Lovász Theta Function, Semidefinite Programs and Algorithms

Parts 1,2

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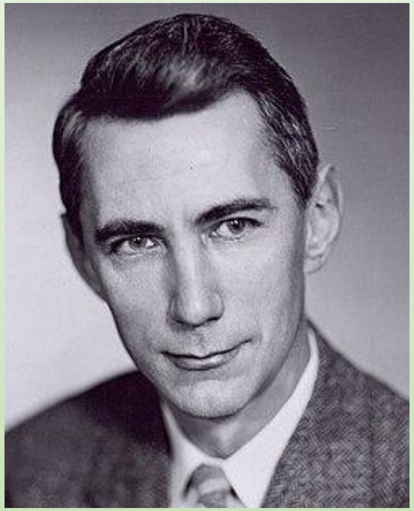
Short Term Program on “Graphs, Matrices and Applications”

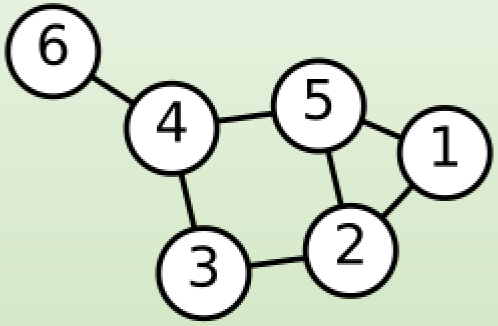
Indian Institute of Technology, Hyderabad

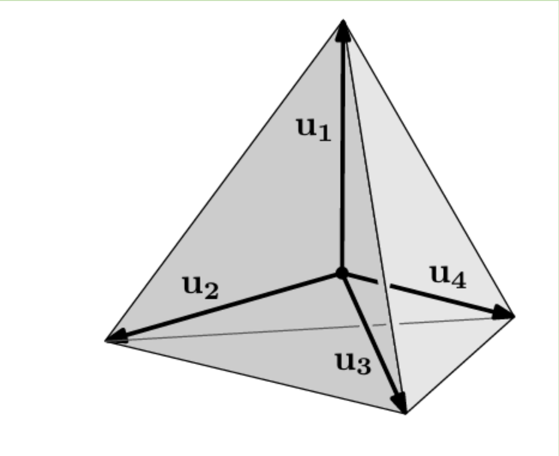
3rd Oct 2025 (Fri)

Shannon, Lovász, Graphs and Geometry



Claude Shannon





László Lovász

Outline for today

- Optimization in CS
- Shannon Capacity
- The Theta Function of a Graph
- Lovasz Bound
- Shannon capacity of the 5-cycle
- Linear and Semidefinite Programs

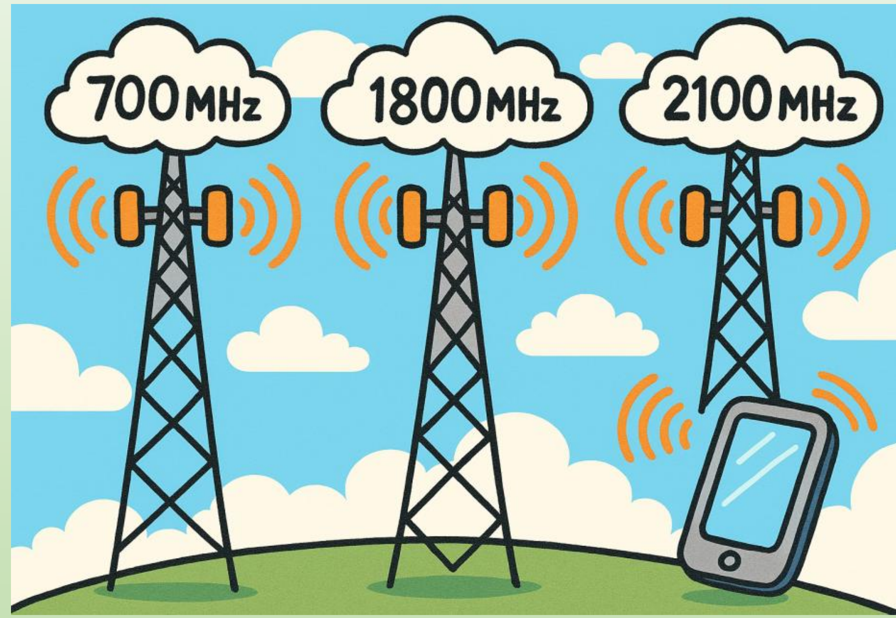
Parts 3,4

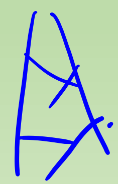
- Semidefinite programs for the Theta function
- Sandwich theorem and perfect graphs
- Relaxations and Rounding: Combinatorial optimization. Examples.
- Goemans Williamson Max-Cut algorithm

Broadcasting Problem

- A telecom company rolling out radio towers in a city.
- Two towers conflict if their signals interfere when they use the same frequency.
- Which towers can broadcast simultaneously on the same channel?

Towers close by should use diff frequencies.





