Algebraic Connectivity of Graphs

Sasmita Barik

Indian Institute of Technology Bhubaneswar

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Outline

- Algebraic connectivity
- Fiedler vectors and graph structure
- Spectral drawing
- Related problems and references

Union, join, and complement

Theorem Let G and H be two vertex disjoint graphs on n and m vertices respectively. Let $S(G) = (0, \lambda_2, \ldots, \lambda_n)$ and $S(H) = (0, \mu_2, \ldots, \mu_m)$. Then

(i) Laplacian eigenvalues of $G \cup H$ are $0, 0, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m$.

(ii) Laplacian eigenvalues of $G \vee H$ are
$0, \lambda_2 + m, \ldots, \lambda_n + m, \mu_2 + n, \ldots, \mu_m + n, n + m$.

(iii) Laplacian eigenvalues of G^c are $0, n - \lambda_2, \ldots, n - \lambda_n$.

Proof. Let $L(G)x_i = \lambda_i x_i$ and $L(H)y_j = \mu_j y_j$.

- $L(G \cup H) = \begin{bmatrix} L(G) & \mathbf{0} \\ \mathbf{0} & L(H) \end{bmatrix}$. Choose vectors $\begin{bmatrix} x_i \\ \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{0} \\ y_j \end{bmatrix}$.

- $L(G \vee H) = \begin{bmatrix} L(G) + mI_n & -J_{n\times m} \\ -J_{m\times n} & L(H) + nI_m \end{bmatrix}$. Choose vectors $\mathbf{1}_{m+n}$, $\begin{bmatrix} x_i \\ \mathbf{0} \end{bmatrix}$ $(i = 2, \ldots, n)$, $\begin{bmatrix} \mathbf{0} \\ y_j \end{bmatrix}$ $(j = 2, \ldots, m)$, and $\begin{bmatrix} n\mathbf{1}_n \\ -m\mathbf{1}_m \end{bmatrix}$.

- $L(G) + L(G^c) = L(K_n)$

Algebraic connectivity

Lemma For a disconnected graph G, the multiplicity of 0 as an eigenvalue of $L(G)$ is equal to the number of connected components in G.

Lemma $\lambda_n = n$ if and only if G^c is disconnected.

[1] M. Fiedler, Algebraic connectivity of graphs, *Czechoslov. Math. J.*, 23(2):298-305 (1973)

Algebraic connectivity

Theorem If T is a tree, then $a(T) \leq 1$, with equality if and only if T is a star.

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Thus, $a(T) \leq 1$.

- $S(S_n) = (0, 1, \ldots, 1, n)$.

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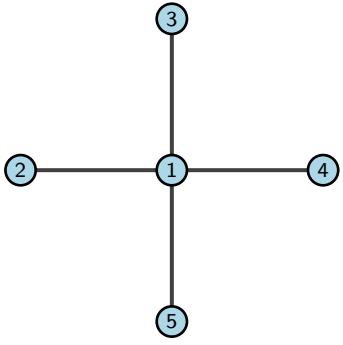
Algebraic connectivity and graph structure

Theorem If T is a tree on n vertices, then $a(P_n) \leq a(T) \leq a(S_n) = 1$.

P_5



S_5

Figure: $a(P_5) = 0.3820, \quad a(S_5) = 1.$

Fiedler vectors and graph structure

Theorem Let G be a graph on n vertices and $\delta(G)$ be the minimum vertex degree in G. Then

$$a(G) \leq \frac{n}{n-1}\delta(G).$$

Fiedler vectors and graph structure

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Then X is indexed by $V(G)$ and thus it gives a labeling of $V(G)$. We label vertex i by x_i. We call vertex i positive, negative or zero according as $x_i > 0$, $x_i < 0$ or $x_i = 0$, respectively.

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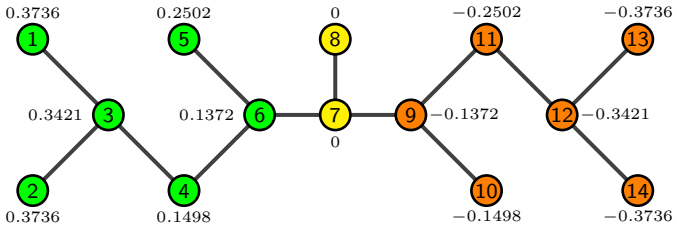
Let

$$V^+ = \{i : x_i \geq 0\}, V^- = \{i : x_i \leq 0\}.$$

Fiedler vectors and graph structure

$$X = \begin{pmatrix} 0.3736 \\ 0.3736 \\ 0.3421 \\ 0.2502 \\ 0.1498 \\ 0.1372 \\ 0 \\ 0 \\ -0.1372 \\ -0.1498 \\ -0.2502 \\ -0.3421 \\ -0.3736 \\ -0.3736 \end{pmatrix}$$



$$a(T) = 0.0844, \quad V^+ = \{1, 2, 3, 4, 5, 6, 7, 8\}, \quad V^- = \{7, 8, 9, 10, 11, 12, 13, 14\}.$$

