Lovász Theta Function, Semidefinite Programs and Algorithms

Parts 3,4

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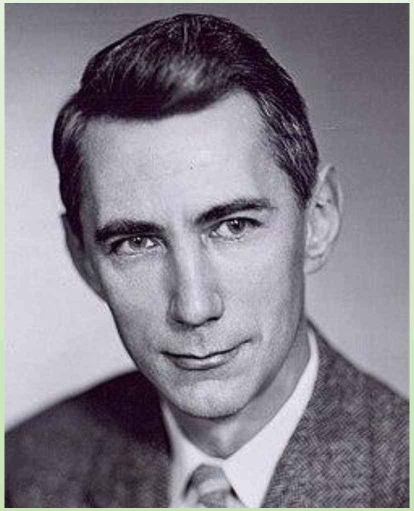
Short Term Program on “Graphs, Matrices and Applications”

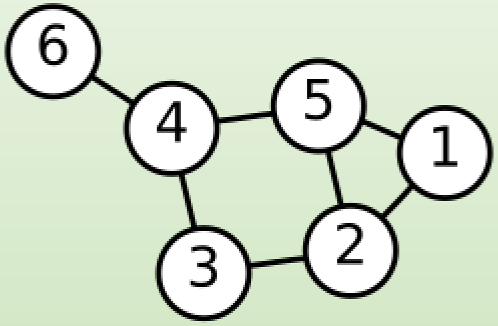
Indian Institute of Technology, Hyderabad

4th Oct 2025 (Sat)

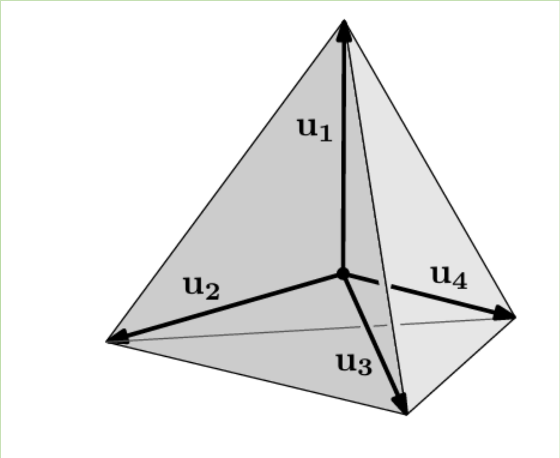
Shannon, Lovász, Graphs and Geometry



Claude Shannon



László Lovász



Recap from Lectures 1,2

- Optimization in CS
- Shannon Capacity
- The Theta Function of a Graph
- Lovasz Bound
- Shannon capacity of the 5-cycle

Outline for Today

- Linear and Semidefinite Programs
- Semidefinite programs for the Theta function
- Sandwich theorem and perfect graphs
- Relaxations and Rounding: Combinatorial optimization. Examples.
- Goemans Williamson Max-Cut algorithm
- SDPs for Coloring

Quick Recap

- Independence number $\alpha(G)$ and chromatic number $\chi(G)$ are hard-to-compute quantities of G, but important from both theoretical and practical standpoints
- Given a graph G, we are interested in finding the value of
$$S(G) \coloneqq \sup_{\{k \in \mathbb{N}\}} \alpha\left(G^k\right)^{\frac{1}{k}}$$
  - This quantity characterizes the measure of information that can be sent across a channel per symbol when edges of G show which alphabets cannot be sent together
- Lovasz formulated the *theta* function $\vartheta(G)$, that satisfies:
$$S(G) \leq \vartheta(G)$$
- $\vartheta(G)$ is a function that utilizes *orthonormal representations* of G, and is easier to analyze than $S(G)$
  - For instance, $\vartheta\left(G^k\right) \leq \vartheta(G)^k$
- Using this, Lovasz showed that $\vartheta(C_5) = \sqrt{5}$, implying $S(G) = \sqrt{5}$

Recap: OR and theta function

$V = \{1, 2, \ldots n\}$

- **Orthonormal Representation (OR)** for G: A set of unit vectors $\{u_1, \ldots, u_n\}$ satisfying:
  - $u_i^T u_j = 0 \cancel{\Leftrightarrow}$ if $\{i, j\} \in \bar{E}$

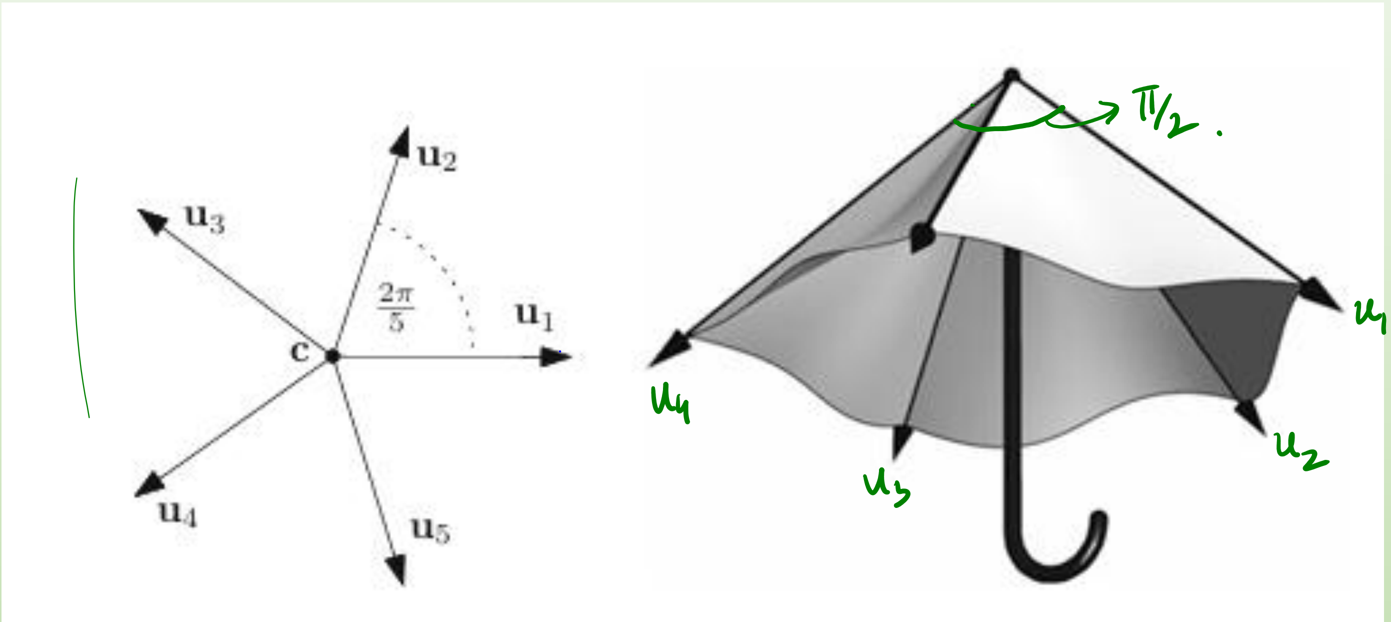
- The **value** of U is defined as

$$\vartheta(U) := \min_{c:\|c\|=1} \max_i \frac{1}{(c^T u_i)^2}$$

c = "handle" of the OR.