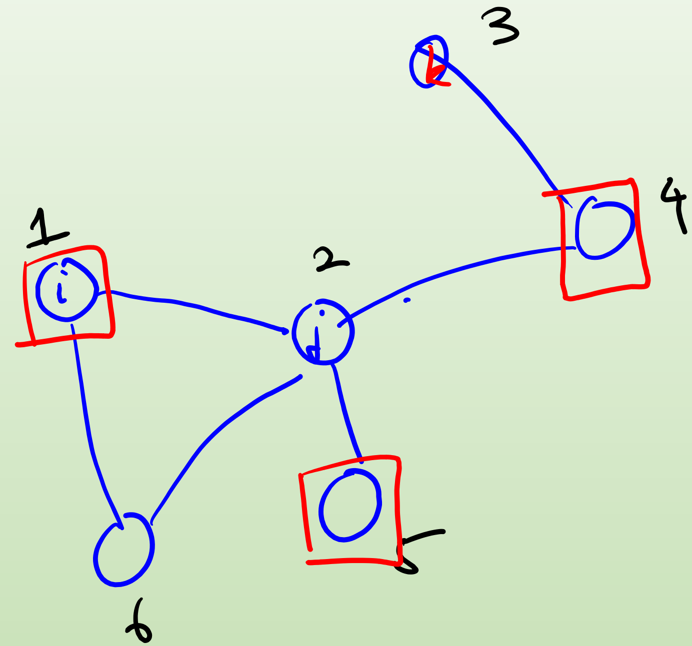
Modelling as a graph problem

$G = (V, E)$

Towers : represent vertices

Edge between two towers i, j

if they may conflict



Finding: a subset $S \subseteq V$ s.t.

$\forall\ i,j \in S \quad \{i,j\} \notin E$

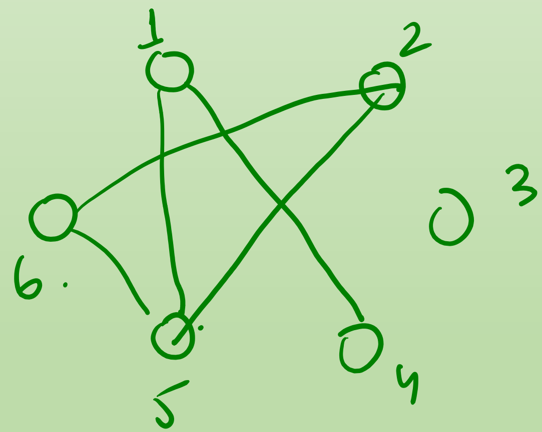
Such a set is called an independent set.

Q: what is $\max\limits_{S:\,\text{independent}} |S|$

Scheduling Tasks on Shared Machines

- In a high-performance computing cluster: tasks need machines
- Some pairs of tasks can't be run at the same time because they require the same resource.
- Find the largest batch of tasks that can run in parallel without conflicts.

Goal: Find max indep set.



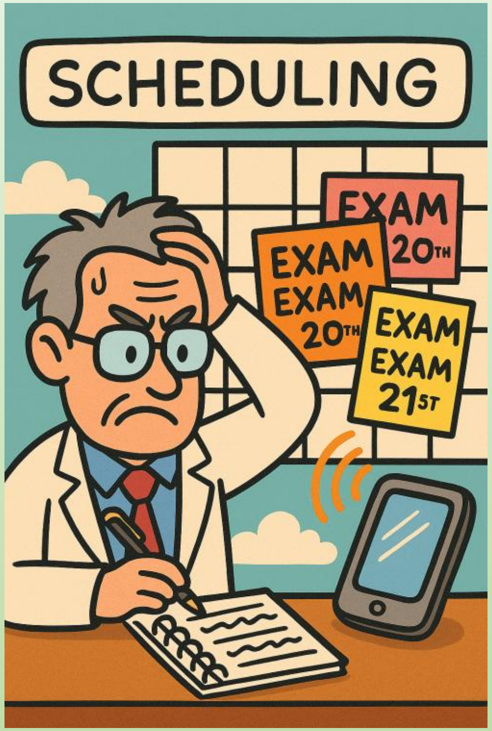
edge between i,j if they conflict.

Scheduling Exams

- A professor has to schedule n exams

- Two exams cannot be scheduled in the same slot, if some student has registered for both of them

- Find the maximum number of exams that can be scheduled simultaneously in the same slot



- Minimum number of days required to schedule all exams?

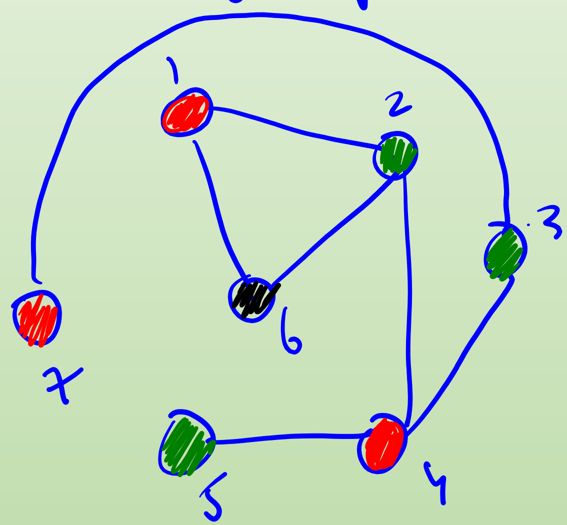
Exam Scheduling as a graph problem

$G = (V, E)$: conflict graph.

Vertices $V \equiv$ exams.

edge $\{i, j\} \in E$ if exams i, j conflict (can't be scheduled together).

