Lecture-1

Adjacency matrices of graphs

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Graphs

Definition

A **graph** $G = (V(G), E(G))$ consists of two sets:

- $V(G)$ the vertex set of graph which is a non-empty set of elements called **vertices**,
- $E(G)$ the edge set of graph which is a possibly empty set of elements called **edges**, such that each edge e is assigned an unordered pair of vertices (u, v) called the end vertices of the edge e.

$|V(G)|$ – finite set

undirected – no direction for the edges

simple – no loops & no multi edges

Graph Examples

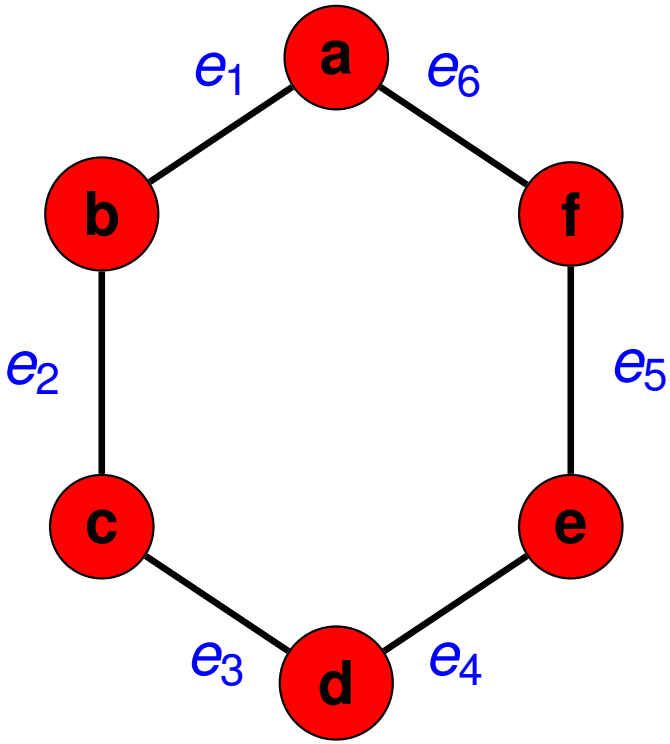


Figure: Cycle C_6

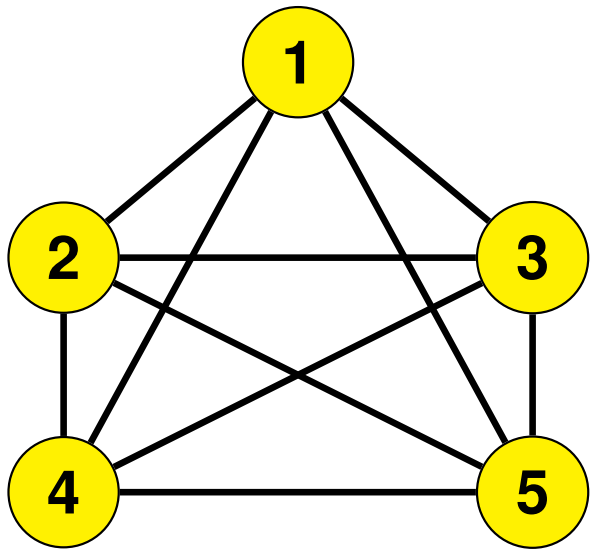


Figure: Clique K_5

Complete graph

Graph Examples Contd

Figure: Path P_5

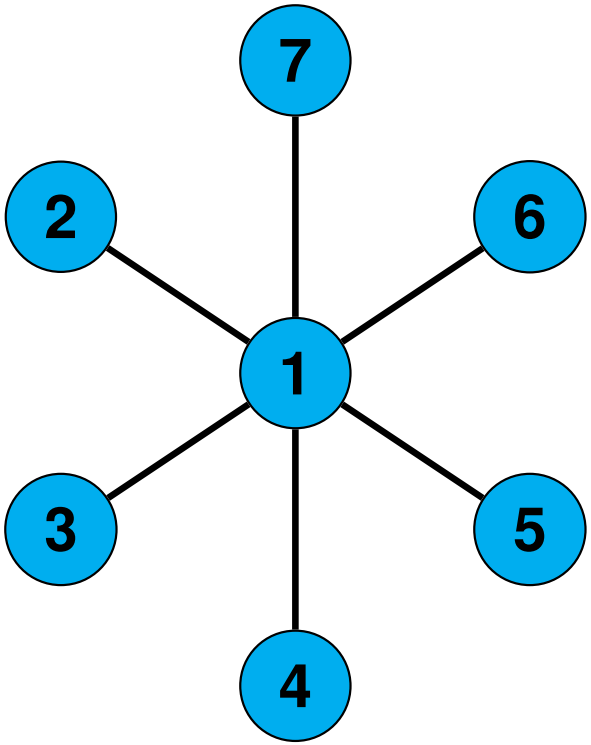


Figure: Star Graph $K_{1,6}$

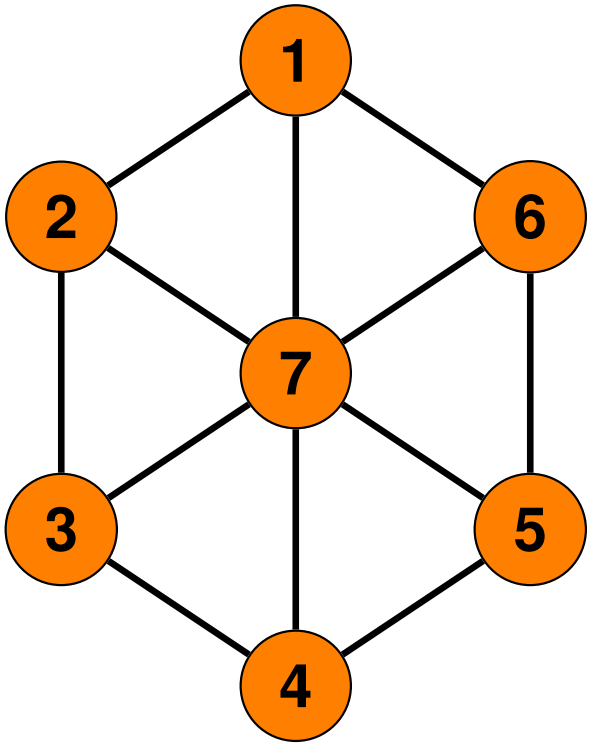


Figure: Wheel W_7

Graph Examples Contd

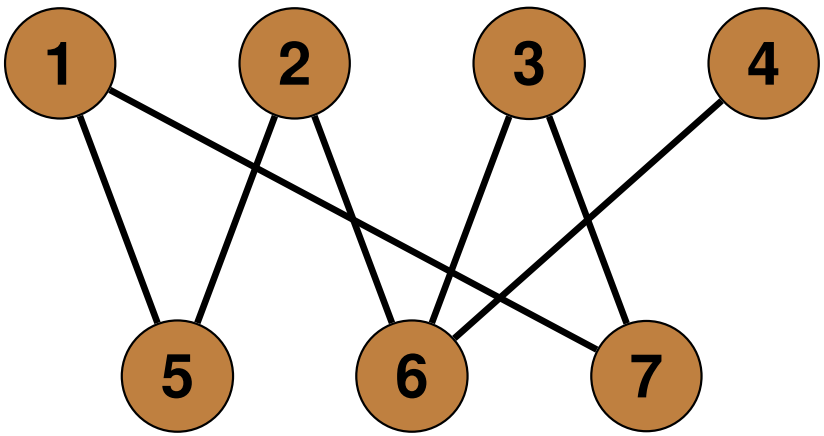


Figure: Bipartite Graph

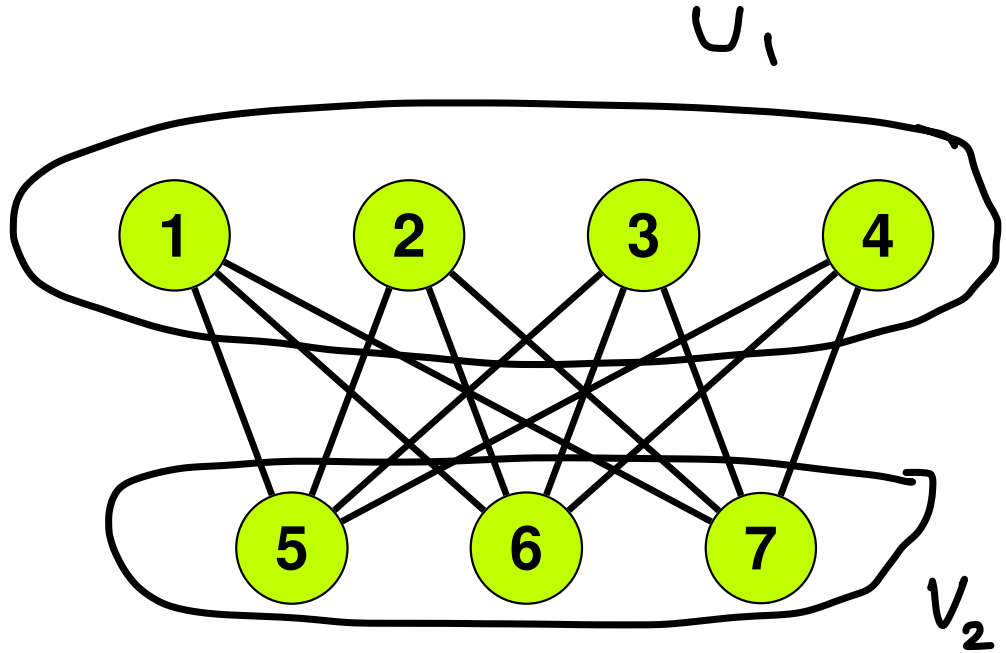


Figure: Complete bipartite graph $K_{4,3}$