- $\vartheta(G) := \min_{U:\text{OR for } G} \vartheta(U)$

Illustration of OR for C_5



$$\boxed{\vartheta(u) = \frac{1-\cos 4\pi/5}{-\cos 4\pi/5} = \sqrt{5}}$$

At satisfying point of OR:

$$u_i = \left(\cos\frac{2\pi i}{5},\ \sin\frac{2\pi i}{5},\ z\right), \text{ for } i = 1, 2, 3, 4, 5$$

$$v_5^T u_2 = 0. \Rightarrow (1, 0, z)^T \begin{pmatrix} \cos 4\pi/5 \\ \sin 4\pi/5 \\ z \end{pmatrix} = 0$$

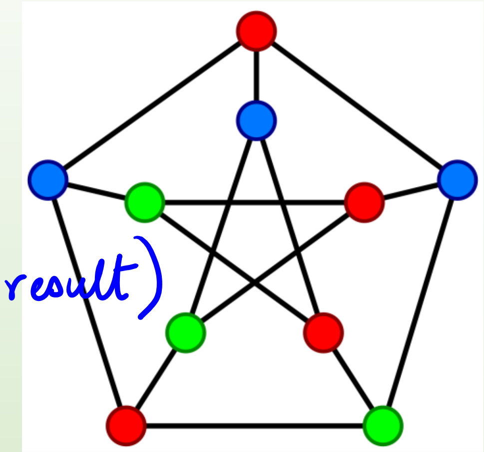
$$\Rightarrow \cos\frac{4\pi}{5} + z^2 = 0.$$

Some definitions

- $\omega(G)$: Size of the maximum clique in G

$\alpha(\bar{G}) = \omega(G)$

The Sandwich Theorem



Theorem [Lovasz, 1979]: For all graphs G, (Main result)

$$\omega(\bar{G}) \underbrace{\le}_{\text{Shown yesterday}} \vartheta(G) \overbrace{\le}^{\text{Today}} \chi(\bar{G})$$

Shown yesterday: $\underset{\omega(\bar{G})}{\underset{\parallel}{\alpha(G)}} \le \vartheta(G)$.

(Notice: $\chi(\bar{G}) \geq \omega(\bar{G})$).

$\rightarrow$ Go through "computational aspects of $\vartheta(G)$"

Computation of $\vartheta(G)$

- $\vartheta(G)$ is a *relaxation* of $\alpha(G)$

- $\vartheta(G)$ is an *optimization* problem

min over all OR's $\mathcal{U}$

of $\vartheta(\mathcal{U})$.

- Can a solution to this optimization problem be *computed* efficiently?

General form of an optimization problem

$$Z^* := \quad \min \quad \text{or} \max f(x)$$
$$\text{subject to}: \quad g_1(x) = 0$$
$$g_2(x) \geq 0$$
$$\ldots$$

x: variables
$x \in \mathbb{R}^n$ or similar

- Some optimization problems are "easy" computationally, others are hard
- If $f, g_{\mathrm{i}}'s$ are "simple", then the optimization problem may be efficiently solved
  - Will assume that an optimum exists

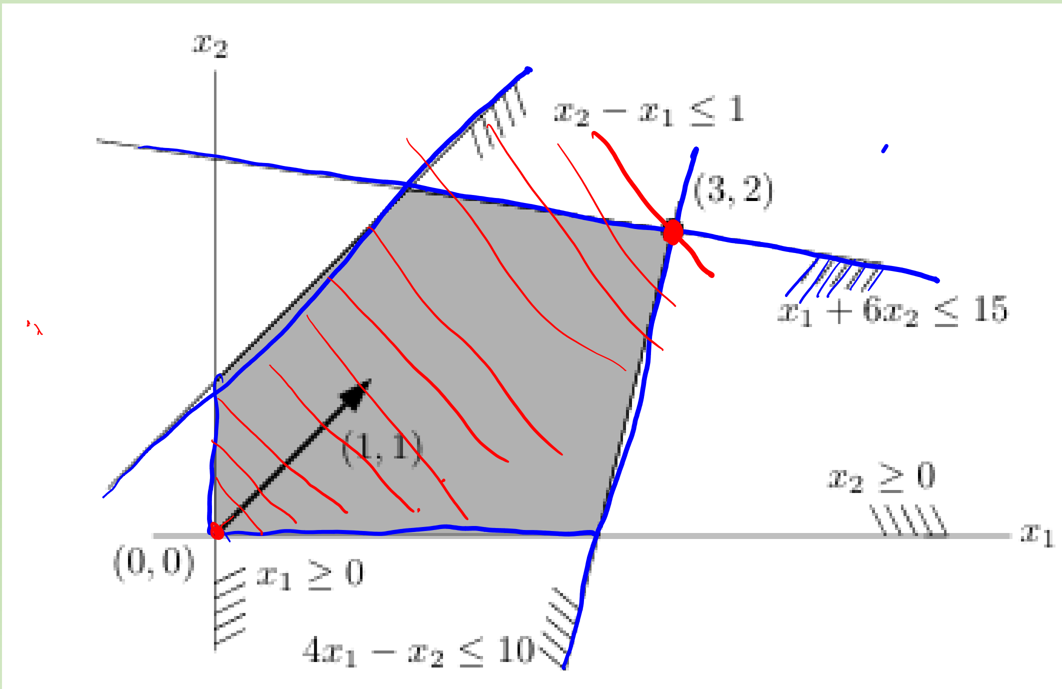
Example: f, g linear

Maximize $x_1 + x_2$ objective (2-variables)

Subject to:

$$\left.\begin{aligned} x_1, x_2 &\geq 0 \\ x_2 - x_1 &\leq 1 \\ x_1 + 6x_2 &\leq 15 \\ 4x_1 - x_2 &\leq 10 \end{aligned}\right\} \text{constraints.}$$

Feasible solutions: $(x_1, x_2) \in \mathbb{R}^2$ s.t. they satisfy all the constraints.