<u>To find</u>: min # days



: Monday

: Tuesday :

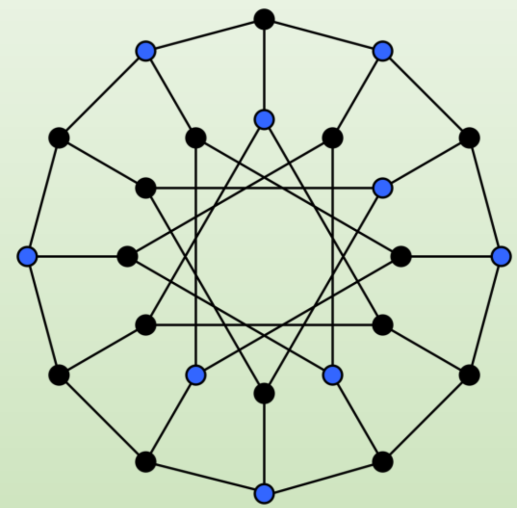
: Wed.

Assigning colors to nodes s.t. no two adjacent nodes have the same color.

while <u>minimizing</u> the number of colors.

Fundamental Graph Quantities

- $G = (V, E)$ $|V| = n$.



- Independent Set

  $S \subseteq V : \forall i, j \in S, \{i, j\} \notin E$.

Brute force / trivial procedure to find a max Independent set:

Try all $S \subseteq V$, check if independent

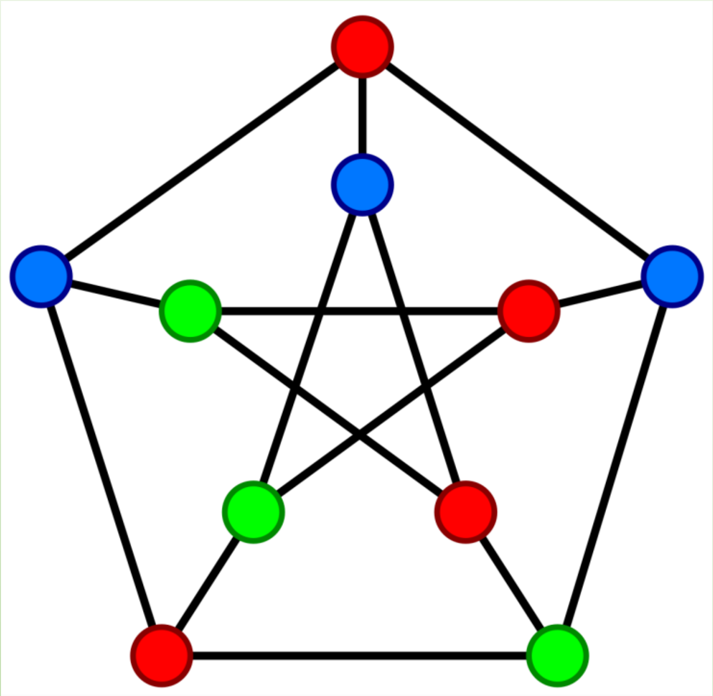
→ Takes 2^n tries for checking all $S \subseteq V$.

- Maximum Independent Set: $\underbrace{\alpha(G)}_{= \max_{S:\text{ indep set}} |S|}$.

Chromatic Number

- Proper Colouring of a graph For $k \in \mathbb{N}$,

$\chi : V \to \{1, \dots, k\}$ is a proper-k-coloring if $\forall \{i, j\} \in E : \chi(i) \neq \chi(j)$



- Chromatic Number $\chi(G)$

$\doteq \min_{k \in \mathbb{N}}$ G has a proper k-coloring

$\chi(G)$ $\alpha(G)$.

NP-Hardness

- Both $\alpha(G)$ and $\chi(G)$ are believed to be *hard to compute* for any input graph G

If $P \neq NP$, $\alpha(G)$ and $\chi(G)$ can't be computed in polynomial time
(eg: $O(n^3)$
or $O(n^{100})$)

- But they turn up frequently in practical applications!

– "Fix" : approximate the solution

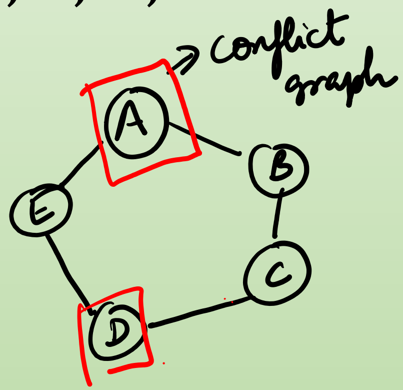
eg when $n = 10,00,000$. If $\alpha(G) = 10000$
can output S : indep, $|S| \geq \frac{\alpha(G)}{2}$

Information Transmission

- Suppose we send symbols across a noisy channel. Some symbols might be misread as others — they 'conflict.'

- Example: We have a communication channel with five possible symbols: A, B, C, D, E.