↳ $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 \in \phi$

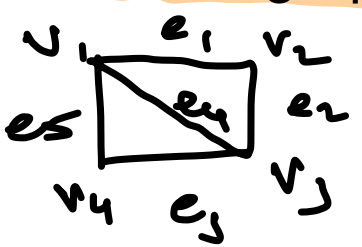
$e \in E(G)$

$e = uv$, $u \in V_1$ $v \in V_2$

Definitions

$G_1 = (V(G), E(G))$

- A subgraph H of G is such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- A spanning subgraph H of G is subgraph with $V(H) = V(G)$ and $E(H) \subseteq E(G)$.
- A vertex v_i is incident with an edge e_k, if $v_i \cap e_k \neq \emptyset$. $e_k = \{v_i, v_j\}$
- Two vertices v_i and v_j are adjacent, if $(v_i, v_j) \in E$.
- A walk in a graph G is a non-empty alternating sequence of vertices and edges $v_1 e_1 \ldots e_{k-1} v_k$.
- A path is a walk in which all the vertices are distinct.
- A graph G is connected, if between any two vertices of G there exist a path connecting them in G.
- A component is maximal connected subgraph of a graph G.



$w: v_1 e_1 v_2 e_2 v_3 e_6 v_4 e_5 v_1 e_5 v_4$

Definitions

- A cycle is a path in which the starting and the ending vertices are the same.
