# Laplacian Matrix of Graphs and the Matrix-Tree Theorem

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# Outline

- Laplacian matrix
- Laplacian eigenvalues
- Matrix-tree theorem
- Related problems and references

# Laplacian matrix

Let $G$ be a graph on vertices $1, 2, \ldots, n$. The **Laplacian matrix** of $G$ is defined as $L(G) = [l_{ij}]$, where

$$l_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

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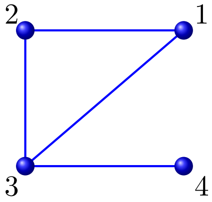


$G$

$$L(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

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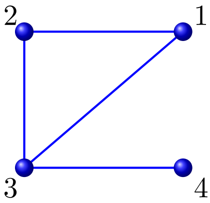


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$$L(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

# Incidence matrix

The vertex-edge **incidence matrix** of an oriented graph of $G$ is defined as an $n \times m$ matrix $Q(G) = [q_{ij}]$, where

$$q_{ij} = \begin{cases} 1 & \text{if } e_j \text{ originates at } i, \\ -1 & \text{if } e_j \text{ terminates at } i, \\ 0 & \text{otherwise.} \end{cases}$$

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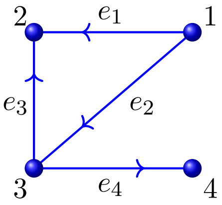


$G$

$$Q(G) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

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• $L(G)$ is positive semidefinite and $0$ is one eigenvalue of $L(G)$ with corresponding eigenvector $\mathbf{1}$.

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**Proof.** Suppose that $x$ is a vector in the left null space of $Q(G)$. Then $x_i - x_j = 0$ whenever $i \sim j$. It follows that $x_i = x_j$ if there is a path between $i$ and $j$.

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$$x^T Q(G) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 & x_1 - x_3 & x_3 - x_2 & x_3 - x_4 \end{bmatrix}$$

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Also, as $Q(G)$ has $n$ rows and they are linearly dependent, therefore rank $Q(G) \leq n - 1$. Thus, rank$Q(G) = n - 1$. ■

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Let $G_1, \ldots, G_k$ be the connected components of $G$. Then

$$Q(G) = \begin{bmatrix} Q(G_1) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & Q(G_2) & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & Q(G_k) \end{bmatrix}.$$

$$\mathrm{rank}(Q(G)) = \mathrm{rank}(Q(G_1)) + \mathrm{rank}(Q(G_2)) + \cdots + \mathrm{rank}(Q(G_k))$$

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For $k = 1$, the statement holds. Assume that the statement holds for $k - 1$ and consider a $k \times k$ submatrix $B$ of $Q(G)$.

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If each column of $B$ has a $1$ and a $-1$, then $\det(B) = 0$. Also, if $B$ has a zero column, then $\det(B) = 0$.

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Suppose that $B$ has a column with only one nonzero entry, which must be $\pm 1$. Expand the determinant of $B$ along that column and use induction hypothesis to conclude that $\det B$ must be $0$ or $\pm 1$. ∎

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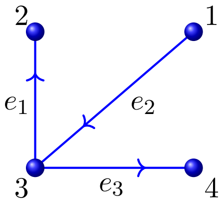


$G$

$$Q(G) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \\ \hline 0 & 0 & -1 \end{array}\right]$$

Thus, if $Y$ denotes the submatrix of $Q(G)$ formed by the rows $1, \dots, n-2, n$, then $\det X = \det Y$. Thus, if $\det X = 0$, then $\det Y = 0$.

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Continuing this way we can show that if $\det X = 0$ then each $(n-1) \times (n-1)$ submatrix of $Q(G)$ must be singular. In fact, if any one of the $(n-1) \times (n-1)$ submatrices of $Q(G)$ is singular, then all of them must be so.

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But, rank$Q(G) = n - 1$. Thus, at least one of the $(n-1) \times (n-1)$ submatrices of $Q(G)$ must be nonsingular. ∎

# Properties of the Laplacian matrix

**Lemma 4.** Let $G$ be a graph with $V(G) = \{1, \dots, n\}$ and $E(G) = e_1, \dots, e_m$. Then the following statements hold.

(i) $L(G)$ is a symmetric, positive semidefinite matrix.

(ii) If $G$ has $k$ connected components, then rank $L(G) = n - k$.

(iii) For any vector $x$,

$$x^T L(G) x = \sum_{i \sim j} (x_i - x_j)^2.$$

(iv) The row and column sums of $L(G)$ are zero.

(v) The cofactors of any two elements of $L(G)$ are equal.