- Possible confusion:
  - A with B, E,
  - C with B, D,
  - E with A, D



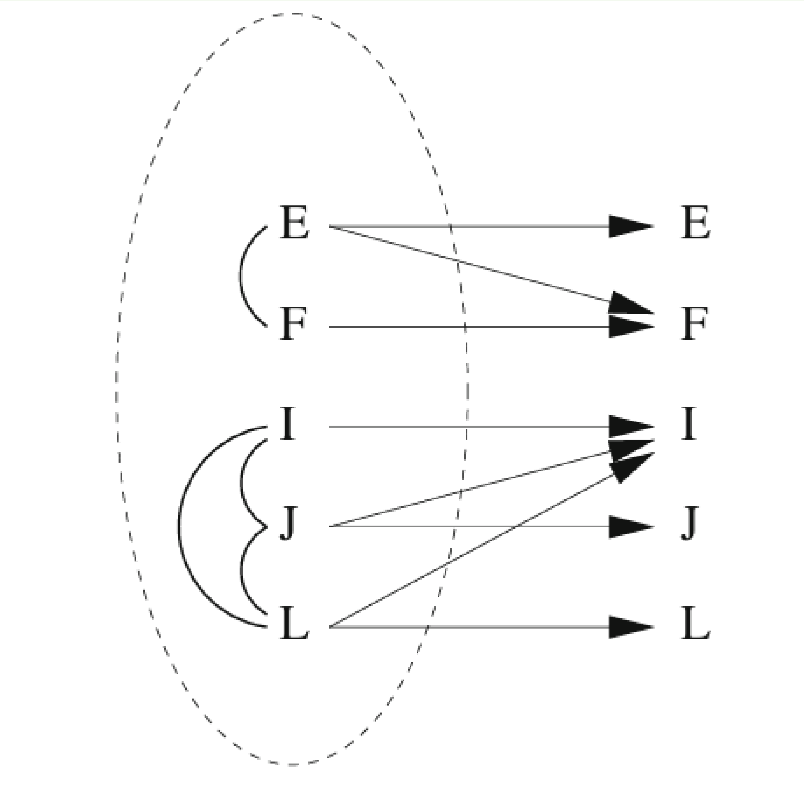
- Goal: Transmit a message without confusion at the receiver's end. At most how many alphabets can you transmit?

Another example

- If you receive "JILL"

- What are the possible words that might have led to it?
  - JILL, JJLL, JLLL may all have been received as JILL
- Conflict graph shows that 2 messages (e.g. fix E and I) can be transmitted without ambiguity from the sender's end

- Can you do something better if you allow to transmit $k = 2$ symbols at a time?

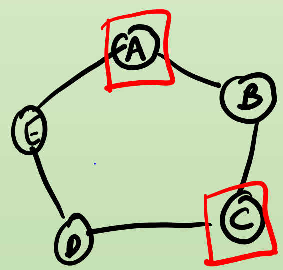


Transmitting 2 symbols at a time.

$\{\underbrace{E, F}, \underbrace{I, J, L}\}$

Every message consists of (two symbols).

5-cycle :
conflict graph



$\alpha = 2$ here.

$k=2$ symbols at a time.

AA, AC, CA, CC.

$\downarrow$

4 messages can be sent by using the channel twice

In fact 5 messages can be sent :

$\{AA, BC, CE, DB, ED\}$

Definition: Similarity Graph G

- $v \sim w$: v is similar to w if $v = w$ or $\{v, w\} \in E$

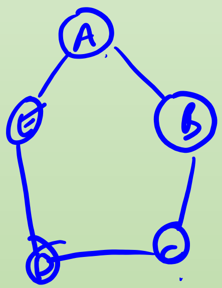
- Similarity-free dictionary on G → a collection of vertices, all of which are pairwise dissimilar.

Graph G^k

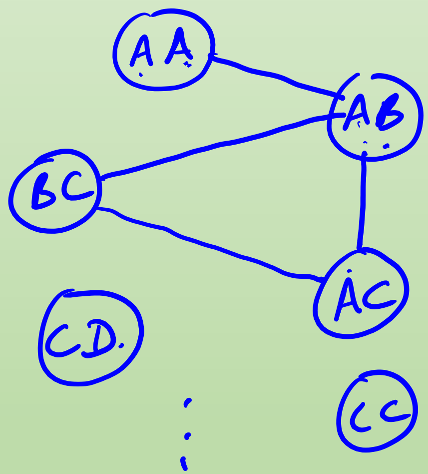
$G^k = (V^k, E_k)$.

$\{(v_1, \ldots v_k), \quad (v_1', \ldots v_k')\} \in E_k$, if $\forall i = 1 \ldots k$,

$v_i \sim v_i'$ (i.e v_i is similar to v_i').

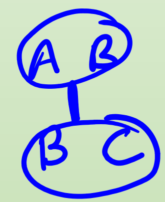


C_5



C_5^3

25 vertices



$\{B, E\}$ → Indep set in base graph

$\{B, E\} \times \{B, E\}$

↓

$\{(B,B), (B,E), (E,B), (E,E)\}$

indep set in squared graph

$\alpha(C_5) = 2$. → $\alpha(C_5^2) \geq 4$

Independent sets in G, G^k

- Theorem: $\alpha(G^{k+\ell}) \geq \alpha(G^k)\alpha(G^\ell)$

$V^{k+\ell}$ V^k V^ℓ.

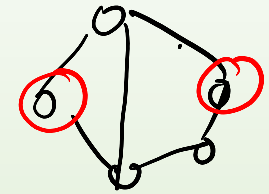
$S \times T$ is Indep set in $G^{k+\ell}$ $\longleftarrow$ Indep set S $\times$ Indep set T.

Example: C_5

$$\alpha(C_5^2) \geq \alpha(C_5) \times \alpha(C_5).$$

$$\underset{\displaystyle 5}{\overset{\shortparallel}{}} \qquad \underset{\displaystyle 2.}{\overset{\shortparallel}{}}$$

Shannon Capacity



- Bits of information transmitted is $\log_2 \alpha(G)$ (definition)

Use graph $G^k \rightarrow \alpha(G^k)$ is # distinct messages we can transmit w/o confusion.