- A graph G is a tree, if there is exactly one path between any two vertices of G (equivalently, connected graph without cycles).
- If v_i and v_j are vertices of a connected graph $\mathcal{G}$, then the distance between the vertices v_i and v_j, denoted by $d(v_i, v_j)$, defined as the length of a shortest path between them.
- The diameter of a graph G is $\mathrm{diam}(G) = \max\{d(v_i, v_j)\}$

Adjacency matrix

Spectral graph theory
- Study of structure of graphs using eigenvalues.

Definition (Adjacency matrix)

The adjacency matrix of a graph G with n vertices is an $n \times n$ matrix, denoted by $A(G) = (a_{ij})$, and is defined by

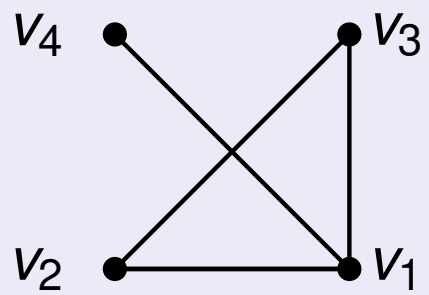
$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$A(G) = \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} \overset{\begin{matrix} v_1 & v_2 & \dots & v_n \end{matrix}}{\begin{bmatrix} & & 1 & & \end{bmatrix}} \in \{0,1\}^{n \times n}$$

Example

Example

Consider the graph G



The **adjacency matrix** of G is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Characteristic Polynomial

