is connected if and only if both $F$ and $H$ are connected and at least one of them is non-bipartite. Furthermore, if both $F$ and $H$ are connected and bipartite, then $F \times H$ has exactly two connected components.

• The direct product a bipartite graph with any bipartite graph is always bipartite.

---

[1] P. M. Weichsel, The Kronecker product of graphs, *Proc. Am. Math. Soc.*, (1962).

**Lemma**[1]**:** Let $F$ and $H$ be two graphs of order $m$ and $n$, respectively. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $F$ and $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $H$, then the eigenvalues of $F \times H$ are

$$\lambda_i \mu_j, \quad i = 1, 2, \ldots, m; \;\; j = 1, 2, \ldots, n.$$

[1] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs: Theory and application*, (1980).

[2] S. Barik, D. Mondal, and S. Pati, Trees with the reciprocal eigenvalue property, *Linear Multilinear Algebra*, (2022).

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**Lemma:** Let $F$ and $H$ be two connected graphs. If $F$ and $H$ both satisfy property (R), then $F \times H$ satisfies property (R).

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(Note: The converse is not true.)

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**Theorem** Let $G$ be a connected graph with property (R). Then $|\det A(G)| = 1$.

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- The $n^n$ eigenvalues of $H$ are of the form

$$\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}, \quad \text{where } i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}.$$

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- By RRT, $\frac{1}{m} = \pm 1 \implies m = \pm 1$.

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**Remark:** Let $G$ be a connected graph on $n$ vertices and $P(G;x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. If $G$ satisfies property (R), then $|a_n| = 1$.

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**Theorem:** Let $T$ a singular tree of order $n \geq 2$. Then its nonzero eigenvalues cannot satisfy the reciprocal eigenvalue property.

[1] S. Barik, D. Mondal and S. Pati, Trees with the reciprocal eigenvalue property, *Linear and Multilinear Algebra*, (2022).

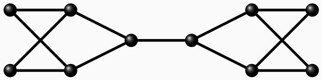
**Can there be a nontrivial graph with weak reciprocal eigenvalue property?**

Let $G$ be a singular graph with $P(G;x) = x^{n-k}(x^k + a_1x^{k-1} + \cdots + a_k)$.

Let $\text{rank}(A(G)) = k$. So, $|a_k| \neq 0$.

**Question:** Can we ever have $|a_k| = 1$?

**Example:**



$G_1$

$$P(G_1;x) = x^4\left(x^6 - 13x^4 + 44x^2 - 16\right).$$

**Theorem:** Let $G$ be a connected singular bipartite graph of order $n \geq 2$. Assume that $A(G)$ has rank $k$. Let $P(G; x) = x^{n-k}(x^k + a_1 x^{k-1} + \cdots + a_k)$ be the characteristic polynomial of $G$. Then $|a_k| \neq 1$.

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- Let $(R, C)$ be a bipartition of $V(G)$ such that $R = \{v_1, \ldots, v_p\}$, $C = \{v_{p+1}, \ldots, v_{p+q}\}$ and $p + q = n$ ($p \leq q$).

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- Then
$$A(G) = \begin{bmatrix} \mathbf{0} & B \\ B^t & \mathbf{0} \end{bmatrix},$$
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- Since $G$ is bipartite, $k$ is even. Let $k = 2r$.

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where $B$ is of order $p \times q$.
- Since $G$ is bipartite, $k$ is even. Let $k = 2r$.
- Then $P(G;x) = x^n + a_2x^{n-2} + a_4x^{n-4} + \cdots + a_{2r}x^{n-2r}$.

[1] S. Barik, D. Mondal, S. Pati, and K. Sarma, Singular graphs and the reciprocal eigenvalue property, *Discrete Math.*, (2024).

- Note that

$$P(BB^t; x) = x^p + a_2 x^{p-1} + a_4 x^{p-2} + \cdots + a_{2r} x^{p-r}$$

and $\mathrm{rank}(B) = \mathrm{rank}(BB^t) = r$.