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$$V^+ = \{1, 2, \ldots, r\}.$$

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Then we may partition $L(G)$ as

$$L(G) = \begin{bmatrix} L_{11} & \mathbf{0} & L_{13} \\ \mathbf{0} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}.$$

Fiedler vectors and graph structure

Partition X conformally and consider the equation

$$\begin{bmatrix} L_{11} & \mathbf{0} & L_{13} \\ \mathbf{0} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = a(G) \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

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Then

$$L_{11}X_1 + L_{13}X_3 = a(G)X_1.$$

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Then

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Since $L_{13} \leq \mathbf{0}$ and $X_3 < \mathbf{0}$, we have $L_{13}X_3 \geq \mathbf{0}$. Since G is connected, L_{13} has a nonzero entry and hence $L_{13}X_3 \neq \mathbf{0}$.

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Thus,

$$(L_{11} - a(G)I)X_1 \leq \mathbf{0} \text{ and } (L_{11} - a(G)I)X_1 \neq \mathbf{0} \implies X_1^T(L_{11} - a(G)I)X_1 \leq 0.$$

Now, if $L_{11} - a(G)I$ is positive semidefinite, then $X_1^T(L_{11} - a(G)I)X_1 \geq 0$. Thus, we have

$$X_1^T(L_{11} - a(G)I)X_1 = 0 \implies (L_{11} - a(G)I)X_1 = 0,$$

a contradiction.

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Hence, if $L_{11} - a(G)I$ is not positive semidefinite.

Thus, L_{11} has an eigenvalue less than $a(G)$.

A similar argument shows that L_{22} has an eigenvalue less than $a(G)$.

Fiedler vectors and graph structure

Thus, the second smallest eigenvalue μ of $\begin{bmatrix} L_{11} & \mathbf{0} \\ \mathbf{0} & L_{22} \end{bmatrix}$ is less than $a(G)$.

[1] Cauchy interlacing theorem: Let A be a symmetric $n \times n$ matrix and let B be a principal submatrix of A of order $n-1$. If $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_{n-1}$ are the eigenvalues of A and B, respectively, then

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Therefore, the subgraph induced by V^+ is connected. It can similarly be proved that the subgraph induced by V^- is also connected. ∎

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Classification of trees

Let T be a tree on vertices $1, 2, \ldots, n$. Let X be a Fiedler vector of T.

Case 1: First, suppose that X has no zero coordinate. Define

$$V^+ = \{i : x_i > 0\}, V^- = \{i : x_i < 0\}.$$

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Then there must be precisely **one edge** such that one of its end-vertices is positive and the other negative. Such an edge is called a **characteristic edge** (with respect to X).

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Any other edge has either both its end-vertices positive or both negative.

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- It is clear that a pendant vertex cannot be a characteristic vertex.

Classification of trees

Theorem Let T be a tree on vertices $1, 2, \ldots, n$. Let X be a Fiedler vector of T and suppose n is a characteristic vertex. Let $T_1, \ldots, T_k$ be the components of $T \setminus \{n\}$. Then for any $j = 1, \ldots, k$, the vertices of $V(T_j)$ are either all positive, all negative or all zero.

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Corollary Let T be a tree on vertices $1, 2, \ldots, n$. Let X be a Fiedler vector of T.Then T has at most one characteristic vertex with respect to X.

Proof. Suppose $i \neq j$ are both characteristic vertices. Then $x_i = x_j = 0$. By the above theorem, all vertices of the components of $T \setminus \{i\}$ that contains j are zero vertices. Then j cannot be connected to a nonzero vertex. ∎

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Proof. Let $L(T)Y = a(T)Y$ where $y_i \neq 0$, for $i = 1, \ldots, n$. Let $E =$diag$(y_1, \ldots, y_n)$ and $C = E(L(T) - a(T)I)E$.

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Then $G(C)$ is a tree and $C\mathbf{1} = \mathbf{0}$. Thus, rank$(C) = n - 1$.

Then rank$(L(T) - a(T)I) = n - 1$, and hence $a(T)$ has multiplicity 1.

