Adjacency matrices of graphs - II

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$A = UDU^T$

Theorem (Spectral theorem for real symmetric matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

Theorem

If $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of an $n \times n$ real symmetric matrix, then the minimal polynomial of is given by $(x - \lambda_1) \ldots (x - \lambda_k)$.

Theorem

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k.

$Ax = \lambda x \quad \Rightarrow \quad A^2 x = A(Ax)$

$x \neq 0$

$= A(\lambda x)$

$= \lambda A x$

$= \lambda^2 x$

$k \in \mathbb{N}$

$\Rightarrow \quad A^k x = \lambda^k x,$

$\mathrm{spec}(A) = \{\lambda_1, \dots, \lambda_n\}$

$\mathrm{spec}(A^k) = \{\lambda_1^k, \dots, \lambda_n^k\}$

$p(x)$ – polynomial real coefficients

$\hookrightarrow a_0 + a_1 x + \dots + a_s x^s$

$p(A) = a_0 I + a_1 A + \dots + a_s A^s$

$$\operatorname{spec}(p(A)) = \{ p(\lambda_i) : \lambda_i \in \operatorname{spec}(A) \}$$

Positive Semidefinite Matrices(PSD)

positive definite $x^T A x > 0$
$\forall x \neq 0$

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is *positive semidefinite*(PSD) if $\underline{x^T A x} \geq 0$ for every $x \in \mathbb{R}^n$.

$= \langle Ax, x \rangle$

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$: + symmetric :

1. *A is PSD.*
2. *All the eigenvalues of A are nonnegative,*
3. *There exists an $n \times k$ real matrix B such that $A = BB^T$,*

$n \times k$ $\quad k \times n$

rank(A) = k

$A = A^T \Rightarrow A = UDU^T$

① $\Rightarrow$ ② $\quad Ax = \lambda x, \quad x \neq 0$

$$x^T A x = \lambda x^T x$$

$$\Rightarrow \lambda = \frac{x^T A x}{x^T x} \geq 0$$

② $\Rightarrow$ ① $\quad A = A^T, \quad A = UDU^T$

$$\downarrow D \geq 0$$

<u>claim:</u> A is psd

$$x^T A x = \langle Ax, x \rangle$$

$$= \langle UDU^T x, x \rangle$$

$$= \langle D U^T x, U^T x \rangle$$

$y = U^T x$

$$\geq 0$$

(i) A is P.S.D.

① $\Longleftrightarrow$ ③

A is PSD & $\text{rank}(A) = k$

$$A = U D U^T$$

$$= U \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & d_k & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} U^T$$

$$\tilde{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & d_k \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{n \times k}$$

$$B = U\sqrt{\tilde{D}}, \quad \text{where } \sqrt{\tilde{D}} = \begin{bmatrix} \sqrt{d_1} & 0 & \cdots & 0 \\ 0 & \sqrt{d_2} & \cdots & 0 \\ 0 & 0 & \cdots & \sqrt{d_k} \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$B^T = (\sqrt{\tilde{D}})^T U^T$$

$$BB^T = U\sqrt{\tilde{D}}(\sqrt{\tilde{D}})^T U^T$$

$$= U\tilde{D}U^T = UDU^T = A$$

$$A = BB^T \qquad B \in \mathbb{R}^{n \times k}.$$

(3) $\Rightarrow$ (1) $\boxed{A = BB^T}$, then A is PSD

$$\langle Ax, x\rangle = \langle BB^T x, x\rangle$$

$$= \langle B^T x, B^T x\rangle$$

$$= \|B^T x\|^2 \geq 0$$

Cone of positive semidefinite matrices

Let $\mathcal{S}^n$ denote the subspace of symmetric matrices in $\mathbb{R}^{n\times n}$.The set of positive semidefinite matrices in $\mathcal{S}^n$ will be denoted by $\mathcal{PSD}_n$. $\mathcal{PSD}_n$ is a convex cone.

convex cone: A subset $C \subseteq \mathbb{R}^{n\times n}$ is convex if

(i) $x, y \in C \Rightarrow x+y \in C$

(ii) $\alpha \geq 0$, $x \in C \Rightarrow \alpha x \in C$

(iii) $C \cap (-C) = \{0\}$ (pointed)

$A, B \in PSD_n \Rightarrow A+B \in PSD_n$

$x^T(A+B)x = x^TAx + x^TBx \geq 0$

(*) $\alpha \geq 0$, $\alpha A \rightarrow x^T(\alpha A)x = \alpha \cdot x^TAx \geq 0$

(**) $A, -A \in PSD_n$

$\underline{\underline{\text{Claim}}}$:- $A = 0$.