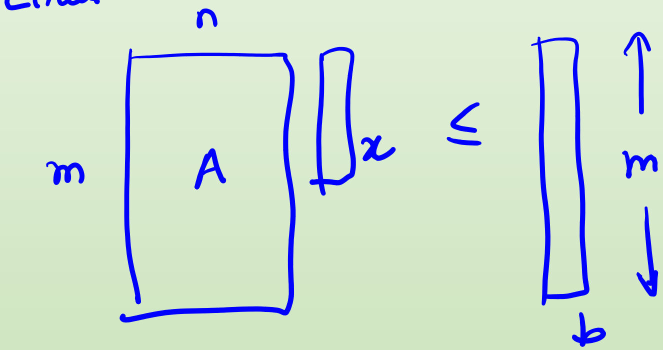
General *Linear Program*

$$\max\ c^T x \rightarrow \sum_{i=1}^{n} c_i x_i$$

$$s.t. \quad Ax \le b \rightarrow \text{Linear contraints}$$

$$x \ge 0$$

$$x \in \mathbb{R}^n$$



Here, $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$

- An optimal solution to a Linear Program can be found efficiently computationally
  - Given inputs c, A, b, we can find an x^* optimizing the above

More constraints

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & \\ & & \ddots & \\ & & & x_{nn} \end{pmatrix}$$

- Say, now, the variables are entries x_{ij} of a symmetric matrix
- The space of variables is, therefore: (n^2 variables)

$$\boxed{\mathrm{SYM}_n = \{X \in \mathbb{R}^{n \times n}: x_{ij} = x_{ji}\}}$$

- Let's generalize the previous Linear Program:

- $\max\ c^T x$

  $s.t. \quad Ax \leq b$

  $\underline{x \geq 0}$

  $x \in \mathbb{R}^n$

$\rightsquigarrow$

$\max \sum_{ij} c_{ij} x_{ij}$ (Linear)

s.t. Linear constraints.

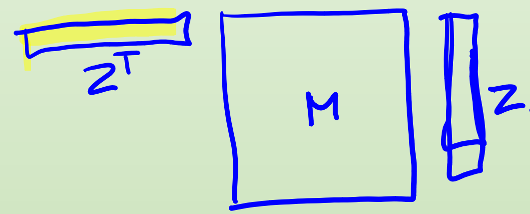
$\sum_{ij} A_{ij}^{(k)} x_{ij} \leq \underline{b^{(k)}} \quad k = 1, \ldots, m$

$X \succeq 0$ (X is psd)

$X \in \mathrm{SYM}_n$.

Positive Semidefinite Matrices

- **Fact:** Let $M \in \mathrm{SYM}_n$. The following are equivalent:
  1. M is positive semidefinite: All the eigenvalues of M are non-negative
  2. $z^T M z \geq 0$ for all $z \in \mathbb{R}^n$
  3. $M = U^T U$, for some matrix U



$\geq 0.$

$\forall z$

(3) ⇒ (2) : $z^T M z = z^T U^T U z$

$= \|Uz\|^2 \geq 0.$

Semidefinite Program (SDP)

max/min $\sum_{ij} c_{ij} x_{ij}$

s.t. Linear constraints on x_{ij}'s. (eg: $4x_{11} + 3x_{21} + 5x_{23} \geq 10$)

$X \succeq 0.$

$X \in SYM_n.$

- Omitting technical conditions*, we can *efficiently* find optimal solutions* to Semidefinite Programs!

A slight caveat

- max $-x_{11}$
- s.t. $x_{12} = 1$
- $X \underbrace{\succcurlyeq}_{X \text{ is psd.}} 0, X \in \mathrm{SYM}_2$

- What is the optimum? → Exists, but can't be attained

→ Solvers give approximately optimal solutions

Back to the Theta Function

- $\vartheta(G)$ as an optimization problem

Find vectors $u_1 \cdots u_n$: $u_i^T u_j = 0$ for $\{i,j\} \in \bar{E}$
and handle 'c'

and $\min_{\substack{c, \\ u_1 \cdots u_n}} \max_i \frac{1}{(c^T u_i)^2}$

s.t. OR constraint.

For any feasible
$u_1 \cdots u_n$
and c,
the optimum t
$= \min_{i \in [n]} (c^T u_i)$

Claim: Consider the optimization problem

$$
\begin{aligned}
Z^* := \max \quad & t \\
\text{s.t.} \quad & u_i^T u_j = 0 \quad \forall \{i,j\} \in \bar{E} \\
& c^T u_i \geq t \quad \forall i \in \{1, \cdots n\}. \\
& \forall i : \|u_i\| = 1, \ \|c\| = 1
\end{aligned}
$$

$\{u_i \in \mathbb{R}^n, \ c \in \mathbb{R}^n, \ t\} \rightarrow$ variables.

$$\boxed{Z^* = \frac{1}{\sqrt{\vartheta(G)}}}$$

or, $\vartheta(G) = \frac{1}{(Z^*)^2}$.

Previous is actually a SDP

$$\max \ t$$

s.t. $u_i^T u_j = 0 \quad \forall \{i,j\} \in \overline{E}$

$c^T u_i \geq t$

$\|u_i\|^2 = 1, \quad \|c\|^2 = 1.$

→ A "vector program"

(variables are vectors).

Introduce $Y \in SYM_{n+1}$

$$\begin{array}{c|c} & j \\ \hline \begin{array}{c} 0 \\ \vdots \\ 2 \\ \vdots \\ i \\ \vdots \\ n \end{array} & \begin{pmatrix} \cdot & \cdot & \cdot \\ & y_{ij} & \\ & & \end{pmatrix} \end{array}$$

$y_{ij} := u_i^T u_j \quad i, j > 0.$

$y_{0,j} := c^T u_j \quad \forall j > 0.$

equiv

$$\max \ t$$

s.t. $y_{ij} = 0 \quad \forall \{i,j\} \in \overline{E}$

$y_{ii} = 1 \quad \forall i$

$y_{0j} \geq t \quad \forall j$

$Y \succeq 0$ →

$t \in \mathbb{R}.$

Why does $Y \succeq 0$ make sense?