- For k- letter words, average information *per ~~bit~~* symbol is:

For C_5
#1 A A
#2 B C
#3 C E
#4 D B
#5 E D

$$\frac{1}{k}\log_2(\alpha(G^k))$$

For C_5 $k=1$, $\frac{1}{1}\log 2$

$k=2$, $\frac{1}{2} \times \log_2 5 > 1 \approx 1.161.$

$5 \downarrow \alpha(C_5^2)$.

Part 2

Lovász Theta Function and Shannon Capacity of C_5

Quick Recap of Part 1

- Independence number $\alpha(G)$ and chromatic number $\chi(G)$ are hard-to-compute quantities of G, but important from both theoretical and practical standpoints

- Alphabet= $\{A, B, C, \dots\}$, each is a vertex of $G = (V, E)$
- $\{i, j\} \in E$ if alphabet at vertex i and j are *similar (i.e. can be confused)*
- $\alpha(G)$ = max subset of non-similar messages

$G^k = (V^k, E')$

$(v_1, \dots v_k) \sim (v_1', \dots v_k')$ if v_i is similar to v_i' for all i.

- $\alpha(G^k)$ = max subset of non-similar messages that can be transmitted using G^k (k symbols at a time used)
- Information *per alphabet symbol* = $\frac{1}{k} \log_2\left(\alpha\left(G^k\right)\right)$
- $\alpha\left(G^k\right) \geq \alpha(G)^k$, inequality may be strict (e.g. $\alpha\left(C_5^2\right) = \underline{5} > 2^2$)

Shannon Capacity: definition

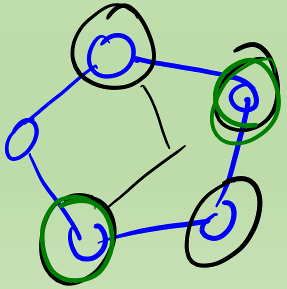
Shannon capacity $\leftarrow$ $$\sigma(G) = \sup\left\{\frac{1}{k}\log \underbrace{\alpha(G^k)}_{\text{size of max indep set in } G^k.} : k \in \mathbb{N}\right\}$$

- Note that C_5 shows that $k = 1$ need not always be the best!

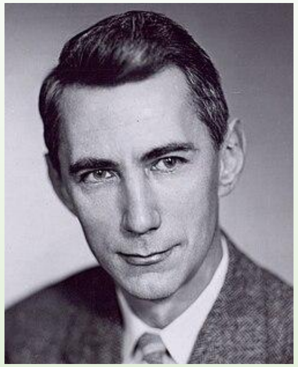
$$\alpha(G^k) \leq |V|^k \Rightarrow \sigma(G) = \frac{1}{k}\log_2 |V|^k \leq \log_2 |V|$$

- $\sigma(G)$ is bounded, and satisfies the limit criterion

$$\sigma(G) = \lim_{k\to\infty} \frac{1}{k}\log \alpha(G^k).$$



Hard to analyze!



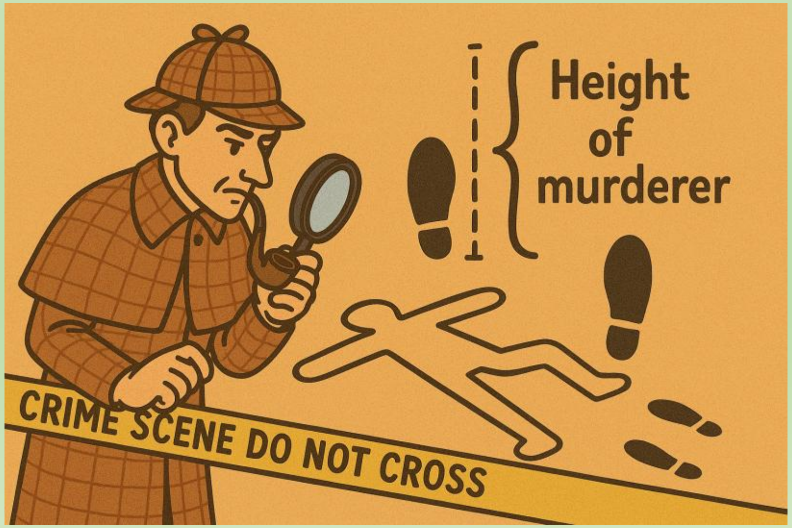
- $\sigma(G)$ is “notoriously” hard to compute even for simple graphs [Shannon, 1956]
- Lovasz [1979] determined $\sigma(C_5)$ via an ingenious “relaxation”
- Open to determine $\sigma(G)$ for many graphs, even C_7

Finding $\sigma(G)$

- Let $S(G) := 2^{\{\sigma(G)\}} = \left(\alpha(G^k)\right)^{\frac{1}{k}}$ $\geq (\alpha(G)^k)^{1/k} = \alpha(G).$

- To find what $S(G)$ is, first look to bound it

- Lovasz's idea: look at a geometrical representation of G, and its properties

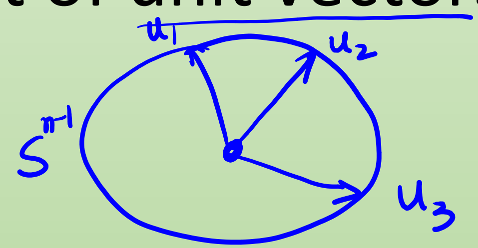


Lovasz's approach: Orthonormal Representations

- Let $G = (V, E)$, with $V = \{1,2, \dots n\}$

  $\{i,j\} : i \neq j$
- Let $\bar{E} = \binom{V}{2} \setminus E$

- **Orthonormal Representation** of G: A set of unit vectors $\{u_1, \dots, u_n\}$ satisfying:



- $u_i^T u_j = 0 \leftrightarrow \{i,j\} \in \bar{E}$

- Dimension? $= n$

Orthonormal Representations examples

- Trivial orthonormal representation in $\mathbb{R}^n$.