A is P.S.D $\Rightarrow$ eigenvalues of A are nonnegative

$-A$ is PSD $\Rightarrow$ " of A " are nonpositive

$\Rightarrow$ eigenvalues of A are zero.

Rayleigh ratio: $A \in \mathbb{R}^{n \times n}$. $x (\neq 0) \in \mathbb{R}^n$

$$\boxed{R(A)(x) = \frac{x^T A x}{x^T x},} \qquad x \in \mathbb{R}^n, x \neq 0.$$

Theorem (**Rayleigh-Ritz**)

If A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and if $\{u_1, u_2, \ldots, u_n\}$ is any orthonormal basis of eigenvectors of A, where u_i is a unit eigenvector associated with λ_i, then

$$\boxed{\max_{x \neq 0} \frac{x^T A x}{x^T x} = \lambda_1}$$

with the maximum attained for $x = u_1$, and

$$\boxed{\min_{x \neq 0} \frac{x^T A x}{x^T x} = \lambda_n}$$

with the maximum attained for $x = u_n$.

$A = A^T$, $\quad A = UDU^T$, $\quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \lambda_n \end{bmatrix}$

$\exists$ an orthonormal basis for $\mathbb{R}^n$ consists of the eigenvectors of A.

$\{u_1, \dots, u_n\}$ be ~~the~~ O.N.B

Let $x \in \mathbb{R}^n$.

$$x = \sum_{i=1}^{n} \alpha_i u_i$$

$$Ax = A\left(\sum_{i=1}^{n} \alpha_i u_i\right)$$

$$= \sum_{i=1}^{n} \alpha_i A u_i$$

$$= \sum_{i=1}^{n} \alpha_i \lambda_i u_i \quad \left[\because A u_i = \lambda_i u_i\right]$$

$$\langle Ax, x\rangle = \left\langle \sum_{i=1}^{n} \alpha_i \lambda_i u_i , \sum_{j=1}^{n} \alpha_j u_j \right\rangle$$

$$= \sum_{i=1}^{n} \alpha_i \lambda_i \left\langle u_i , \sum_{j=1}^{n} \alpha_j u_j \right\rangle = \sum_{i=1}^{n} \cancel{\sum_{j=1}^{n}} \alpha_i \lambda_i \alpha_j \langle u_i , u_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \lambda_j \alpha_i \overset{=1}{\langle u_i , u_i \rangle}$$

$$= \sum_{i=1}^{n} \alpha_i^2 \lambda_i$$

$$\underline{\langle x, x\rangle = \sum_{i=1}^{n} \alpha_i^2}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$x \neq 0$

$$\boxed{\frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \frac{\sum_{i=1}^{n} \alpha_i^2 \lambda_i}{\sum_{i=1}^{n} \alpha_i^2}} \quad — \quad (*)$$

$$\leq \lambda_1 \left(\frac{\cancel{\sum_{i=1}^{n} \alpha_i^2}}{\cancel{\sum_{i=1}^{n} \alpha_i^2}} \right)$$

$$= \lambda_1$$

$$\frac{\langle Ax, x\rangle}{\langle x, x\rangle} \leq \lambda_1 \quad \forall x \in \mathbb{R}^n, \ x \neq 0$$

$$\Rightarrow \boxed{\max_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \leq \lambda_1}$$

$\because\ Au_1 = \lambda_1 u_1 \Rightarrow \dfrac{\langle Au_1, u_1\rangle}{\langle u_1, u_1\rangle} = \lambda_1.$

$$\boxed{\therefore\ \lambda_1 = \max_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}}$$

From ⊛

$$\frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \frac{\sum \alpha_i^2 \lambda_i}{\sum \alpha_i^2}$$

$$\geq \lambda_n \quad \forall x$$

$$\Rightarrow \min_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \geq \lambda_n$$

$$\Rightarrow \quad \lambda_n = \min_{x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}. \quad ??$$

$\left[\because A u_n = \lambda_n u_n\right]$

<u>Theorem:</u> G - graph

$A(G)$ - adjacence matrix

Then $\lambda_1 \geq \frac{2m}{n}$, where

m is the number of edges &

n is the number of vertices.