Ans: $Y \succeq 0 \iff \exists$ matrix U s.t.

$Y = U^T U$. (defn of psd)

$$\begin{bmatrix} - u_1 - \\ - u_2 - \\ \vdots \\ - u_n - \end{bmatrix} \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}.$$

Summary: $\dfrac{1}{\sqrt{\vartheta(G)}}$ is the optimal value of a Semidefinite Program.

SDP #2 for the Theta Function

$G = (V, E)$

$V = \{1, \dots n\}$.

$$\min \quad t$$

$$\text{s.t.} \quad u_i^T u_j = -1 \quad \forall \{i,j\} \in E$$

$$\|u_i\|^2 = t-1. \quad \forall i.$$

Can convert into $\rightarrow$ (SDP) by introducing $y_{ij} := u_i^T u_j$

Variables: vectors $u_1, u_2, \dots u_n \in \mathbb{R}^n$.

$t \in \mathbb{R}$.

- Why is the optimal of this equal to $\vartheta(G)$?

Let Z_2^* be the optimal of the above optimization problem.

Lemma: $Z_2^* = \vartheta(G)$.

Proof that $\vartheta(G) = Z_2^*$

$$Z_2^* = \min \ t \quad \Big\} \text{ SDP2.}$$

$$\text{s.t.} \quad v_i^T v_j = -1 \quad \forall \{i,j\} \in \bar{E}$$

$$\|v_i\|^2 = t-1.$$

$$\text{vars: } t \in \mathbb{R}, \ v_i\text{'s} \in \mathbb{R}^n.$$

- Part 1: $Z_2^* \leq \underline{\underline{\vartheta(G)}}$

- Given an optimal OR and handle c, construct a feasible solution to SDP2

Let $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$. $\rightarrow$ optimal OR, handle c.

A Feasible soln to SDP2:

$$v_i := c - \frac{u_i}{(c^T u_i)}.$$

For $i,j \in \bar{E}$: $$v_i^T v_j = c^T c - \frac{c^T u_i}{c^T u_i} - \frac{c^T u_j}{(c^T u_j)} + \cancel{\frac{u_i^T u_j}{(c^T u_i)(c^T u_j)}} \rightarrow 0.$$

$$= 1 - 1 - 1 = -1.$$

Part 2: $Z_2^* \geq \vartheta(G)$

- From an optimal SDP solution, construct an OR for G

(skip the proof here).

Till now

- Optimization formulation (SDP) for the Theta Function

- Goal: To show that $\vartheta(G) \leq \chi(\bar{G})$

shown $\left\{ \begin{array}{c} \text{\rotatebox{90}{$\leq$}} \\ \omega(\bar{G}) \end{array} \right.$

- Need to relate it to a coloring in the complement graph

Proof strategy

To show: $\boxed{\vartheta(G) \le \chi(\bar{G})}$.

- Show that if $\bar{G}$ has a k-coloring, then SDP2 has a feasible solution with value at most k $\Rightarrow \vartheta(G) \le k$.

SDP2:

$Z2 = \min t$

s.t. $y_{ij} = -1$ forall $i, j \in \bar{E}$

$y_{ii} = 1$

$Y \succcurlyeq 0$

$\equiv$

SDP 2 : (vector form)

$Z_2^* = \min \quad t$

s.t. $\quad v_i^T v_j = -1 \quad \forall \{i,j\} \in \bar{E}$

$\|v_i\|^2 = 1$

Variables: v_i's and $t \in \mathbb{R}$

vectors.

k - colorings and Vector k-colorings

$G' = (V, E')$

k-coloring of $G' = (V, E')$: $\chi : V \to \{1, \dots k\}$.

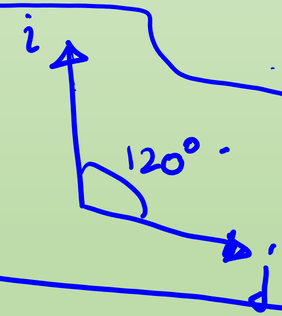
s.t. $\forall \{i, j\} \in E'$, $\chi(i) \neq \chi(j)$.

Definition: A vector k-coloring is an assignment of unit vectors to vertices

$$\gamma : V \to S^{n-1}$$

$$\text{s.t. } \forall \{i, j\} \in E', \quad \gamma(i)^T \gamma(j) = -\frac{1}{k-1}.$$

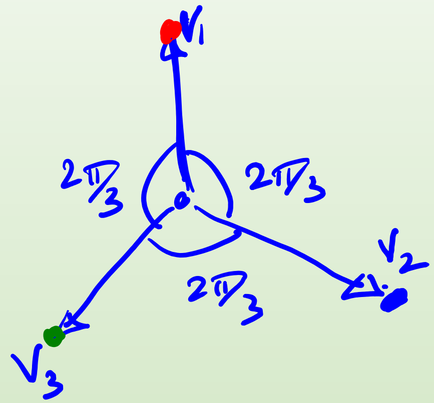
eg: $k=3$. $\gamma(i)^T \gamma(j) = -\frac{1}{2}$.



Claim: If G' has a k-coloring, it has a vector k-coloring.

Want : $r(i) \in \mathbb{R}^n$ s.t. $\forall \{i,j\} \in G'$: $r(i)^T r(j) = \frac{-1}{k-1}$.

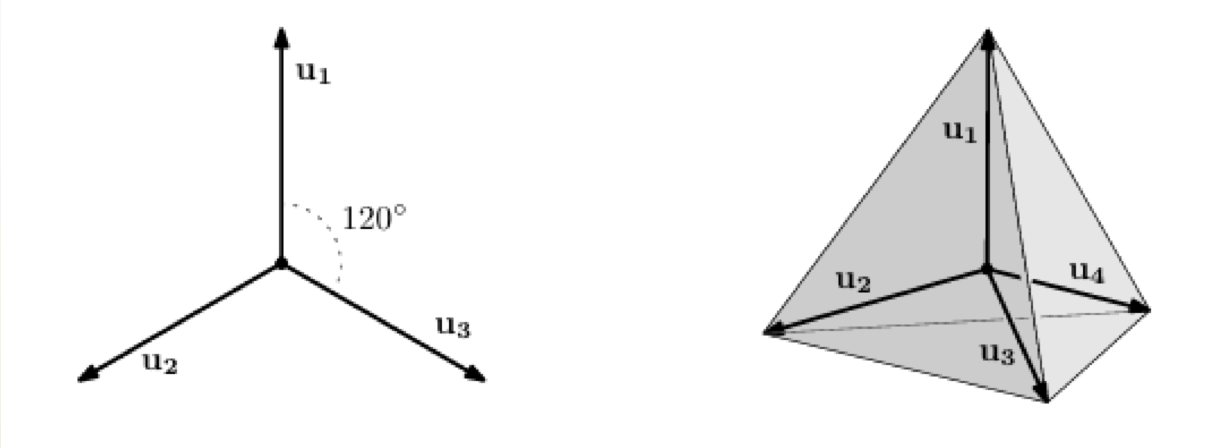
eg: $k=3$.