**Proof of (iii).**

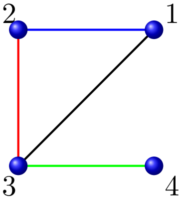
$$x^T L(G)x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 - x_2 - x_3 & -x_1 + 2x_2 - x_3 & -x_1 - x_2 + 3x_3 - x_4 & -x_3 + x_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= 2x_1^2 - x_1x_2 - x_1x_3 - x_1x_2 + 2x_2^2 - x_2x_3 - x_1x_3 - x_2x_3 + 3x_3^2 - x_3x_4 - x_3x_4 + x_4^2$$

# Properties of the Laplacian matrix

$$
\begin{aligned}
&= x_1^2 - 2x_1x_2 + x_2^2 + x_1^2 - 2x_1x_3 + x_3^2 + x_2^2 - 2x_2x_3 + x_3^2 + x_3^2 - 2x_3x_4 + x_4^2 \\
&= (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2.
\end{aligned}
$$

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# Properties of the Laplacian matrix

**Proof of (v).** Note that all the row and column sums of $L = L(G)$ are zero. Let $L(i|j) :=$ the matrix obtained by deleting row $i$ and column $j$ of $L$.

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**Proof of (v).** Note that all the row and column sums of $L = L(G)$ are zero. Let $L(i|j) :=$ the matrix obtained by deleting row $i$ and column $j$ of $L$.

$$\det(L(1|1)) = \det \begin{bmatrix} l_{22} & l_{23} & \dots & l_{2n} \\ l_{32} & l_{33} & \dots & l_{3n} \\ \vdots & \ddots & \vdots & \\ l_{n2} & l_{n3} & \dots & l_{nn} \end{bmatrix} = \det \begin{bmatrix} l_{22} + \cdots + l_{2n} & l_{23} & \dots & l_{2n} \\ l_{32} + \cdots + l_{3n} & l_{33} & \dots & l_{3n} \\ \vdots & \ddots & \vdots & \\ l_{n2} + \cdots + l_{nn} & l_{n3} & \dots & l_{nn} \end{bmatrix}$$

$$= \det \begin{bmatrix} -l_{21} & l_{23} & \dots & l_{2n} \\ -l_{31} & l_{33} & \dots & l_{3n} \\ \vdots & \ddots & \vdots & \\ -l_{n1} & l_{n3} & \dots & l_{nn} \end{bmatrix} = \det(L(1|2))$$

A similar argument shows that the cofactor of $l_{ij}$ equals that of $l_{ik}$, for any $i, j, k$. ∎

# Laplacian spectrum

$S(G) = \left(\lambda_1, \lambda_2, \ldots, \lambda_n\right)$, where $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $L(G)$.

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The complete graph $K_n$ which may be regarded as "highly connected" has

$$\lambda_2 = \cdots = \lambda_n = n.$$

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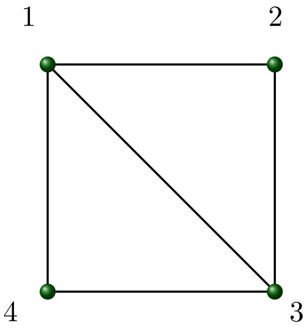
**Theorem 2.** The number of spanning trees of a graph $G$ equals

$$\frac{\lambda_2 \lambda_3 \cdots \lambda_n}{n}.$$

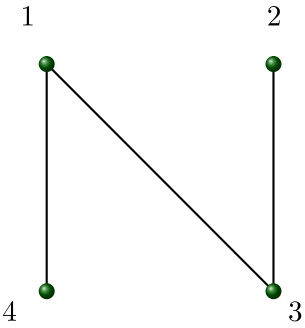
# Laplacian spectrum



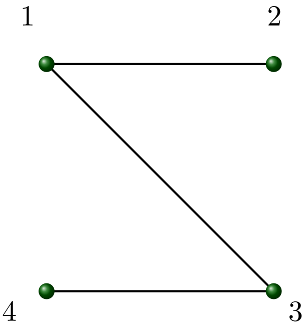

$G$

$S(G) = (0, 2, 4, 4)$. The number of spanning trees$= \frac{1}{4}.2.4.4 = 8$.