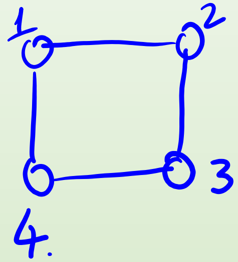
$u_i^T u_j = 0 \Leftrightarrow \{i,j\} \in \overline{E}$

Vertex $i \longmapsto e_i \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \to i$

- Complete graph K_n : (A non-trivial orthonormal repn):

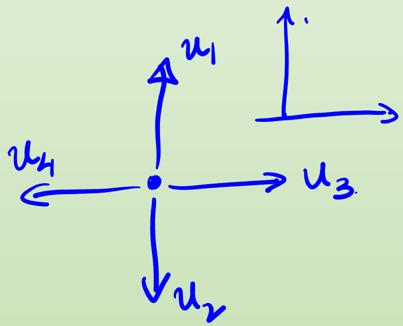
Let all u_i's be the same (no constraint since all edges are present).

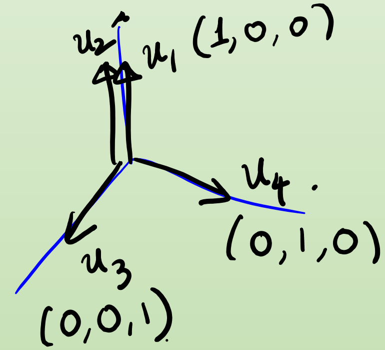
4-Cycle C_4



$u_1 \quad u_2 \quad u_3 \quad u_4$

s.t. $\boxed{\begin{matrix} u_1 \perp u_3 \\ u_2 \perp u_4 \end{matrix}}$





Value of an OR for G OR = orthonormal rep^n.

- Let $U = \{u_1, \dots, u_n\}$ be an Orthonormal Representation of G

- The **value** of U is defined as

$$\vartheta(U) := \min_{\underbrace{c:\|c\|=1}_{c \text{ that minimizes this is called the "handle".}}} \left(\max_i \frac{1}{(c^T u_i)^2} \right)$$

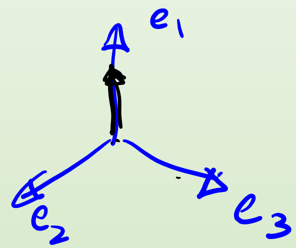
- Note: ϑ is read as "theta"

eg. Trivial rep^n

$\{e_1 , e_2 \dots e_n\}$.

$$\vartheta(U) := \min_{c : \|c\|_2 = 1} \max_{i \in [n]} \frac{1}{(c^T u_i)^2}.$$

Take $c = \frac{1}{\sqrt{n}} (1, 1, 1, \dots 1)$



If $c = e_1$, say,

$$\max_{i \in [n]} \frac{1}{(c^T e_2)^2} = \infty.$$

$\vartheta(U) = n$.

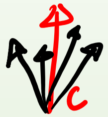
Complete graph K_n : $\{u, u, u, \dots u\}$.

$c = u$, then $\max \frac{1}{(c^T u)} = 1$

$\therefore \vartheta(U) = 1$.

Ex: for C_4.

Theta function $\vartheta(G)$ for G



$$\vartheta(G) := \min_{U:\text{OR for } G} \vartheta(U)$$

Intuitively:
Want all vectors to fit in a small cap around c.

- Note: Minimum exists (continuous function over a compact set)

- Why consider the quantity $\vartheta(G)$?

Lovász Bound

$(\alpha(G^k))^{1/k}$

- **Theorem:** $S(G) \le \vartheta(G)$

Proof uses two lemmas:

- Lemma A: $\vartheta(G \cdot H) \le \vartheta(G)\vartheta(H)$
  - Corollary: $\vartheta(G^k) \le \vartheta(G)^k$
- Lemma B: For all graphs G': $\alpha(G') \le \vartheta(G')$

Fix any $k \in \mathbb{N}$. Apply Lemma B to $G' = G^k$.

$$\alpha(G^k) \underbrace{\le}_{\text{Lem B}} \vartheta(G^k) \underbrace{\le}_{\text{Lem A}} \vartheta(G)^k$$

$$\boxed{\Rightarrow (\alpha(G^k))^{1/k} \le \vartheta(G)} \Rightarrow \boxed{S(G) \le \vartheta(G)} \quad \square$$

Lemma A

- **Lemma A**: $\vartheta(G \cdot H) \leq \vartheta(G)\vartheta(H)$

Definition (Strong Graph Product): $G \cdot H$

$G = (V, E)$, $H = (W, F)$

$G \cdot H$ has vertex set $V \times W$

Edge between (v, w) and (v', w') if

$\underbrace{v \sim v'}_{\text{in } G}$ and $\underbrace{w \sim w'}_{\text{in } H}$.

Proof strategy: Use OR for G and OR for H to give a valid OR for $G \cdot H$, with value $= \vartheta(G) \cdot \vartheta(H)$.