3-colorable graph

$$r(i) = V_{\chi(i)}.$$

Solutions for $k = 3,4$

- 

In general: $\quad r(i) := \dfrac{e_{x(i)} - \frac{1}{k}\sum_{\ell=1}^{k} e_\ell}{\left\| e_{x(i)} - \frac{1}{k}\sum_{\ell=1}^{k} e_\ell \right\|}$

Wanted to show: $\vartheta(G) \leq \underbrace{\chi(\bar{G})}_{=k}$. then it has a vector k-coloring.

$$\min \quad t$$
$$v_i^T v_j = -1 \quad \forall \{i,j\} \in \bar{E}$$
$$\|v_i\|^2 = t-1$$

$\wr\!\!\downarrow$!!!

$\leq k$.

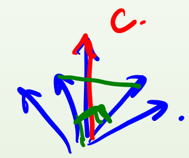
$$\min \quad t$$
$$v_i^T v_j = -\frac{1}{t-1} \quad \forall \{i,j\} \in \bar{E}$$
$$\|v_i\|^2 = 1.$$

$\Rightarrow$ constraints for v_i's to be a vector t-coloring!

$\Rightarrow \text{opt} \leq k$.

$\bar{G}$ has a k-coloring $\Rightarrow$ it has a vector k coloring $\Rightarrow$ there is a feasible solution to the SDP with value k

$$\left(\alpha(G^k)\right)^{1/k} \le \vartheta(G) =$$



$$\min_{c} \max_{i \in V} \frac{1}{(c^T u_i)^2}.$$

Recap

max $f(x)$ $\quad x$: variables

s.t. $g_1(x) \geq 0$ $\quad = (x_1, \cdots, x_n)$

$g_2(x) \geq 0$

- An SDP (Semidefinite Program) is an optimization problem where the n^2 variables are entries x_{ij} of a symmetric psd matrix $X \in \mathrm{SYM}_n$, and the objective and constraints are linear in $x_{ij}'s$

- $\vartheta(G)$ can be expressed as a minimization SDP: i.e, as an optimization problem which can be solved efficiently

- If $\bar{G}$ has a k-coloring, then we can use it to find a feasible solution to the SDP with value k.

- This gives us: $\omega(G) \leq \vartheta(G) \leq \chi(\bar{G})$

Perfect Graphs

Sandwich Thm: $\forall G: \omega(\bar{G}) \leq \vartheta(G) \leq \chi(\bar{G})$

- **Perfect Graphs** are graphs G where $\omega(G') = \chi(G')$ for all vertex-induced subgraphs G' of G

- By sandwich theorem, can compute $\omega(G)$ and $\chi(G)$ for all perfect graphs G efficiently (since by sandwich theorem, both are $= \vartheta(\bar{G})$)

- Examples: Bipartite graphs, Chordal graphs, Interval Graphs

Perfect graph theorems

- **Weak Perfect Graph Theorem** (Lovasz 1972)

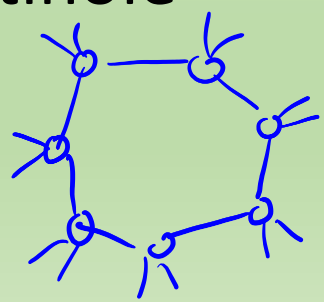
G is a perfect graph if and only if $\bar{G}$ is perfect

$\Rightarrow$ Can compute $\chi(G)$, $\alpha(G)$, $\omega(G)$ from $\vartheta(G)$ and $\vartheta(\bar{G})$.

- **Strong Perfect Graph Theorem** (CRST, 2006, Annals of Math)

A graph is perfect iff it contains no odd hole (odd induced cycle of length ≥ 5) and no odd antihole

- Smallest non-perfect graph is C_5



SDPs and Designing Algorithms

(Part 4)

Easy and hard problems

- “Easy” problems: Problems for which there are efficient (polynomial-time) algorithms
  - Sorting n numbers
  - Matching
  - Finding a minimum Spanning tree
  - Min-cut
  - Max-Flow

- “Hard” problems: NP-hard, optimal solutions likely cannot be found efficiently
  - Minimum Vertex-Cover
  - Max-Cut
  - Minimum Set-Cover
  - Maximum Independent Set

Approximation

- Area of approximation algorithms: design efficient algorithms that *provably* find an *approximately* optimal solution

- Example:
  - Input: Graph G
  - Output: Independent Set S
  - Objective: Maximize |S|

- A β-approximation algorithm ($\beta \leq 1$):
  - If OPT is the optimal value output an answer $|S|$ with the *guarantee* that $|S| \geq \beta \cdot \mathrm{OPT}$

- Definition applies to all maximization problems: if ALG is a solution given by the algorithm, want $ALG \geq \beta \cdot \mathrm{OPT}$

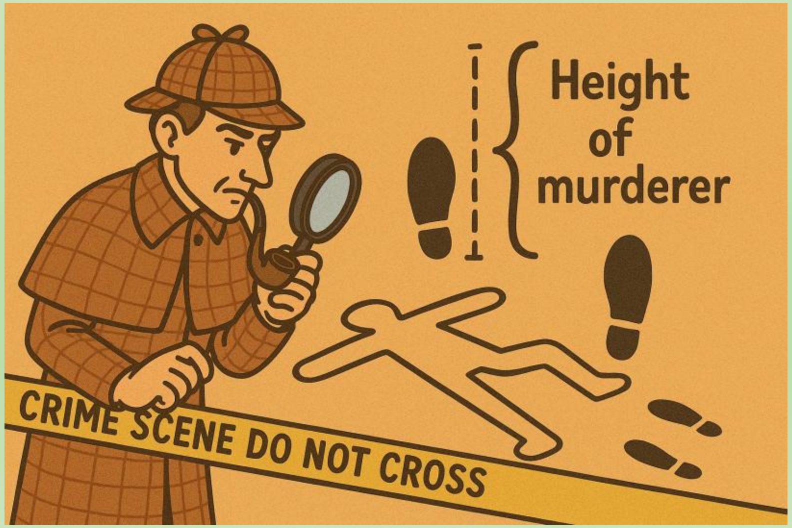
- For minimization problems, want to ensure that we get a solution not too larger than the optimal, i.e. $\mathrm{ALG} \leq \Delta \cdot \mathrm{OPT}$ (for $\Delta \geq 1$).