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- A tree is called **Type II** if it has a characteristic edge with respect to a Fiedler vector.
- Every tree must be one of the two types, not both.
- If $a(G)$ has multiplicity more than 1, then it is necessarily of Type I. But, the converse is not true.

Monotonicity of Fiedler vectors

Theorem (Fiedler): Let T be a tree on vertex set $\{1, 2, \ldots, n\}$. Let Y be a Fiedler vector of T. Then one of the following cases occur.

(i) No component of Y is zero (T is of Type II). In this case, there is a unique edge $\{u, v\}$ such that $Y(u) > 0$ and $Y(v) < 0$. Further, along any path in T that starts at u and does not contain v, the entries of Y increase, while along any path in T that starts at v and does not contain u, the entries of Y decrease.

(ii) Some component of Y is zero (T is of Type I). In this case, the subgraph of T induced by the zero vertices (the vertices for which the components of Y are zero) is connected. There is a unique vertex v such that $Y(v) = 0$ and v is adjacent to a nonzero vertex. Further, along any path in T that starts at v, the entries of Y either increase or decrease.

Monotonicity of Fiedler vectors

$$Y = \begin{pmatrix} 0.3681 \\ 0.3387 \\ 0.2821 \\ 0.2029 \\ 0.2206 \\ 0.2206 \\ 0.0722 \\ -0.0643 \\ -0.0699 \\ -0.1901 \\ -0.3006 \\ -0.3268 \\ -0.3609 \\ -0.3923 \end{pmatrix}$$

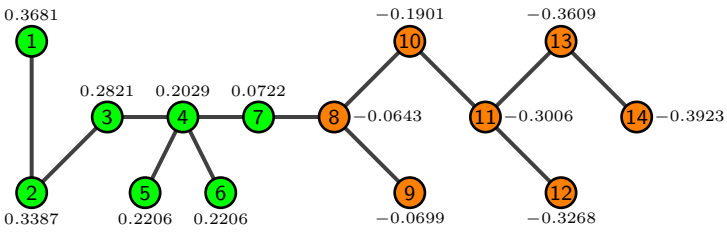


Figure: A tree T on 14 vertices with $a(T) = 0.0801$.

Monotonicity of Fiedler vectors

$$Y = \begin{pmatrix} 0.3736 \\ 0.3736 \\ 0.3421 \\ 0.2502 \\ 0.1498 \\ 0.1372 \\ 0 \\ 0 \\ -0.1372 \\ -0.1498 \\ -0.2502 \\ -0.3421 \\ -0.3736 \\ -0.3736 \end{pmatrix}$$

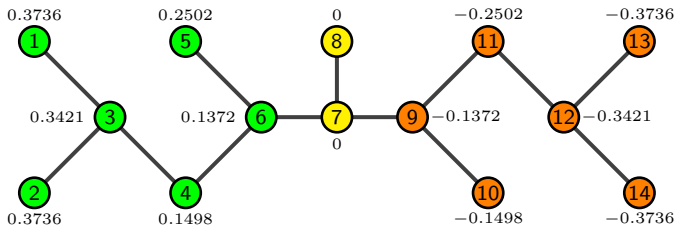


Figure: A tree T on 14 vertices with $a(T) = 0.0844$.

Applications

Embedding graph in plane

Consider eigenvectors $(Y,\ Y \perp \mathbb{1})$ of $L(G)$ corresponding to the nonzero eigenvalues. Using them best possible embedding can be made.

