Adjacency matrices of graphs - III

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Perron-Frobenius : Graph version

Theorem

Let G be a connected graph adjacency matrix A. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues with the corresponding the eigenvectors $x_1, \ldots, x_n$, respectively. Then,

1. $\lambda_1 \geq -\lambda_n$.
2. $\lambda_1 > \lambda_2$.
3. *There exists a positive eigenvector x_1.*

Spectrum and bipartiteness

Theorem

Let G be a graph. Then, G is bipartite if and only if the following holds: whenever λ is an eigenvalue of $A(G)$, $-\lambda$ is an eigenvalue of $A(G)$. Further, if G is connected and $\lambda_1 = -\lambda_n$, then G is bipartite.

($\Rightarrow$) $A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$

($\Leftarrow$)

$\underline{\lambda_1^k + \cdots + \lambda_n^k}$ — # of closed walks of length k.

$\lambda_1 + \lambda_n = 0 \Rightarrow G$ is bipartite

$\lambda_1 + \lambda_n \neq 0 \Rightarrow G$ is close bipartite ??

~ 0

$$\boxed{\lambda_1 = -\lambda_n}$$ claim: G is bipartite

$Ax = \lambda_n x$, $x \neq 0$, $x^T x = 1$

$y = |x| := (|x_1|, |x_2|, \ldots, |x_n|)$,

$y^T y = 1$

$$\lambda_n = x^T A x$$
$$= \sum_{i,j} a_{ij} x_i x_j$$

$$\Rightarrow |\lambda_n| \leq \sum_{i,j} |a_{ij}| \cdot |x_i| \cdot |x_j|$$
$$= \sum_{i,j} a_{ij} y_i y_j$$
$$= y^T A y$$
$$\leq \lambda_1$$

$$\left[\because \ \lambda_1 = \max_{z \neq 0} \frac{z^T A z}{z^T z} \geq y^T A y \right]$$

$\because \lambda_1 = -\lambda_n$,

$\therefore$ all the inequalities in the above are equalities.

$$\lambda_1 = y^T A y$$

$\because$ G is connected $\Rightarrow$ $Ay = \lambda_1 y$, $\quad y_i > 0$

$$\boxed{\Rightarrow x_i \neq 0 \quad \forall i}$$

$$\left| \sum a_{ij} x_i x_j \right| = \sum_{i \sim j} |a_{ij}| \cdot |x_i| \cdot |x_j|$$

All the products $x_i x_j$ $(i \sim j)$ have the same sign

Note that, $\lambda_n = x^T A x < 0$

So each $x_i \cdot x_j < 0$ whenever $i \sim j$.

$V_1 = \{ i : x_i < 0 \}$
$V_2 = \{ i : x_i > 0 \}$

$V = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$
$\Rightarrow$ G is a bipartite graph.

Theorem (**Courant-Fischer**)

Let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and let $\{u_1, u_2, \ldots, u_n\}$ be any orthonormal basis of eigenvectors of A, where u_i is a unit eigenvector associated with λ_i. If $\mathcal{V}_k$ denotes the set of subspaces of $\mathbb{R}^n$ of dimension k, then

$$\lambda_k = \min_{W \in \mathcal{V}_k} \max_{x \in W, x \neq 0} \frac{x^T A x}{x^T x},$$

$$\lambda_k = \max_{W \in \mathcal{V}_{n-k+1}} \min_{x \in W, x \neq 0} \frac{x^T A x}{x^T x}.$$

$$\Rightarrow \begin{cases} \lambda_1 = \min\limits_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \dfrac{x^T A x}{x^T x} \\ \lambda_1 = \min\limits_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \dfrac{x^T A x}{x^T x} \end{cases}$$

Interlacing inequalities

Theorem

Let A be a symmetric $n \times n$ matrix and let B be a principal submatrix of A of order $n-1$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are the eigenvalues of A and B, respectively, then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \cdots \leq \mu_{n-1} \leq \lambda_n$$

$$A = \begin{bmatrix} B & y \\ y^T & c \end{bmatrix}$$

Interlacing inequalities

Theorem (Poincaré separation theorem)

Let A be an $n \times n$ matrix partitioned as

$$A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$$

such that B and D are square matrices order m and $n - m$, respectively. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of A and B, respectively, then $\lambda_i \leq \mu_i \leq \lambda_{n-m+i}$ for $i = 1, \ldots, m$

$$\lambda_1 \leq \mu_1 \leq \lambda_{(n-m)+1}$$

$$\vdots$$

$$\lambda_m \leq \mu_m \leq \lambda_{(n-m)+m} = \lambda_n$$

Wilf's theorem

Lemma

Let G be a graph with n vertices and H be a induced *subgraph of G on p vertices. Then,* $\lambda_{\max(G)} \geq \lambda_{\max(H)}$ *and* $\lambda_{\min(G)} \leq \lambda_{\min(H)}$.

