Laplacian Integral Graphs

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Outline

- Graphs determined by the spectrum and cospectral graphs
- Laplacian integral graphs
- Related problems and references

Graphs determined by the spectrum

- A graph G is said to be **determined by its spectrum (DS)** if every graph cospectral with G is necessarily isomorphic to G.

- A graph is non-DS $\implies$ Existence of cospectral graphs

Cospectral pairs

Figure: Two non-isomorphic, adjacency cospectral graphs

Spectrum: $(-2, 0, 0, 0, 2)$.

Path is determined by the spectrum

Theorem. The path on n vertices is determined by the adjacency spectrum.

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- Thus, G is a tree.

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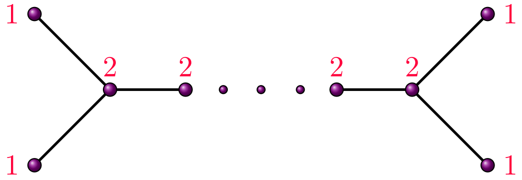
- Further, any tree with the following structure has an eigenvalue 2 (can be seen from the given eigenvector).



- Thus, G is a tree with no vertex of degree greater than 3, and at most one vertex of degree 3.

- Two graphs G and H are cospectral, then trace$(A(G)^i)$ =trace$(A(H)^i)$. So, G and H have the same number of closed walks of length i.

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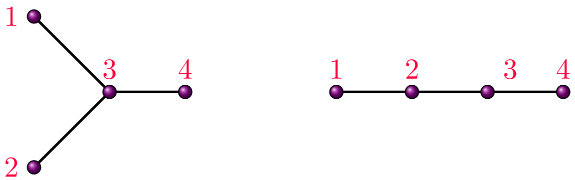
- Two graphs G and H are cospectral, then trace$(A(G)^i)$ =trace$(A(H)^i)$. So, G and H have the same number of closed walks of length i.
- Supoose v is a vertex of degree 3. Moving one branch at v to an endpoint of G, changes G to P_n.
- Since G and P_n are cospectral, this operation should not change the number of closed walks of length 4.



- But clearly the number of closed walks of length 4 are not same in G and P_n.

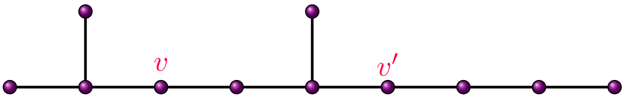
- Hence, G has no vertex of degree 3, so G is isomorphic to P_n.

Construction of cospectral graphs

Theorem (Schwenk): Let G and H be two adjacency cospectral graphs. Suppose that $v \in V(G)$ and $v' \in V(H)$ are such that the vertex-deleted subgraphs $G \setminus \{v\}$ and $H \setminus \{v'\}$ are also cospectral. Let Γ be any graph with a fixed vertex u. Then the coalescence of G and Γ with respect to v and u is cospectral with the coalescence of H and Γ with respect to v' and u.

Example:

Let $G = H$ be as given below. Then $G \setminus \{v\}$ and $H \setminus \{v'\}$ are isomorphic and hence, cospectral.



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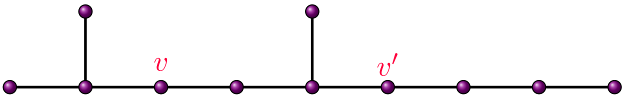

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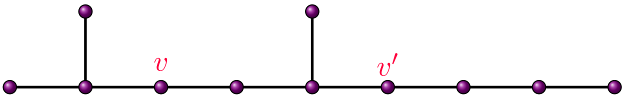

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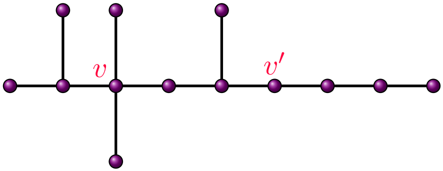

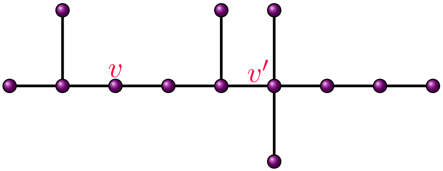

GM switching

Theorem (Schwenk). With respect to the adjacency matrix almost all trees are non-DS.

GM switching

- **Godsil and McKay (GM) switching**: A more general version of Seidel switching that gives precise conditions under which the adjacency spectrum remains unchanged.

- Although GM switching was designed for the adjacency matrix, the same idea also works for the Laplacian and signless Laplacian matrices.

[1] C. D. Godsil and B. D. McKay, Constructing cospectral graphs, *Aequationes mathematicae*, 25:257–268, (1982).