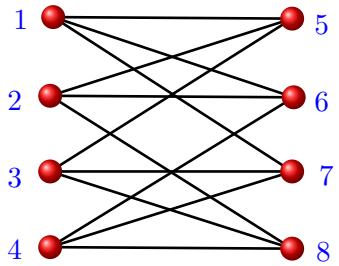


Figure: Graph G

Spectral graph drawing

$$S(G) = (0, 2, 2, 2, 4, 4, 4, 6).$$

Eigenvectors:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -0.1524 \\ 0.5585 \\ -0.4061 \\ 0.4061 \\ -0.5585 \\ 0.1524 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.2759 \\ -0.4052 \\ 0.2228 \\ 0.4583 \\ -0.4583 \\ -0.2228 \\ 0.4052 \\ 0.2759 \end{pmatrix}, \begin{pmatrix} 0.5467 \\ -0.4331 \\ -0.1162 \\ 0.0027 \\ -0.0027 \\ 0.1162 \\ 0.4331 \\ -0.5467 \end{pmatrix}, \begin{pmatrix} 0 \\ -0.4455 \\ -0.0953 \\ 0.5408 \\ 0.5408 \\ -0.0953 \\ -0.4455 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.5675 \\ 0.3272 \\ -0.0249 \\ 0.2652 \\ 0.2652 \\ -0.0249 \\ 0.3272 \\ -0.5675 \end{pmatrix}, \begin{pmatrix} 0.2302 \\ 0.2635 \\ -0.6044 \\ 0.1106 \\ 0.1106 \\ -0.6044 \\ 0.2635 \\ 0.2302 \end{pmatrix}, \begin{pmatrix} -0.3536 \\ -0.3536 \\ -0.3536 \\ -0.3536 \\ 0.3536 \\ 0.3536 \\ 0.3536 \\ 0.3536 \end{pmatrix}.$$

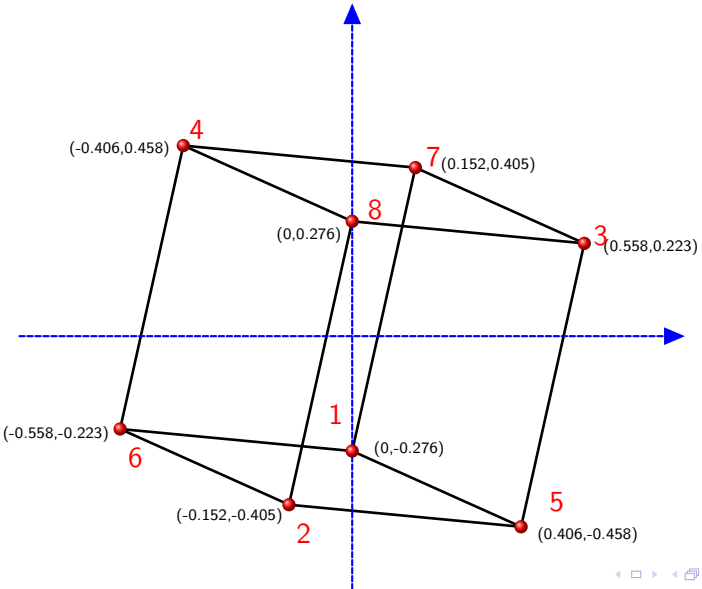
Spectral graph drawing

Take the two eigenvectors corresponding to $a(G) = 2$.

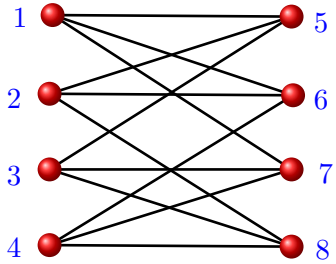
$$\begin{pmatrix} 0 \\ -0.1524 \\ 0.5585 \\ -0.4061 \\ 0.4061 \\ -0.5585 \\ 0.1524 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -0.2759 \\ -0.4052 \\ 0.2228 \\ 0.4583 \\ -0.4583 \\ -0.2228 \\ 0.4052 \\ 0.2759 \end{pmatrix}.$$

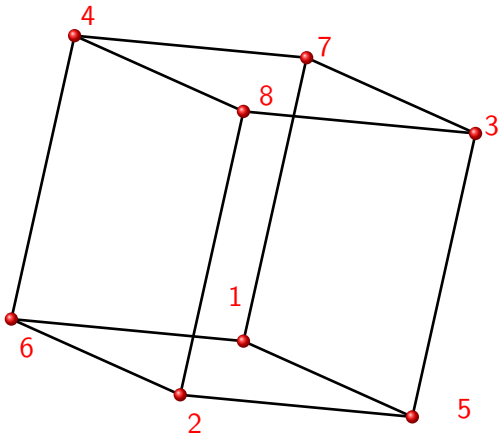
Take the 8 points
$(0, -0.2759), (-0.1524, -0.4052), (0.5585, 0.2228), (-0.4061, 0.4583),$
$(0.4061, -0.4583), (-0.5585, -0.2228), (0.1524, 0.4052), (0, 0.2759)$ and plot in xy-plane.

Spectral drawing of the graph G



Spectral drawing of the graph G





Home work

Problem 1. Let P_n be the path with n vertices, where $n \geq 3$ is odd. Show that the central vertex is a characteristic vertex.

Problem 2. Let G be a connected graph and let X be a Fiedler vector. If $x_i > 0$, then show that there exists a vertex $j \sim i$ such that $x_i > x_j$.

Problem 3. Is it true that the algebraic connectivity necessarily decreases when a vertex is deleted?

References

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THANK YOU