Let A be an $n \times n$ matrix. Then the *characteristic polynomial* of the matrix A is defined as:

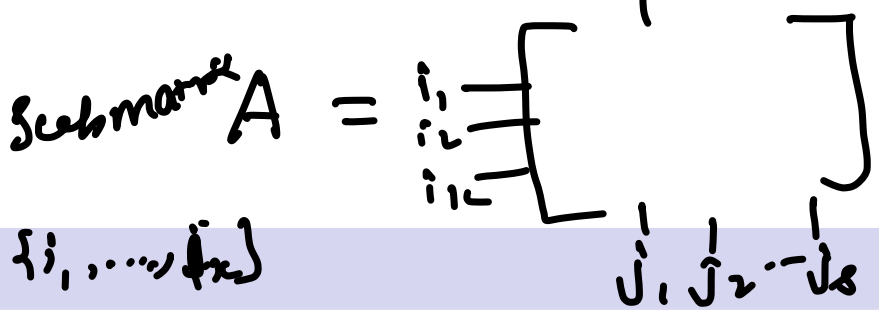
$$\det(A - \lambda I),$$

and the *characteristic equation* of A is

$$\det(A - \lambda I) = 0.$$

The roots of the characteristic equation are called the eigenvalues of the matrix A. By the Fundamental Theorem of Algebra, every $n \times n$ matrix (real/complex) has n eigenvalues.

Characteristic polynomial : Characterization



Definition

A principal submatrix of a square matrix is a submatrix formed by a set of rows and the corresponding set of columns. A principal minor of A is the determinant of a principal submatrix. The sum of its principal minors of size k is denoted by $E_k(A)$.

Theorem

The characteristic polynomial of a matrix A is given by :

$$\det(A - \lambda I) = \lambda^n - E_1(A)\lambda^{n-1} + \cdots + (-1)^{n-1} E_{n-1}(A)\lambda + (-1)^n E_n(A).$$

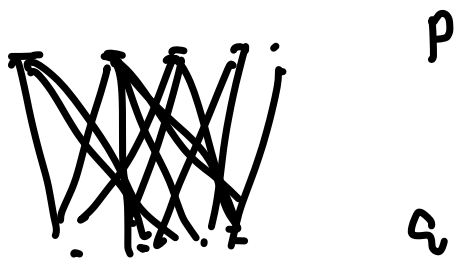
Examples

Theorem

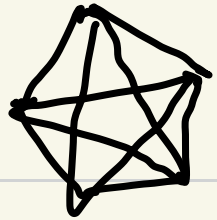
For any positive integer n, the eigenvalues of K_n are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$.

Theorem

For any positive integers p and q, the eigenvalues of $K_{p,q}$ are $\sqrt{pq}$, $-\sqrt{pq}$ and 0 with multiplicity $p+q-2$.



K_n → spectrum.



$$A(K_n) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & & \cdots & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \rightarrow \text{e-values of } J = \{ n^{(1)}, \underline{0}^{(n-1)} \}$$

$e^T = (1, \cdots, 1)$

$Je = n \cdot e$

$A(K_n) = J - I$

→ eigenvalue

$\text{Spec}(A(K_n))$

$= \{ (n-1)^{(1)}, -1^{(n-1)} \}$

$X + I$

$V = V_1 \cup V_2$

$K_{p,q}$:

$|V_1| = p$

$|V_2| = q$

$n = p+q$

$$A(K_{p,q}) = \begin{array}{c} \\ V_1 \\ V_2 \end{array} \begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \end{array}$$

J : $(q-2)$ zeros

* $A(K_{p,q})$ → rank is 2.