# Spanning trees

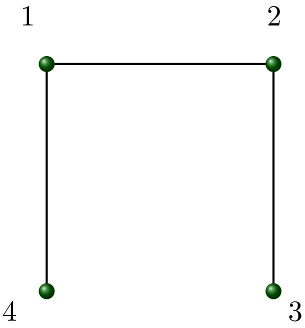



$T_1$

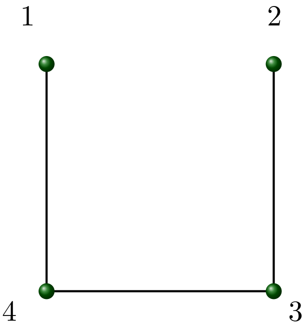



$T_2$

# Spanning trees




$T_3$




$T_4$

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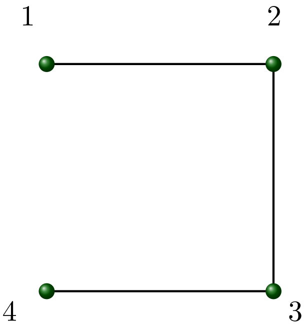
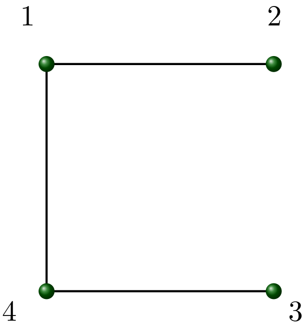
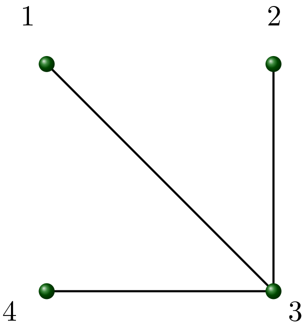



$T_5$

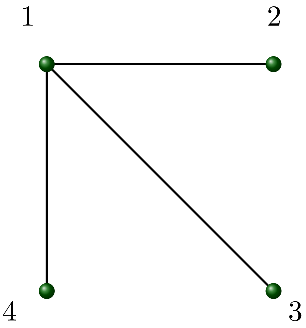



$T_6$

# Spanning trees




$T_7$




$T_8$

# Proof of matrix-tree theorem

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\begin{aligned}
&\det(L(1|1)) \\
&= \det((QQ^T)(1|1)) \\
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**Example:** Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \\ 4 & 5 \end{bmatrix}$, then

$$
\det(AB) = \det \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \det \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}
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Notice that $\det Q[2, 3 \ldots, n|Z]$ is nonsingular if and only if the corresponding edges form a tree. Hence, there is a one-to-one correspondence between nonsingular submatrices of $Q$ of size $n-1$ and spanning trees of $G$. ■

# The number of spanning trees

**Theorem.** The number of spanning trees of a graph $G$

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$$\det L(1|1) + \det L(2|2) + \cdots + \det L(n|n) = \lambda_1 \cdots \lambda_{n-1} + \cdots + \lambda_2 \cdots \lambda_n.$$

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$$\det L(1|1) + \det L(2|2) + \cdots + \det L(n|n) = \lambda_1 \cdots \lambda_{n-1} + \cdots + \lambda_2 \cdots \lambda_n.$$

Thus,

$$n \times t(G) = \lambda_2 \cdots \lambda_n.$$

# The number of spanning trees

**Theorem.** The number of spanning trees of a graph $G$

$$t(G) = \frac{\lambda_2\lambda_3\cdots\lambda_n}{n}.$$

**Proof.** Recall that the sum of the principal minors of $L(G)$ of order $n-1$ equals the sum of the products of the eigenvalues, taken $n-1$ at a time.

$$\det L(1|1) + \det L(2|2) + \cdots + \det L(n|n) = \lambda_1\cdots\lambda_{n-1} + \cdots + \lambda_2\cdots\lambda_n.$$

Thus,

$$n \times t(G) = \lambda_2\cdots\lambda_n.$$

Since $\lambda_1 = 0$, the sum of the products of the eigenvalues, taken $n-1$ at a time, equals $\lambda_2\cdots\lambda_n$. ∎

# Generalization

A generalization of the matrix-tree theorem was obtained by Kel'mans which gives a combinatorial interpretation to all the coefficients of $\phi(L(G); x)$ in terms of the numbers of certain subforests of the graph.

**Theorem.** Let $G$ be a graph on $n$ vertices and the characteristic polynomial of $L(G)$ be

$$\phi(L(G); x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x.$$

Then

$$c_i = (-1)^i \sum_{\substack{S \subset V(G) \\ |S| = n-i}} t(G_s),$$

where $t(G_s)$ is the number of spanning trees of $H$, and $G_S$ is obtained from $G$ by identifying all vertices of $S$ to a single vertex.

# Home work

**Problem 1.** Let $G$ be a graph on $n$ vertices and $L$ be the Laplacian matrix of $G$. Then show that

$$t(G) = \frac{1}{n^2}\det(L + J),$$

where $J$ is the $n \times n$ matrix with all entries equal to 1.