$\vartheta(G\cdot H) \leq \vartheta(G)\cdot \vartheta$

<u>Want</u>: $\vartheta(G\cdot H) \leq \vartheta(G)\cdot\vartheta(H)$

Let $P = \{u_1, u_2, \dots u_{|V|}\}$ be an optimal OR for G with handle vector c.

Let $Q = \{v_1, v_2 \dots v_{|W|}\}$ be an optimal OR for H with handle vector d.

$i \in V$, $j \in W$ $\quad (G = (V, E), \quad H = (W, F))$.

$\boxed{(i,j)} \in V(G\cdot H) \longrightarrow u_i \otimes v_j$ $\rightarrow$ Tensor product

$$x \otimes y = \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ \vdots \\ x_i y_j \\ \vdots \\ x_n y_m \end{pmatrix} \in \mathbb{R}^{mn}.$$

$x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

<u>Fact</u>: $\boxed{(x\otimes y)^T (x' \otimes y') = (x^T x') \times (y^T y')}$

Above is a valid OR for $G\cdot H$ since:

$(u_i \otimes v_j)^T (u_{i'} \otimes v_{j'})$

$= (u_i^T u_{i'}) \times (v_j^T v_{j'})$

$= 0$ iff $(i, i') \in \bar{E}$ or $(j, j') \in \bar{F}$

$\equiv (i,j) \not\sim (i', j')$ in $G\cdot H$.

$\underbrace{\vartheta(P \otimes Q)}_{\downarrow}$.

vectors $(u_i \otimes v_j)$.

Use handle vector $(c \otimes d)$

$$\vartheta(P \otimes Q) \leq \max_{(i,j) \in V \times W} \frac{1}{\left((c \otimes d)^T (u_i \otimes v_j)\right)^2}$$

$$= \max_{(i,j) \in V \times W} \frac{1}{(c^T u_i)^2 \cdot (d^T v_j)^2}.$$

$$= \max_i \frac{1}{(c^T u_i)^2} \times \max_j \frac{1}{(d^T v_j)^2}.$$

$$= \vartheta(G) \times \vartheta(H).$$

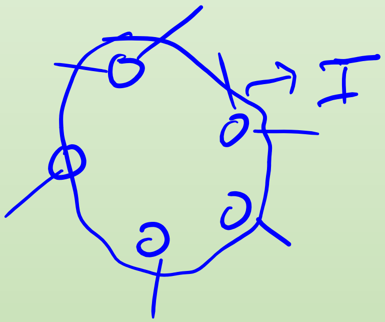
$\square$

Lemma B

Lemma B: For all graphs G': $\alpha(G') \leq \vartheta(G')$

Proof: Let $\{u_1, \ldots, u_n\}$ be an optimal OR for G', with handle vector c.

Let I be a maximum independent set in G',



(a) $\|c\|^2 = c^T c \geq \sum_{i\in I} (c^T u_i)^2$

$$\Rightarrow 1 \geq \sum_{i\in I} (c^T u_i)^2 \geq \left(\min_{i\in I} (c^T u_i)^2\right) \times |I|$$

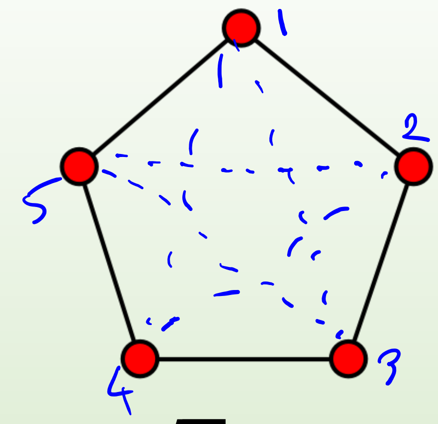
$$\Rightarrow \frac{1}{\min_{i\in I} (c^T u_i)^2} \geq |I| = \alpha(G)$$

$$\Rightarrow \alpha(G) \leq \max_{i\in I} \frac{1}{(c^T u_i)^2} \leq \max_{i\in V} \frac{1}{(c^T u_i)^2} = \vartheta(G')$$

$$S(G) \leq \underbrace{\vartheta(G)}_{\text{Theta function}}.$$

$$\left(\alpha(G^k)\right)^{1/k}.$$

Shannon capacity of C_5: Lovász's umbrella construction



Lemma: C_5 has an OR with value $= \sqrt{5}$. Thus, $\vartheta(C_5) = \sqrt{5}$

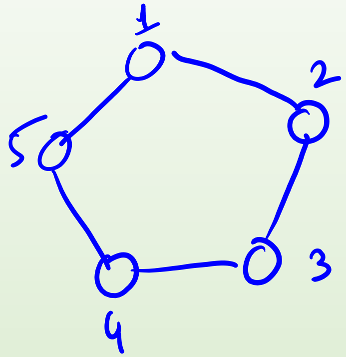
Recall that we had shown that $S(C_5) \geq \sqrt{5}$. Combined with the above, we infer that $S(C_5) = \sqrt{5}$.

when we set $k=2$, we have

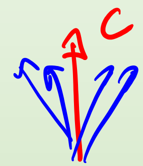
$\alpha(C_5^2) = 5$

$\Rightarrow (\alpha(C_5^2))^{1/2} = \sqrt{5}$

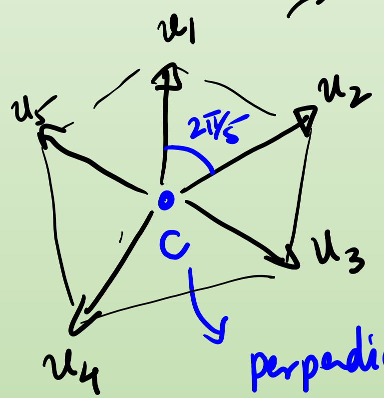
$\Rightarrow S(C_5) \geq \sqrt{5}$



$\left\{ \begin{array}{c} \vartheta(u) \\ \downarrow \\ \text{OR} \end{array} \right.$ is small if vectors "cluster around" the handle vector c.