Theorem (Godsil-McKay): Let N be a $(0,1)$ matrix of size $b \times c$ whose column sums are 0, b, or $b/2$.

- Replace each column v of N with $b/2$ ones by its complement $\mathbf{1} - v$ to obtain N'.
- Let B be a symmetric $b \times b$ matrix with constant row (and column) sums.
- Let C be a symmetric $c \times c$ matrix.

Then the matrices $M = \begin{bmatrix} B & N \\ N^T & C \end{bmatrix}$ and $M' = \begin{bmatrix} B & N' \\ N'^T & C \end{bmatrix}$ are cospectral.

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Proof. Let $Q = \begin{bmatrix} \frac{2}{b}J - I_b & \mathbf{0} \\ \mathbf{0} & I_c \end{bmatrix}$. Then $M' = QMQ^{-1}$.

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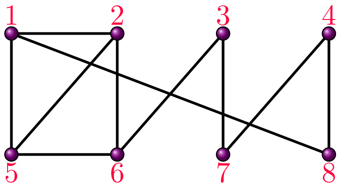
- If one wants to apply GM switching to the Laplacian matrix $L(G)$ of a graph G, define $M = -L$.

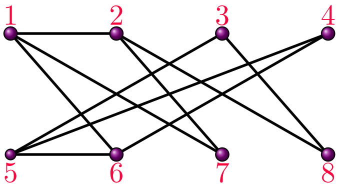
- The requirement that B has constant row sums means that N must have constant row sum, that is, the vertices of B all have the same number of neighbours in C.

Ecample:

The graph H can be obtained from the graph G through **GM switching** and are Laplacian cospectral.



G



H

$$-L(G) = \begin{bmatrix} -3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}, -L(H) = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

Union, join and complement

Theorem: Let G and H be two vertex disjoint graphs on n and m vertices respectively. Let $\sigma_L(G) = (0, \lambda_2, \ldots, \lambda_n)$ and $\sigma_L(H) = (0, \mu_2, \ldots, \mu_m)$. Then

(i) Eigenvalues of $L(G \cup H)$ are $0, 0, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m$.

(ii) Eigenvalues of $L(G \vee H)$ are
$0, \lambda_2 + m, \ldots, \lambda_n + m, \mu_2 + n, \ldots, \mu_m + n, n + m.$

(iii) Eigenvalues of $L(G^c)$ are $0, n - \lambda_2, \ldots, n - \lambda_n$.

Laplacian integral graphs

Definition A graph G is said to be Laplacian integral if the spectrum of $L(G)$ consists entirely of integers.

- The complete graph K_n is Laplacian integral for each $n \geq 1$.

$$S(K_n) = (0, n, \dots, n)$$

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- A tree is Laplacian integral if and only if it is a star S_n.
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- Both K_n and S_n are uniquely determined by the Laplacian spectrum.

Cographs

A graph is called a cograph if it is constructed using the following rules:

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- The complement of a cograph is a cograph.
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- **Cographs are Laplacian integral**.

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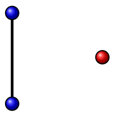

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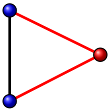

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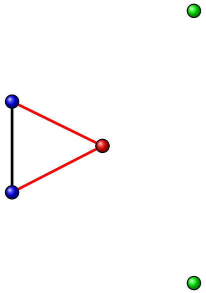

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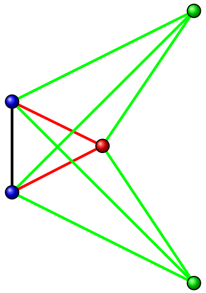

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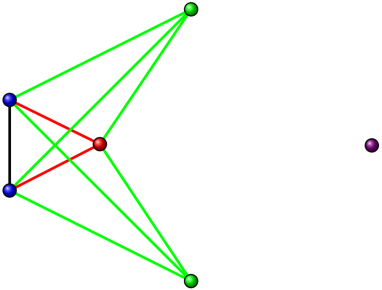

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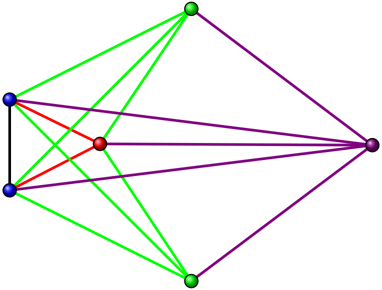

Constructing Laplacian integral graphs

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Let G be a graphs on vertices $1, \ldots, n$. Let i, j be two nonadjacent vertices of G, and let $G + e$ be the graph obtained from G by adding the edge $e = \{i, j\}$. Then

$$L(G + e) = L(G) + e_{ij}e_{ij}^T.$$

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Thus,

$$0 = \lambda_1(G) = \lambda_1(G + e) \leq \lambda_2(G) \leq \lambda_2(G + e) \leq \cdots \leq \lambda_n(G) \leq \lambda_n(G + e).$$