**Problem 2.** Let $G$ be a graph on $n$ vertices and $m$ edges. Then show that

$$t(G) \leq \frac{1}{n}\left(\frac{2m}{n-1}\right)^{n-1}.$$

# Brouwer's Conjecture

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Let the Laplacian eigenvalues of $G$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$.

For $k \in \{1, \ldots, n\}$, let

$$S_k(G) = \sum_{i=1}^{k} \lambda_i(G).$$

**Conjugate degrees:** $d_k^* = \big|\{v \in V(G) : d_v \geq k\}\big|$

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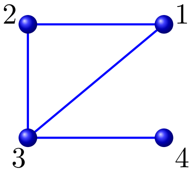


$d(G) = (d_1 = 2, d_2 = 2, d_3 = 3, d_4 = 1)$. $d_1^* = 4, d_2^* = 3, d_3^* = 1, d_4^* = 0, \ldots$

# Brouwer's Conjecture

**Grone-Merris[1] Conjecture:** For any graph $G$ on $n$ vertices and for each $k \in \{1, \dots, n\}$

$$S_k(G) \leq \sum_{i=1}^{k} d_i^*(G)$$

It is currently known as the **Grone-Merris-Bai** theorem.

**Conjecture [Brouwer[3], 2012]:** For any graph $G$ on $n$ vertices and for each $k \in \{1, \dots, n\}$

$$S_k(G) \leq |E(G)| + \binom{k+1}{2}.$$

[1] R. Grone and R. Merris, The Laplacian spectrum of a graph II, *SIAM J. Discrete Math.*, 7:221–229, (1994).
[2] H. Bai, The Grone-Merris conjecture, *Trans. Amer. Math. Soc.*, 363:4463–4474, (2011).
[3] A. E. Brouwer and W. H. Haemers, Spectra of graphs, *Springer-Verlag*, New York, 2012.

# Brouwer's conjecture

- $\lambda_1 \leq n$. Thus, the conjecture is true for $k = 1$ for any connected graph.

[1] W. H. Haemers, A. Mohammadian, and B. Tayfeh-Rezaie, On the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, 432:2214–2221, (2010).

[2] X. Chen, On Brouwer's conjecture for the sum of k largest Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, 578:402–410, (2019).

# Brouwer's conjecture

- $\lambda_1 \le n$. Thus, the conjecture is true for $k = 1$ for any connected graph.
- $S_{n-1} = S_n = 2e(G) \le e(G) + \binom{n}{2}$. Thus, the conjecture is true for $k = n$ and $k = n - 1$.

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- When $k = 2$, **Haemers et al.**[1] proved that the conjecture holds.

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- $\lambda_1 \le n$. Thus, the conjecture is true for $k = 1$ for any connected graph.
- $S_{n-1} = S_n = 2e(G) \le e(G) + \binom{n}{2}$. Thus, the conjecture is true for $k = n$ and $k = n - 1$.
- When $k = 2$, **Haemers et al.**[1] proved that the conjecture holds.
- **Chen**[2] showed that if the conjecture holds for all graphs when $k = p$ $(1 \le p \le \frac{n-1}{2})$, then it also holds for $k = n - p - 1$.
- Thus, the conjecture also holds for all graphs when $k = n - 2$ and $k = n - 3$.

[1] W. H. Haemers, A. Mohammadian, and B. Tayfeh-Rezaie, On the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, 432:2214–2221, (2010).

[2] X. Chen, On Brouwer's conjecture for the sum of k largest Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, 578:402–410, (2019).

# Brouwer's conjecture

The Brouwer's conjecture is true for several classes of graphs (for all $k \in \{1, \ldots, n\}$):

- Trees and threshold graphs (**Haemers**[1])
- Unicyclic graphs and bicyclic graphs (**Du and Zhou** [2])
- Regular graphs and split graphs (**Mayank**[3]).

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[1] W. H. Haemers, A. Mohammadian, and B. Tayfeh-Rezaie, On the sum of Laplacian eigenvalues of graphs, *Linear Algebra Appl.*, 432:2214–2221, (2010).

[2] Z. Du, B. Zhou, Upper bounds for the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl., 436:3672–3683, (2012).

[3] Mayank, On Variants of the Grone-Merris Conjecture, Master's thesis, Eindhoven University of Technology, 2010.

# Suggested Books


1. R. B. Bapat, *Graphs and Matrices*, (2010).

2. A. Brouwer, *Spectra of Graphs*, (2011).

3. D. M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs*, Academic Press, (1979).

**THANK YOU**