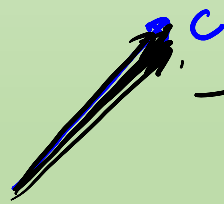
→ "open umbrella"



perpendicular to the plane of paper

Angle between $u_1, u_3 = \frac{4\pi}{5} > \frac{\pi}{2}$.

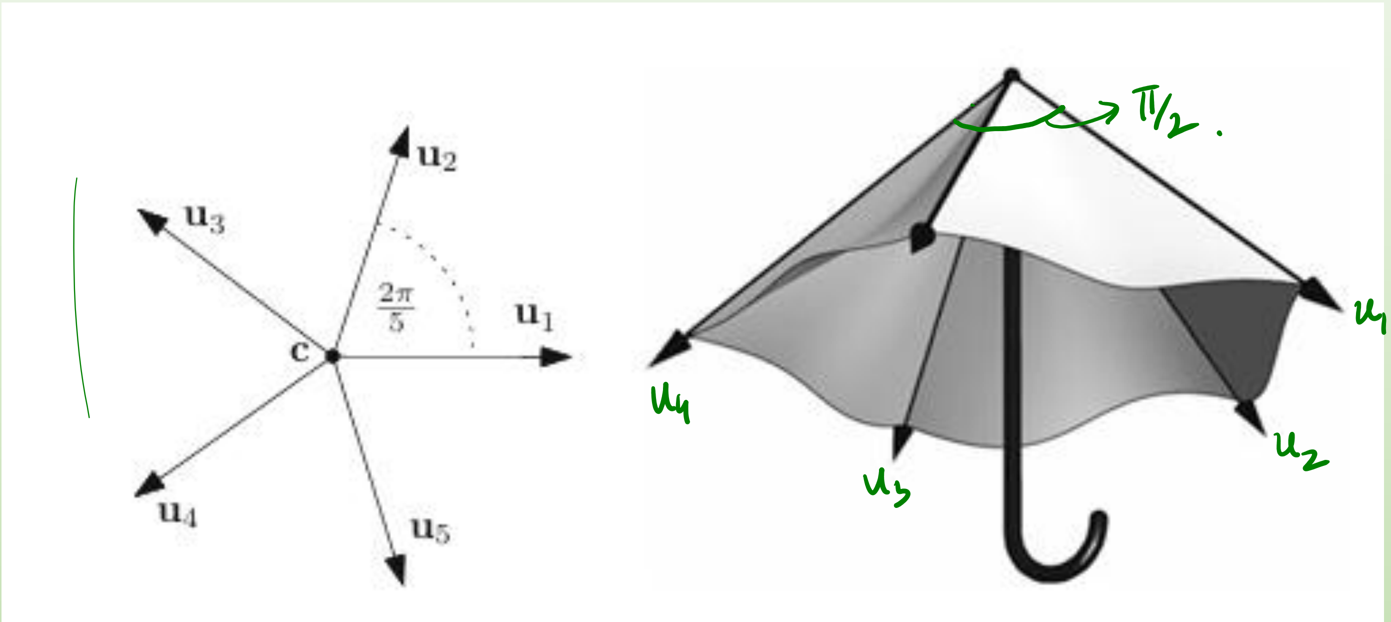
Finally, we have:



→ closed "umbrella"

<u>Obs^n</u>: At some intermediate point, orthonormality is satisfied

Illustration of OR for C_5



$$\vartheta(u) = \frac{1-\cos 4\pi/5}{-\cos 4\pi/5} = \sqrt{5}$$

At satisfying point of OR:

$$u_i = \left(\cos\frac{2\pi i}{5},\ \sin\frac{2\pi i}{5},\ z\right),\ \text{for } i = 1,2,3,4,5$$

$$u_5^T u_2 = 0. \Rightarrow (1, 0, z)^T \begin{pmatrix}\cos 4\pi/5 \\ \sin 4\pi/5 \\ z\end{pmatrix} = 0$$

$$\Rightarrow \cos\frac{4\pi}{5} + z^2 = 0.$$

To analyze $\underbrace{S(G)}_{\left(\alpha(G^k)\right)^{1/k}}$, used a "proxy" $\leq \vartheta(G)$.

$$\left(\underbrace{\alpha(G^k)}_{\text{hard to analyze}}\right)^{1/k}$$

$$\vartheta(G^k) \underset{=}{\leq} \left(\vartheta(G)\right)^k$$

$\square$

More properties of $\vartheta(G)$

- For a graph G and its complement $\bar{G}$, $\vartheta(G)\vartheta(\bar{G}) \geq n$

Relaxations for combinatorial quantities

- Linear Relaxation for Independent Set

References

- B. Gartner and J. Matousek, Approximation Algorithms and Semidefinite Programming, Springer, 2012