Lemma

For a graph G, $\delta(G) \leq \lambda_{\max}(G) \leq \Delta(G)$, *where* $\Delta(G)$ *and* $\delta(G)$ *are the maximum and minimum vertex degrees of G, respectively.*

Theorem (Wilf's)

$\chi(G) \leq 1 + \lambda_{\max}(G)$, *where* $\chi(G)$ *is the chromatic number of G.*

$\chi(G) \leq 1 + \Delta(G) \rightarrow$ (Brookes theorem)

$G \longrightarrow A(G)$ — $n \times n$ matrix

$H \longrightarrow A(H)$ — $p \times p$ - matrix.

$A(H)$ principal submatrix $A(G)$

$$\left.\begin{aligned} \Rightarrow \lambda_{max}(G) &\geq \lambda_{max}(H) \\ \lambda_{min}(G) &\leq \lambda_{min}(H) \end{aligned}\right\} \text{Poincare separation theorem!}$$

——— x ———

$$\delta(G) \leq \lambda_{max}(G) \leq \Delta(G)$$

$$A(G)\,x = \lambda_{max} \cdot x$$

$$\sum_{j=1}^{n} a_{ij} \cdot x_j = \lambda_{max} \cdot x_i \quad \forall\, i = 1, \dots, n$$

$x_k > 0$ be the maximum coordinate of x.

$$\Rightarrow \lambda_{max} x_k = \sum_{j=1}^{n} a_{kj} \cdot x_j$$

$$\leq x_k \cdot \left(\overbrace{\sum_{j=1}^{n} a_{kj}}^{\deg(k)} \right)$$

$$\leq x_k \cdot \Delta(G)$$

$$\Rightarrow \boxed{\lambda_{max} \leq \Delta(G)}$$

$$\lambda_{max} = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

$$e = (1, \cdots, 1)$$

$$\Rightarrow \lambda_{max} \geq \frac{2m}{n}.$$

$$2m = \sum_{i=1}^{n} d_{v_i} \geq \delta \cdot n$$

$$\Rightarrow \lambda_{max} \geq \frac{\delta \cdot n}{n} = \delta$$

$$\boxed{\delta \leq \lambda_{max} \leq \Delta}$$

Proof of wilf's theorem : $\boxed{\chi(G) \leq 1 + \lambda_{max}^{(G)}}$

$\chi(G) = 1 \Rightarrow \lambda_{max}^{(G)} = 0$

$\therefore \chi(G) \leq 1 + \lambda_{max}^{(G)}$

Let $\chi(G) = p$, $p \geq 2$.

Let H be a subgraph of G such that

$\chi(H) = p$ & $\chi(H \setminus \{v_i\}) < p$.

Claim: $\boxed{\delta(H) \geq p-1}$.

Suppose that $\delta(H) < p-1$.

Let $\deg(v_i) < p-1$

$\chi(H \setminus \{v_i\}) < p$

i.e) $H \setminus \{v_i\}$ can be colored with at most $(p-1)$ colors.

Then v_i can be colored with one of the $(p-1)$ colors in H.

i.e) H is $(p-1)$ colorable.

$\Rightarrow \Leftarrow$ to $\chi(H) = p$

ie) $\delta(H) \geq p-1.$

$$\lambda_{max}(G) \geq \lambda_{max}(H) \geq \delta(H) \geq p-1$$

$$\Rightarrow \quad p \leq 1+\lambda_{max}(G)$$

$$\boxed{\chi(G) \leq 1+\lambda_{max}(G)}$$

Block matrices

Lemma

If B and C are symmetric $n \times n$ matrices, then

$$\lambda_{\max}(B + C) \leq \lambda_{\max}(B) + \lambda_{\max}(C)$$

Lemma

Let B be an $n \times n$ positive semidefinite matrix and suppose B is partitioned as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $p \times p$. Then $\lambda_{\max}(B) \leq \lambda_{\max}(B_{11}) + \lambda_{\max}(B_{22})$.

$$\lambda_{max}(B+C) = \max\{x^T(B+C)x : \|x\|_2=1\}$$

$$= \max\{x^TBx + x^TCx : \|x\|_2=1\}$$

$$\le \max\{x^TBx : \|x\|_2=1\} + \max\{x^TCx : \|x\|_2=1\}$$

$$= \lambda_{max}(B) + \lambda_{max}(C)$$

——— x —

B is P.S.D.