But how does it work?

- How can we guarantee that $\mathrm{ALG} \geq \beta \cdot \mathrm{OPT}$, when we have no idea what OPT will be?

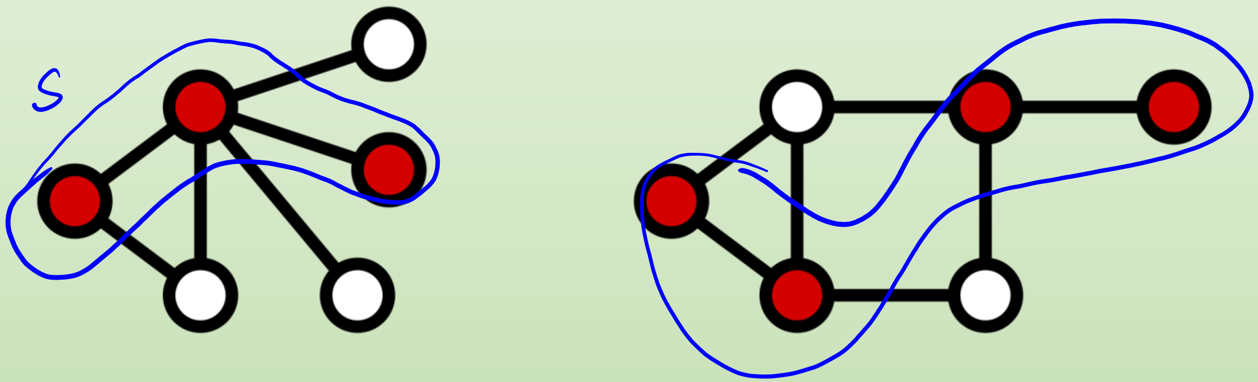
- *Answer:* Use a *proxy* to get an idea of what OPT should be like!

- Very similar to Lovasz's theta function!

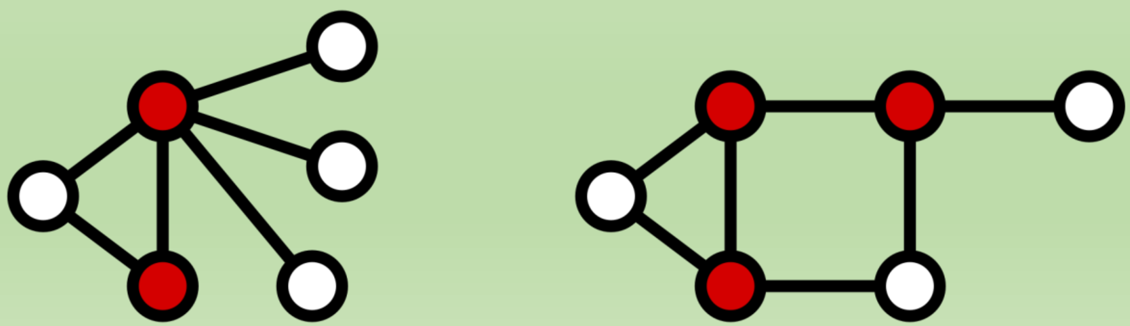


Example: Minimum Vertex-Cover

- A vertex cover is a subset S of vertices that covers (touches) all edges



- Goal: Find a minimum-sized vertex cover in input graph G



Optimization formulation and relaxation

for Vertex Cover

Given $G = (V, E)$, $V = \{1, 2, \ldots n\}$

Let variables be $x_1, x_2, \ldots x_n$.

$$Z = \min \sum_{i \in V} x_i$$

$$\text{s.t.} \quad x_i + x_j \geq 1 \quad \forall \{i,j\} \in E$$

$$x_i \in \{0,1\} \quad \forall i \in V$$

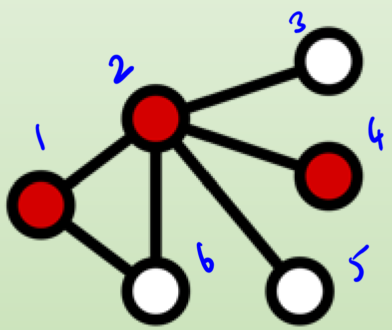
Any feasible soln $x \in \{0,1\}^n$ to this problem is an indicator vector of a vertex cover in G

$\Rightarrow$ optimal solution will be a minimum vertex cover in G

$$\min \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 .$$

$$\text{s.t.} \quad \left.\begin{array}{l} x_1 + x_2 \geq 1 \\ x_1 + x_6 \geq 1 \\ x_2 + x_6 \geq 1 \\ x_2 + x_5 \geq 1 \\ x_2 + x_4 \geq 1 \\ x_2 + x_3 \geq 1 . \end{array}\right\}$$

$$x_i \in \{0,1\} \quad \forall i \in \{1, \ldots 6\}$$



Relaxation

$$Z_{LP} := \min \sum_{i \in V} x_i$$

$$\text{s.t.} \quad x_i + x_j \geq 1 \quad \forall \{i, j\} \in E$$

$$\left.\begin{array}{l} x_i \geq 0 \\ x_i \leq 1 \end{array}\right\} \forall i \in V.$$

$\longrightarrow$ LP, efficiently solvable.

Prev: $x_i \in \{0, 1\}$

$\rightarrow$ Output on solving need not be "integral".

x_i's may be fractional!

eg: $x^* = (0.2, \ 0.3, \ 0.47, \ 0.9, \ 0.6)$

Know. $\sum_{i \in V} x_i^* \leq OPT$ (i.e $Z_{LP} \leq OPT$)

Algorithm

$$\min \sum_{i \in V} x_i$$
$$s.t. \;\; x_i + x_j \geq 1 \quad \forall \{i,j\} \in E$$
$$x_i \geq 0 \quad \forall i \in V$$

① Feed the above into a solver, get a solution $x^* = (x_1^*, \ldots x_n^*)$
with value $Z_{LP}^* = \sum_{i \in V} x_i^* \leq OPT \hookrightarrow$ size of min vertex cover in G.