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- Then

$$a_{2r} = (-1)^r \sum_{\substack{S \subseteq \{1,2,\ldots,p\} \\ |S| = r}} \det(BB^t[S,S]).$$

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- Using Cauchy-Binet formula,

$$a_{2r} = (-1)^r \sum_{\substack{S\subseteq\{1,2,\ldots,p\} \\ |S|=r}} \sum_{\substack{T\subseteq\{1,2,\ldots,q\} \\ |T|=r}} \det(B\,[S,T])\det(B^t\,[T,S]).$$

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$$|a_{2r}| = \sum_{\substack{S\subseteq\{1,2,\ldots,p\}\\|S|=r}} \sum_{\substack{T\subseteq\{1,2,\ldots,q\}\\|T|=r}} (\det(B\,[S,T]))^2.$$

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$$|a_{2r}| = \sum_{\substack{S\subseteq\{1,2,\ldots,p\}\\|S|=r}} \sum_{\substack{T\subseteq\{1,2,\ldots,q\}\\|T|=r}} (\det(B[S,T]))^2.$$

- There exist at least two $r\times r$ submatrices of $B$, say $B_1$ and $B_2$ such that $\det(B_1)\neq 0$ and $\det(B_2)\neq 0$.

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- Thus, $|a_{2r}| > 1$

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**Theorem:** Let $G$ be a connected singular graph of order $n \geq 2$. Assume that $A(G)$ has rank $k$. Let $P(G;x) = x^{n-k}(x^k + a_1x^{k-1} + \cdots + a_k)$ be the characteristic polynomial of $G$. Then $|a_k| \neq 1$.

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- Applying the previous result to $G \times P_2$, we have $|a_{n-k}|^2 \neq 1$ and hence $|a_{n-k}| \neq 1$.

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[1] S. Barik, D. Mondal, S. Pati, and K. Sarma, Singular graphs and the reciprocal eigenvalue property, *Discrete Math.*, (2024).

**Theorem:** Let $G$ be a connected singular graph with $n \geq 2$ vertices. Then it cannot satisfy the weak reciprocal eigenvalue property.

[1] S. Barik, D. Mondal, S. Pati, and K. Sarma, Singular graphs and the reciprocal eigenvalue property, *Discrete Math.*, (2024).

**Theorem:** Let $A$ be symmetric $n \times n$ matrix with integer entries. Suppose that $\mathrm{rank} A = k < n$ and $A$ has at least $k+1$ nonzero rows. Let $P_A(x) = x^{n-k}(x^k + a_1 x^{k-1} + \cdots + a_k)$ be the characteristic polynomial of $A$. Then $|a_k| \neq 1$.

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The pseudo-determinant $\mathrm{Det}(A)$ of a square matrix $A$ is defined as the product of the nonzero eigenvalues of $A$.

**Theorem:** Let $A$ be symmetric $n \times n$ matrix with integer entries. Suppose that $\mathrm{rank} A = k < n$ and $A$ has at least $k + 1$ nonzero rows. Then $\mathrm{Det}^2(A) \neq 1$.

[2] S. Barik, D. Mondal, S. Pati, and K. Sarma, Singular graphs and the reciprocal eigenvalue property, *Discrete Math.*, (2024).

**Does there exist a nontrivial weighted graph with the weak reciprocal eigenvalue property?**

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$w : E(G) \to (0, \infty)$ is the weight function.

Let $T_w$ be a weighted tree whose underline graph $T = P_5$ and the weight of each edge is $\frac{1}{\sqrt[4]{3}}$.

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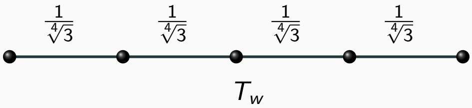

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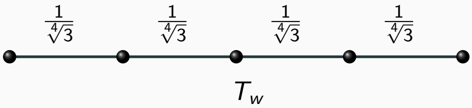


$\sigma(T_w) = \left(-\sqrt[4]{3}, -\frac{1}{\sqrt[4]{3}}, 0, \frac{1}{\sqrt[4]{3}}, \sqrt[4]{3}\right)$ and hence $T_w$ satisfies the weak reciprocal eigenvalue property.

**Theorem:** Let $G$ be a connected singular bipartite graph on $n$ vertices with $\mathrm{rank}(G) = 2$ or 4. Suppose that

$$P(G; x) = x^{n-4}(x^4 + a_2x^2 + a_4).$$

Let $G_w$ be the weighted graph obtained from $G$ by assigning weight

(i) $\frac{1}{\sqrt{-a_2}}$ to each edge if $\mathrm{rank}(G) = 2$ and

(iI) $\frac{1}{\sqrt[4]{a_4}}$ to each edge if $\mathrm{rank}(G) = 4$, respectively.

Then, $G_w$ satisfies the weak reciprocal eigenvalue property.

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$w : E(G) \to [1, \infty)$

**Theorem:** Let $T_w$ be a weighted singular tree such that the weight of each edge is at least 1. Then $T_w$ can not satisfy the weak reciprocal eigenvalue property.

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- Characterization of unicyclic graphs with property (R)

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- Characterization of graphs with the anti-reciprocal eigenvalue property

# THANK YOU