$\exists\, C$ s.t.

$B = CC^T$.

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}_{2\times 1}$$ conformal with B

$$C C^T = \begin{bmatrix} \underline{c_1 c_1^T} & c_1 c_2^T \\ c_2 c_1^T & \underline{c_2 c_2^T} \end{bmatrix} = \begin{bmatrix} \underline{B_{11}} & B_{12} \\ B_{21} & \underline{B_{22}} \end{bmatrix}$$

$$\Rightarrow B_{11} = c_1 c_1^T, \quad B_{22} = c_2 c_2^T$$

$$\begin{aligned} \lambda_{max}(B) &= \lambda_{max}(C C^T) \\ &= \lambda_{max}(C^T C) \end{aligned}$$

$C^T = [\, c_1^T \; c_2^T \,]$

$$= \lambda_{max}(c_1^T c_1 + c_2^T c_2)$$

$C^T C = [\, c_1^T \; c_2^T \,] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{aligned} &\le \lambda_{max}(c_1^T c_1) + \lambda_{max}(c_2^T c_2) \\ &= \lambda_{max}(c_1 c_1^T) + \\ &\quad \lambda_{max}(c_2 c_2^T) \end{aligned}$$

$$= \lambda_{max}(B_{11}) + \lambda_{max}(B_{22})$$

Block matrices - contd.

Lemma

Let B be an $n \times n$ symmetric matrix and suppose B is partitioned as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $p \times p$. Then $\lambda_{\max}(B) + \lambda_{\min}(B) \leq \lambda_{\max}(B_{11}) + \lambda_{\max}(B_{22})$.

Lemma

Let B be a symmetric matrix partitioned as

$$B = \begin{bmatrix} 0 & B_{12} & \dots & B_{1k} \\ B_{21} & 0 & \dots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \dots & 0 \end{bmatrix}_{k \times k}$$

Then $\lambda_{\max}(B) + (k-1)\lambda_{\min}(B) \leq 0$.

$k=2$, follows from

B is Symmetric $\Rightarrow$ $\underline{B - \lambda_{min}^{(B)} I}$ P.S.D.

$$B - \lambda_{min}^{(B)} I = \begin{bmatrix} B_{11} - \lambda_{min}^{(B)} I & B_{12} \\ B_{21} & B_{22} - \lambda_{min}^{(B)} I \end{bmatrix}$$

— P.S.D.

previous lemma,

$$\lambda_{max}(B - \lambda_{min}^{(B)} I) \le \lambda_{max}(B_{11} - \lambda_{min}^{(B)} I) + \lambda_{max}(B_{22} - \lambda_{min}^{(B)} I)$$

$$\Rightarrow \lambda_{max}(B) + \lambda_{min}(B) I \le \lambda_{max}(B_{11}) + \lambda_{max}(B_{22})$$

$\boxtimes$

$$\lambda_{max} + (k-1)\lambda_{min} \le 0$$

Assume that, the holds for $(k-1)$.

C - principal submatrix of B obtained by deleting the last row & coloumn blocks of B.

i.e) C is a $(k-1) \times (k-1)$ block matrix.

$$\therefore \lambda_{max}(C) + (k-2)\lambda_{min}(C) \le 0.$$

By the previous lemma, $\lambda_{max}(B) + \lambda_{min}(B) \le \lambda_{max}(C)$

$$B = \begin{bmatrix} C & * \\ * & 0 \end{bmatrix} \longrightarrow$$

Also, $\lambda_{min}(B) \leq \lambda_{min}(C)$

$$\therefore \quad \lambda_{max}(B) + \lambda_{min}(B) \leq \lambda_{max}(C)$$
$$\leq -(k-2)\lambda_{min}(C)$$

$$\Rightarrow \quad \lambda_{max}(B) + \lambda_{min}(B) + (k-2)\lambda_{min}(B) \leq 0$$

$$\Rightarrow \quad \lambda_{max}(B) + (k-1)\lambda_{min}(B) \leq 0.$$

$\boxtimes$

Hoffman's bound

Theorem (Hoffman's bound)

$\chi(G) \geq 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}$. , (G has at least one edge.)

$\chi(G) = k$, $V = V_1 \cup V_2 \cdots \cup V_k$, V_i: one color class

$$\begin{array}{c c} & \begin{array}{cccc} V_1 & V_2 & \cdots & V_k \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ \vdots \\ V_k \end{array} & \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \end{array}$$

$\lambda_{\max}(G) + (k-1)\lambda_{\min}(G) \leq 0 \Rightarrow k \geq 1 - \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}$

References

$A = \begin{bmatrix} & \cdot \\ \cdot & \end{bmatrix}$

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