② Let $S = \{ i : x_i^* \geq \frac{1}{2} \}$.

<u>Claim</u>: S is a vertex cover in G.

<u>Proof</u>: Take any $\{i,j\} \in E$. We know that $x_i^* + x_j^* \geq 1$

$\Rightarrow$ atleast one endpoint on every edge is in S.

Why is $|S|$ small?

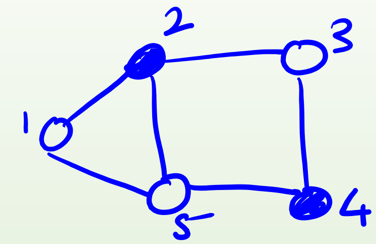
$$|S| = \sum_{i \in S} 1 \leq 2 \sum_{i \in S} x_i^* \quad \left(\text{since } x_i^* \geq \tfrac{1}{2} \quad \forall i \in S\right)$$

$$\leq 2 \left(\sum_{i \in V} x_i^*\right) \quad \left(\text{since } x_i^* \geq 0 \ \forall i\right)$$

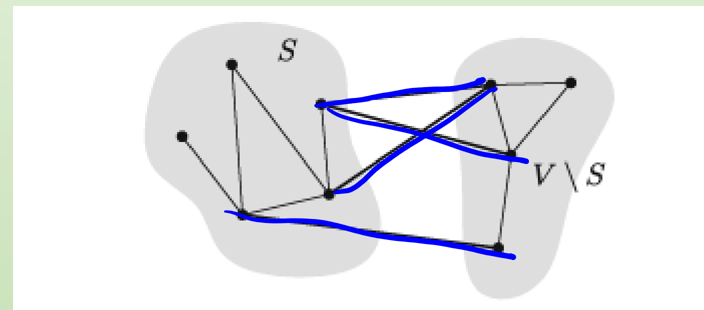
$$= Z_{LP}$$

$$\leq 2 \cdot \underbrace{OPT}_{\text{min vertex cover size.}} \quad \left(\text{since } Z_{LP} \leq OPT\right)$$

The MAX-CUT problem



- Input: $G = (V, E)$ $V = \{1, 2, \cdots n\}$
- Objective: Find a partition (S, S^c) of V that maximizes the number of edges across the cut: $\max\ |E(S, S^C)|$



- Unlike min-cut, NP Hard to find exactly!

- A $\frac{1}{2}$ - approximation was known
- Goemans and Williamson [1994] give an algorithm that guarantees a 0.878 approximation, which introduced the use of SDPs in designing algorithms.

An observation

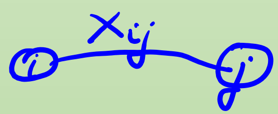
Fact: $\overbrace{OPT}^{\text{Max cut value}} \geq \frac{1}{2}|E|$ $\forall$ graphs G.

Proof: Choose a set $S \subseteq V$ randomly as follows:

For each $i \in V$: choose i to be in S w.p. $\frac{1}{2}$ (independently).

Will analyze: $\mathbb{E}[|E(S, S^c)|]$.

$$X_{ij} = \begin{cases} 1 & \text{if } i,j \text{ lie on opp sides of cut} \\ 0 & \text{if } i,j \text{ lie on same side of cut.} \end{cases}$$



$$\mathbb{E}[|E(S, S^c)|] = \mathbb{E}\Big[\sum_{i,j \in E} X_{ij}\Big]$$

$$= \sum_{\{i,j\} \in E} \mathbb{E}[X_{ij}] = \sum_{i,j \in E} \frac{1}{2}$$

$$= \frac{1}{2}|E|$$

Randomized algorithm:

Choose an S as above.

Output (S, S^c).

$$\mathbb{E}\left[|E(S,S^c)|\right] \geq \frac{1}{2}|E| \geq \frac{1}{2}\underbrace{OPT}_{\text{max-cut}}$$

Optimization formulation

- Linear formulations do not work very well for MAX-CUT

- A *quadratic formulation*

Variables $x_1 \cdots x_n$, x_i indicates if $i \in S$.

$$\max \sum_{\{i,j\} \in E} \left(\frac{1 - x_i x_j}{2} \right)$$

$$x_i^2 = 1 \equiv x_i \in \{-1, 1\} \quad \forall i \in V$$

$x_i = 1$ means $i \in S$

$x_i = -1$ means $i \notin S$

Claim: Optimum of this formulation is exactly MAX-CUT value of G.

Proof: Pick $\{i,j\} \in E$. $\frac{1 - x_i x_j}{2} = 1$ exactly when $x_i \neq x_j$

$= 0$ if $x_i = x_j$

Relaxation of quadratic form to a SDP

$$\max \sum_{\{i,j\} \in E} \left(\frac{1 - x_i x_j}{2} \right) \xrightarrow[\text{scalar variables by vectors}]{\text{replace}} \max \sum_{i,j \in E} \left(\frac{1 - u_i^T u_j}{2} \right)$$

$$\text{s.t.} \quad x_i^2 = 1 \quad \forall i \in V \qquad\qquad \text{s.t.} \quad \|u_i\|^2 = 1, \quad u_i \in \mathbb{R}^n.$$

Let Y = gram matrix of vectors $\{u_i\}_{i=1}^n$;

$$y_{ij} = u_i^T u_j$$

$$Z_{SDP} = \max \sum_{\{i,j\} \in E} \left(\frac{1 - y_{ij}}{2} \right)$$

$$\text{s.t.} \quad y_{ii} = 1, \quad Y \succeq 0$$

$$\boxed{Z_{SDP} \geq \text{MAX-CUT}(G)}$$

Finding a solution

$$\begin{array}{ll} \text{Maximize} & \sum_{\{i,j\}\in E} \frac{1-\mathbf{u}_i^T\mathbf{u}_j}{2} \\ \text{subject to} & \mathbf{u}_i \in S^{n-1}, \quad i = 1, 2, \ldots, n. \end{array} \Bigg\} \text{SDP\#1}$$

$\hookrightarrow \|u_i\|^2 = 1$

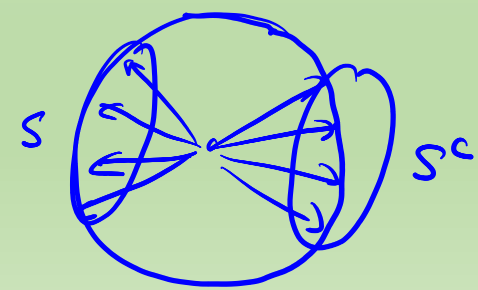
Algo:

① Write out SDP#1, feed it into a solver

Get solution $u_1^*, \ldots u_n^*$ in S^{n-1}

w/ objective function value $= Z_{SDP} \geq \text{MAX-CUT}(G)$.