$$\lambda_1 = \max_{x \neq 0} \frac{x^T A(G) x}{x^T x}.$$

$$e^T = (1, 1, \dots, 1)$$

$$e^T A(G) e = 2 \cdot m$$

$$e^T e = n$$

$$\lambda_1 \geq \frac{2m}{n}.$$

— x —

Theorem

If A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and if $\{u_1, u_2, \ldots, u_n\}$ is any orthonormal basis of eigenvectors of A, where u_i is a unit eigenvector associated with λ_i, then

(i) $$\boxed{\max_{x \perp \mathrm{span}\{u_1,\ldots,u_{k-1}\}} \frac{x^T A x}{x^T x} = \lambda_k}$$

with the maximum attained for $x = u_k$, and

(ii) $$\boxed{\min_{x \perp \mathrm{span}\{u_{k+1},\ldots,u_n\}} \frac{x^T A x}{x^T x} = \lambda_k}$$

with the minimum attained for $x = u_k$.

(i) $\{u_1, u_2, \dots, u_n\}$ ONB e.vectors

$2 \le k \le n$ span $\{u_1, \dots, u_{k-1}\}$ — subspace of $\mathbb{R}^n$

$x \perp \text{span}\{u_1, \dots, u_{k-1}\}$

$\Rightarrow \langle x, u_i \rangle = 0 \quad \Rightarrow \langle x, \sum_{j=1}^{k-1} \alpha_i u_i \rangle = 0$

<u>proof:</u>

$x \in \text{span}\{u_1, \dots, u_{k-1}\}^\perp$

$= \text{span}\{u_k, \dots, u_n\}$

$x = \alpha_k u_k + \dots + \alpha_n u_n$

$Ax = \alpha_k A u_k + \dots + \alpha_n A u_n$

$= \alpha_k \lambda_k u_k + \dots + \alpha_n \lambda u_n$

$$\langle Ax, x\rangle = \sum_{i=k}^{n} \alpha_i^2 \cdot \lambda_i$$

$$\langle x, x\rangle = \sum_{i=k}^{n} \alpha_i^2 .$$

$$\boxed{\frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \frac{\sum_{i=k}^{n} \alpha_i^2 \lambda_i}{\sum_{i=k}^{n} \alpha_i^2}}$$

$$\leq \lambda_k \quad \forall x \in \operatorname{span}\{u_1, \ldots, u_{k-1}\}^{\perp}$$

$$\max_{x \in \operatorname{span}\{u_1, \ldots, u_{k-1}\}^{\perp}} \left\{ \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \right\} \leq \lambda_k .$$

Take $x = u_k$. $\dfrac{\langle Ax, x\rangle}{\langle x, x\rangle} = \lambda_k$

& $u_k \in \operatorname{span}\{u_1, \dots, u_{k-1}\}^\perp$

$$\boxed{\max_{x \in \operatorname{span}^\perp} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \lambda_k}$$

(ii) $x \in \operatorname{span}\{u_{k+1}, \dots, u_n\}^\perp = \operatorname{span}\{u_1, \dots, u_k\}$

$$x = \sum_{i=1}^{k} \alpha_i u_i$$

$$\frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \frac{\sum_{i=1}^{k} \alpha_i^2 \cdot \lambda_i}{\sum_{i=1}^{k} \alpha_i^2}$$

$\geq \lambda_k$. $\forall x (\neq 0) \in$

$$\Rightarrow \min_{\operatorname{span}\{u_{k+1}, \dots, u_n\}^\perp} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \geq \lambda_k.$$

Equality holds, as $Au_k = \lambda_k u_k$

$$\Rightarrow \frac{\langle Au_k, u_k \rangle}{\langle u_k, u_k \rangle} = \lambda_k$$

Theorem (**Courant-Fischer**)

Let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and let $(u_1, u_2, \ldots, u_n)$ be any orthonormal basis of eigenvectors of A, where u_i is a unit eigenvector associated with λ_i. If $\mathcal{V}_k$ denotes the set of subspaces of $\mathbb{R}^n$ of dimension k, then

$$\lambda_k = \max_{W \in \mathcal{V}_{n-k+1}} \min_{x \in W, x \neq 0} \frac{x^T A x}{x^T x},$$

$$\lambda_k = \min_{W \in \mathcal{V}_k} \max_{x \in W, x \neq 0} \frac{x^T A x}{x^T x}.$$