$\operatorname{tr}(A(K_{p,q})) = 0$

$\lambda \neq 0 \in \operatorname{Spec}(A(K_{p,q}))$

$\Rightarrow -\lambda \in \operatorname{Spec}(A(K_{p,q}))$

# of edges in $K_{p,q} = pq$

$A x = \lambda x$,

$x^T = (x_1, \cdots, x_p, x_{p+1}, \cdots, x_{p+q})$

$\Rightarrow$ $x_{p+1} + \cdots + x_{p+q} = \lambda x_1$

$x_{p+1} + \cdots + x_{p+q} = \lambda x_2$

$\vdots$

$x_{p+1} + \cdots + x_{p+q} = \lambda x_p$

$x_1 + \cdots + x_p = \lambda x_{p+1}$

$x_1 + \cdots + x_p = \lambda x_{p+2}$

$\vdots$

$x_1 + \cdots + x_p = \lambda x_{p+q}$

} q equations

Sum first p-equations,

$$p\,(x_{p+1} + \cdots + x_{p+q}) = \lambda(x_1 + \cdots + x_p)$$

$$= \lambda \cdot \lambda \cdot x_{p+1}$$

$$= \lambda^2 \cdot x_{p+1}$$

$$p(x_{p+1} + \cdots + x_{2p}) = \lambda^2 y_{p+2}$$

$$\vdots$$

$$p(x_{(q-1)p+1} + \cdots + x_{qp}) = \lambda^2 x_{p+q}$$

p. q $(x_{p+1} + \cdots + x_{2p}) = \lambda^2 (x_{p+1} + \cdots x_{p+q})$

$\neq 0$

$$\Rightarrow \quad \lambda^2 = pq$$

$$\Rightarrow \quad \lambda^2 - pq = 0$$

$$\Rightarrow \lambda = \pm\sqrt{pq}.$$

$$\text{Spec}(A(K_{p,q})) = \left\{ \sqrt{pq}^{(1)}, -\sqrt{pq}^{(1)}, 0^{n-2} \right\}$$

Examples

For a positive integer $n \geq 2$, let Q_n be the full-cycle permutation matrix of order n. Thus, the $(i, i+1)$-element of Q_n is 1, $i = 1, 2, \ldots, n-1$, the $(n, 1)$-element of Q_n is 1, and the remaining elements of Q_n are zero.

Lemma

For $n \geq 2$, the eigenvalues of Q_n are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{\frac{2\pi i}{n}}$, is the primitive n^{th} root of unity.

Theorem

For $n \geq 2$, the eigenvalues of C_n are $2\cos\frac{2\pi k}{n}$, $k = 1, 2, \ldots, n$.

Graphs & matrices
by R. B. Bapat.

Symmetric matrices

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Theorem

Eigenvalues of any Hermitian matrix are real numbers. (Eigenvalues of any real symmetric matrix are real numbers)

$$Ax = \lambda x, \quad x \neq 0$$

$$\lambda \langle x, x \rangle = \langle \lambda x, x \rangle$$

$$= \langle Ax, x \rangle$$

$$= \langle x, Ax \rangle \quad (\because A = A^T)$$

$$= \langle x, \lambda x \rangle$$

$$= \bar{\lambda} \langle x, x \rangle$$

$$\Rightarrow \lambda = \bar{\lambda}$$

Symmetric matrices

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Theorem

Eigenvalues of any Hermitian matrix are real numbers. (Eigenvalues of any real symmetric matrix are real numbers)

A real matrix A is said to be orthogonal if $AA^T = A^T A = I$, and a complex matrix A is said to be unitary if $AA^* = A^* A = I$.

Theorem (Spectral theorem for real symmetric matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

$A = U D U^T$

U - ortho-

D - diagonal matrix

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- Let λ be a real eigenvalue of A and x be a corresponding eigenvector with unit length.
- Let P be an orthogonal matrix with x as the first column.
- Then $P^{-1}AP = \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix}$, for some vector y^T and some $(n-1) \times (n-1)$ matrix C.

→ Home work !

$$AB = [Ab_1 \cdots Ab_n]$$

$$B = [b_1 \cdots b_n]$$

Proof continued...

- Since $P^{-1} = P^T$, we have $y^T = 0$ and C is symmetric.

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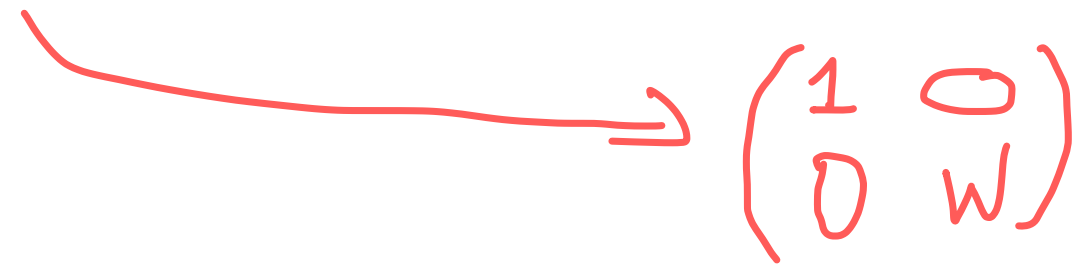
- Set $Q = diag(1, W)$, then Q and PQ are diagonal matrices.