Constructing Laplacian integral graphs

Note that

$$\sum_{i=1}^{n} (\lambda_i(G+e) - \lambda_i(G)) = 2.$$

Thus, if G is Laplacian integral, then $G+e$ will also be Laplacian integral if either

- $n-1$ eigenvalues of $L(G)$ and $L(G+e)$ coincide and one eigenvalue of $L(G)$ increases by 2, or
- $n-2$ eigenvalues of $L(G)$ and $L(G+e)$ coincide, and two eigenvalues of $L(G)$ increase by 1 each.

Spectral integral variation

Let G be a general graph of order n, and i and j are two nonadjacent vertices in G. Let $e = \{i, j\}$. We say that

- the spectral integral variation of G occurs at one place by adding e if exactly one eigenvalue of $L(G)$ increases by 2 when the edge e is added to G.
- the spectral integral variation of G occur at two places by adding e if exactly two eigenvalues of $L(G))$ increase by 1 each when e is added to G.

Spectral integral variation at one place

Theorem (Fan). Let $G = (V, E)$ be a connected graph of order n, $e = \{i, j\}$, $i \neq j$, be an edge not in G. Then the following conditions are equivalent.

- The spectral integral variation of G occurs in one place by adding e.
- One eigenvector of $L(G)$ is the Faria vector e_{ij}.
- $N(i) = N(j)$.

- d_i is a Laplacian eigenvalue of G and d_i increases to $d_i + 2$.

Laplacian integral graphs

Observation Let G be a connected graph of order n and $\sigma_L(G) = (0, \lambda_2, \ldots, \lambda_n)$. Then $1 \leq \lambda_i \leq n$.

- **Question-1** Characterize the graphs with n as Laplacian eigenvalue.
- **Question-2** Characterize the graphs with 1 as Laplacian eigenvalue.

Answer to Question 1

Theorem

Let G be connected graph on n vertices. Then n is an eigenvalue of $L(G)$ if and only if G is the join of two graphs.

Question 2 is still open for graphs.

Graphs with distinct integer Laplacian eigenvalues

Consider the set of integers

$$\{0, 1, 2, \ldots, n\}.$$

Let S be a subset of $\{0, 1, 2, \ldots, n\}$ such that $|S| = n$.

We say S is Laplacian realizable if there is a graph G on n vertices whose Laplacian specrum is S.

For $n \geq 2$ and $1 \leq i \leq n$, let

$$S_{i,n} = \{0, 1, 2, \ldots, i-1, i+1, \ldots, n\}.$$

Graphs with distinct integer Laplacian eigenvalues

Question-4 Does there exist a graph with the Laplacian spectrum $S_{i,n}$.? That is, whether $S_{i,n}$ is Laplacian realizable for each i, $1 \leq i \leq n$?

If yes, the which are those graphs?

[1]S. M. Fallat, S. J. Kirkland, J. J. Molitierno, and M. Neumann, On graphs whose Laplacian matrices have distinct integer eigenvalues, *Journal of Graph Theory*, 50:162–174, (2005).

Examples

- K_2, with Laplacian spectrum $S_{1,2} = \{0, 2\}$.
- P_3, with Laplacian spectrum $S_{2,3} = \{0, 1, 3\}$.
- $K_1 \vee (K_1 \cup K_1)$, with Laplacian spectrum $S_{2,4} = \{0, 1, 3, 4\}$.
- $(K_1 \cup K_1) \vee (K_1 \cup K_2)$, with spectrum $S_{1,5} = \{0, 2, 3, 4, 5\}$.
- $K_1 \vee (K_1 \cup P_3)$, with spectrum $S_{3,5} = \{0, 1, 2, 4, 5\}$.

$S_{i,n}$ is Laplacian realizable if $i \neq n$

Theorem: Suppose that G is a graph on $n \geq 6$ vertices. Then G realizes $S_{1,n}$ if and only if G is formed in one of the following two ways:

(i) $G = (K_1 \cup K_1) \vee (K_1 \cup G_1)$, where G_1 is a graph on $n-3$ vertices that realizes $S_{n-4,n-3}$;

(ii) $G = K_1 \vee H$, where H is a graph on $n-1$ vertices that realizes $S_{n-1,n-1}$.