Perron-Frobenius : Graph version

$A(G)$

Theorem

Let G be a connected graph adjacency matrix A. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues with the corresponding the eigenvectors $x_1, \ldots, x_n$, respectively. Then,

$\lambda_1 > 0$

1. $\lambda_1 \geq -\lambda_n$.
2. $\lambda_1 > \lambda_2$.
3. *There exists a positive eigenvector x_1.*

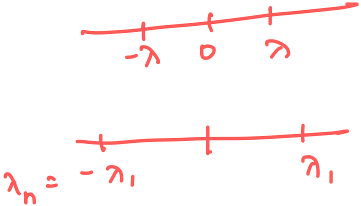
$A x_1 = \lambda_1 x_1$

$x_1 > 0$

Spectrum and bipartiteness

Theorem

Let G be a bipartite graph. Then λ is an eigenvalue of $A(G)$ if and only if $-\lambda$ is an eigenvalue of $A(G)$. Further, if G is connected and $\lambda_1 = -\lambda_n$, then G is bipartite.



G - bipartite $\Rightarrow$ $$A(G) = \begin{array}{cc} & \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \end{array}$$

$$\boxed{Ax = \lambda x} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} Bx_2 \\ B^T x_1 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad Bx_2 = \lambda x_1$$

$$B^T x_1 = \lambda x_2$$

$$\boxed{y = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}} \neq 0$$

$$Ay = \begin{bmatrix} Bx_2 \\ -B^T x_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ -\lambda x_2 \end{bmatrix}$$

$$\boxed{Ay = -\lambda y}$$

converse.

$(A^k)_{ii}$ — # of walks of length k starting & ends at i.

$\sum_{i=1}^{n} (A^k)_{ii}$ = total number of closed walks of length k in G.

$= \lambda_1^k + \dots + \lambda_n^k$

k – odd, $\lambda_1^k + \dots + \lambda_n^k = 0$

$\because$ λ is e.value of $A(G)$

$\iff -\lambda$ is e.value of $A(G)$

$\Rightarrow$ G does not have any odd cycle

$\Rightarrow$ G is bipartite.

Interlacing inequalities

Theorem (Cauchy interlacing theorem)

Let A be a symmetric $n \times n$ matrix and let B be a principal submatrix of A of order $n-1$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are the eigenvalues of A and B, respectively, then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \cdots \leq \mu_{n-1} \leq \lambda_n$$

Theorem

Let A and B be symmetric $n \times n$ matrices such that $B = A + xx^T$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of A and B, respectively, then

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \cdots \leq \lambda_n \leq \mu_n$$

Interlacing inequalities

Theorem (Poincaré separation theorem)

Let A be an $n \times n$ matrix partitioned as

$$A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$$

such that B and D are square matrices order m and $n - m$, respectively. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_m$ are the eigenvalues of A and B, respectively, then $\lambda_i \leq \mu_i \leq \lambda_{n-m+i}$ for $i = 1, \ldots, m$

Wilf's theorem

Theorem

$\chi(G) \leq 1 + \lambda_1(G)$, *where* $\chi(G)$ *is the chromatic number of G.*

Proof of Wilf's theorem

Block matrices

Lemma

If B and C are symmetric $n \times n$ matrices, then

$$\lambda_1(B+C) \leq \lambda_1(B) + \lambda_1(C)$$

Lemma

Let B be an $n \times n$ positive semidefinite matrix and suppose B is partitioned as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $p \times p$. Then $\lambda_1(B) \leq \lambda_1(B_{11}) + \lambda_1(B_{22})$.

Block matrices - contd.

Lemma

Let B be an $n \times n$ symmetric matrix and suppose B is partitioned as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $p \times p$. Then $\lambda_1(B) + \lambda_n(B) \leq \lambda_1(B_{11}) + \lambda_1(B_{22})$.

Lemma

Let B be a symmetric matrix partitioned as

$$B = \begin{bmatrix} 0 & B_{12} & \dots & B_{1k} \\ B_{21} & 0 & \dots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \dots & 0 \end{bmatrix}$$

Then $\lambda_1(B) + (k-1)\lambda_n(B) \leq 0$.

Hoffman's bound

Theorem (Hoffman's bound)

$\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$.

References

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