Intuitively, expect if $\{i,j\} \in E$, u_i, u_j would have large angle between them.

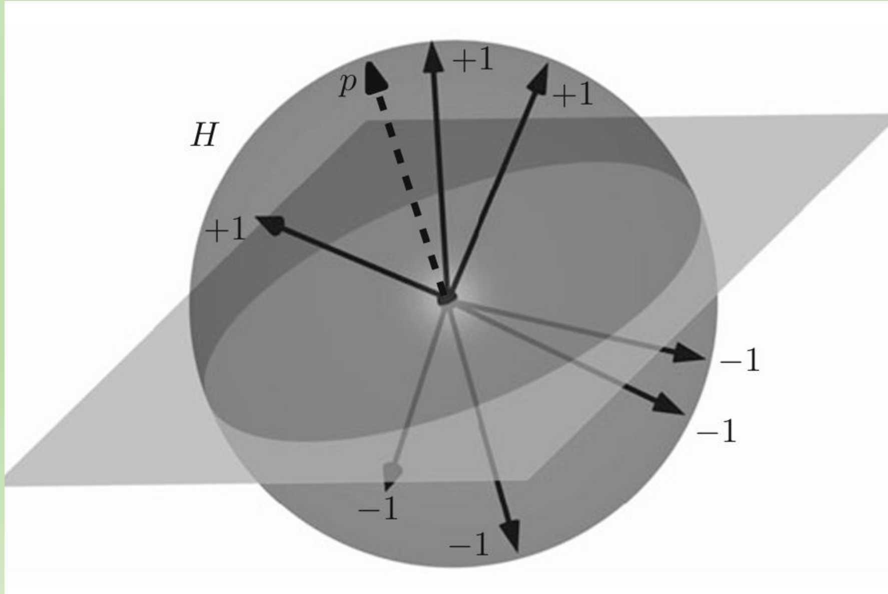


The randomized rounding algorithm

- Choose a $\underbrace{\text{random hyperplane}}_{\text{specify normal vector } p}$ through the origin that cuts S^{n-1}. Vectors $\underbrace{\text{on one side}}_{S=\{i:\, p^T u_i > 0\}}$ give the set S ($\underbrace{\text{and other side is}}_{S^c = \{j:\, p^T u_j \le 0\}}$ S^c).

Remaining : To bound $\mathbb{E}[|E(S,S^c)|]$

[GW'94] : $\mathbb{E}[|E(S,S^c)|] \geq 0.878$ MAX-CUT.

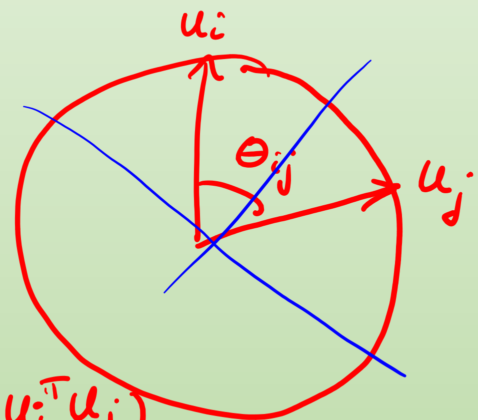


Analysis: Pick any edge $\{i,j\} \in E$

Let $X_{ij} = 1$, if i,j lie on opposite sides of the cut

$$\mathbb{E}\left[\,|E(S,S^c)|\,\right] = \sum_{i,j \in E} \mathbb{E}\left[X_{ij}\right].$$

i $\qquad$ j



$$\Pr(X_{ij}=1) = \frac{\cos^{-1}(u_i^T u_j)}{\pi}$$

$$= \frac{\theta_{ij}}{\pi}.$$

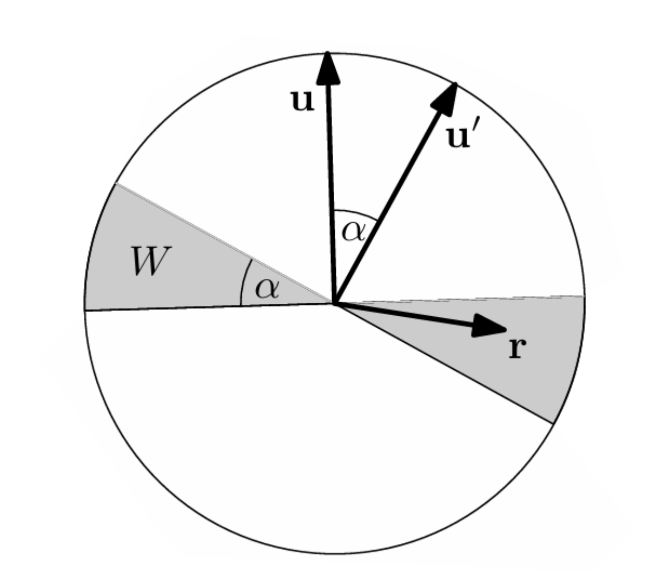
Main Lemma

- Let $u, u' \in S^{n-1}$. Then the probability that u, u' go to different halves is at least:

$$\frac{1}{\pi}\cos^{-1} u^T u'$$

(shown in prev slide)

Probability of separating u, u'



The final bound

- Lemma: $\frac{1}{\pi}\cos^{-1} z \geq 0.87856\ \frac{1-z}{2}$

Set $z = u_i^T u_j$

$\Rightarrow \Pr(x_{ij}=1) \geq 0.87856 \cdot \frac{(1-u_i^T u_j)}{2}$

$\Rightarrow \mathbb{E}[|E(S,S^c)|] = \sum_{i,j \in E} \mathbb{E}[x_{ij}] \geq 0.87856 \times \sum_{i,j \in E} \frac{(1-u_i^T u_j)}{2}$

$= 0.87856 \times Z_{SDP}$

$\geq 0.87856 \times \text{MAX-CUT}(G).$

Closing remarks

- The power of SDPs in approximation is a topic of very active research

References

Most of the material covered can be found in the excellent book:

- B. Gartner and J. Matousek, Approximation Algorithms and Semidefinite Programming, Springer, 2012