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Theorem Suppose that G is a graph on $n \geq 6$ vertices. Then

(a) G realizes $S_{n-1,n}$ if and only if G is formed in one of the following two ways:

(i) $G = K_1 \vee (K_1 \cup G_1)$, where G_1 is a graph on $n-3$ vertices that realizes $S_{2,n-3}$;

(ii) $G = K_1 \vee (K_1 \cup H)$, where H is a graph on $n-2$ vertices that realizes $S_{n-2,n-2}$.

(b) For $2 \leq i \leq n-2$, G realizes $S_{i,n}$ if and only if $G = K_1 \vee (K_1 \cup H^*)$, where H^* is a graph on $n-2$ vertices that realizes $S_{i-1,n-2}$.

[1] S. M. Fallat, S. J. Kirkland, J. J. Molitierno, and M. Neumann, On graphs whose Laplacian matrices have distinct integer eigenvalues, *Journal of Graph Theory*, 50:162–174, (2005).

Laplacian realizability of $S_{n,n}$

$$\mathbf{S_{n,n} = \{0, 1, 2, \ldots, n-1\}}.$$

If a graph G exists that realizes $S_{n,n}$, then the characteristic polynomial of $L(G)$ is the falling factorial

$$P(x) = x(x-1)(x-2)(x-3)\cdots(x-(n-1)),$$

whose coefficients are the Stirling numbers of the first kind.

Laplacian realizability of $S_{n,n}$

Conjecture 1: The set $S_{n,n}$ is not Laplacian realizable for any $n \geq 2$.

Conjecture 2: For $n \geq 2$ and for each admissible i where $1 \leq i \leq n-1$, the set $S_{i,n}$ is realized by a unique graph.

[1] S. M. Fallat, S. J. Kirkland, J. J. Molitierno, and M. Neumann, On graphs whose Laplacian matrices have distinct integer eigenvalues, *Journal of Graph Theory*, 50:162–174, (2005).

Progress on the conjecture 1

Theorem The Conjecture 1 is true for $n \leq 11$.

Theorem The Conjecture 1 is true for $n \geq 6,649,688,933$.

[1] S. M. Fallat, S. J. Kirkland, J. J. Molitierno, and M. Neumann, On graphs whose Laplacian matrices have distinct integer eigenvalues, *Journal of Graph Theory*, 50:162–174, (2005).

[2] A. Goldberger and M. Neumann, On a conjecture on a Laplacian matrix with distinct integral spectrum, *Journal of Graph Theory*, 72(2):178-208, (2013).

Necessary conditions for graphs realising Sn, n

- If $S_{n,n}$ is Laplacian realizable, then n is not a prime.
- If $S_{n,n}$ is Laplacian realizable, then $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$.
- If $S_{n,n}$ is Laplacian realizable, then $2 \leq \delta(G) \leq \Delta(G) \leq n - 3$.
- If $S_{n,n}$ is Laplacian realizable, then both G and G^c have diameter 3.

[1] S. M. Fallat, S. J. Kirkland, J. J. Molitierno, and M. Neumann, On graphs whose Laplacian matrices have distinct integer eigenvalues, *Journal of Graph Theory*, 50:162–174, (2005).

[2] K. C. Das, S.-G. Lee, and G.-S. Cheon, On the conjecture for certain Laplacian integral spectrum of graphs, *Journal of Graph Theory*, 63:106-113, (2010).

Laplacian integral graphs

Consider

$$\begin{array}{ccccc} & & 0 & & \\ & 1 & & 1 & \\ 2 & & 2 & & 2 \\ & 3 & & 3 & \\ & & 4 & & \end{array}$$

Let $S_4 = \{0, 1, 1, 2, 2, 2, 3, 3, 4\}$.

Does there exist a graph on 9 vertices that realizes S_4?

Laplacian integral graphs

Consider

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 2 & & & \\
 & & \vdots & & & \vdots & & & \vdots & & \\
 & n-2 & & n-2 & & \dots & & n-2 & & n-2 & \\
n-1 & & n-1 & & & \dots & & & n-1 & & n-1 \\
 & n & & n & & \dots & & n & & n & \\
 & & \vdots & & & \vdots & & & \vdots & & \\
 & & & 2n-4 & & 2n-4 & & 2n-4 & & & \\
 & & & & 2n-3 & & 2n-3 & & & & \\
 & & & & & 2n-2 & & & & &
\end{array}
$$

Laplacian integral graphs

In general consider the set

$$S_{2n-2} = \{0, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4 \ldots, 2n{-}4, 2n{-}4, 2n{-}4, 2n{-}3, 2n{-}3, 2n{-}2\}.$$

Does there exist a graph on n^2 vertices that realizes S_{2n-2}?

References

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4. S. Kirkland, A characterization of spectral integral variation in two places for laplacian matrices, *Linear and Multilinear Algebra*, 52(2):79-98, (2004).
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THANK YOU