Proof continued...

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- By induction, we have $C = W^{-1}DW$, where D is a diagonal matrix and W is an orthogonal matrix.
- Set $Q = diag(1, W)$, then Q and PQ are diagonal matrices.
- $(PQ)^{-1}A(PQ) = \begin{pmatrix} 1 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & D \end{pmatrix}$.

Linear Algebra

A. Ramachandra Rao

Bhimasankaram.

□

Theorem

If $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of an $n \times n$ real symmetric matrix, then the minimal polynomial of is given by $(x - \lambda_1) \ldots (x - \lambda_k)$.

$A = U D U^T$, $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

Properties

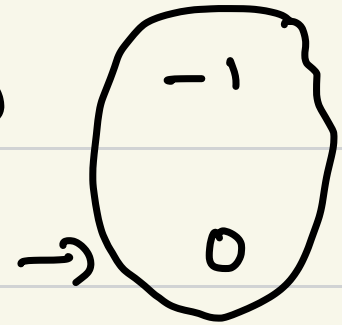
Let G be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. A is symmetric.
2. Sum of the 2×2 principal minors of A equals to $-|E(G)|$.
3. Sum of the 3×3 principal minors of A equals to twice the number of triangles in the graph.
4. $(i, j)^{th}$ entry of the matrix A^k equals the number of walks of length k from the vertex i to the vertex j.
5. If v_i and v_j are vertices of G with $d(v_i, v_j) = m$, then the matrices $I, A, \ldots, A^m$ are linearly independent.
6. Let G be a connected graph with k distinct eigenvalues and let d be the diameter of G. Then $k > d$.

$k \geq d+1$

②

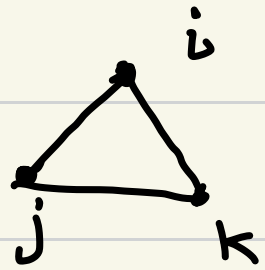
$$\begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} - i \sim j \quad \rightarrow$$



$$\begin{array}{c} \\ i \\ j \end{array}\begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} - i \not\sim j$$

③

$$\begin{array}{c} \\ i \\ j \\ k \end{array}\begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$



no edge - 0

one edge - 0

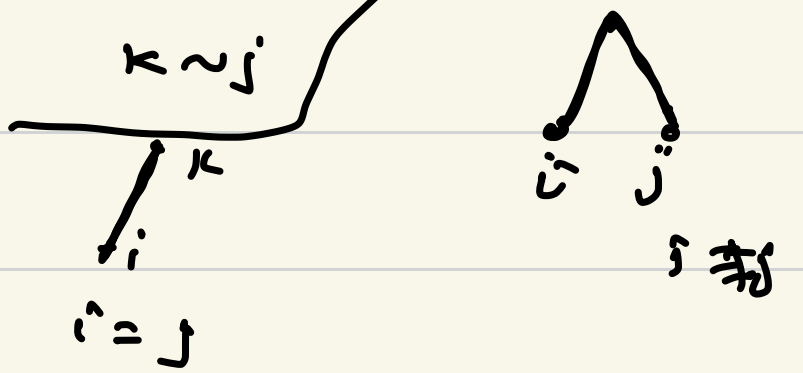
two edges - 0

three edges - 2

④ $(A \cdot A)_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} \neq 0$

$a_{ik} \neq 0 \quad a_{kj} \neq 0$

$i \sim k \qquad k \sim j$



$A^k = A^{k-1} \cdot A$

$(i,j)^{th}$ - walk of length $(k-1)$ from i to j

$I, A, \dots, A^k, \dots \in \mathbb{R}^{n\times n}$, vector space over $\mathbb{R}$

$d(v_i, v_j) = k$

$\{I, A, \dots, A^k\}$ – Linearly independent set in $\mathbb{R}^{n\times n}$

$$\alpha_1 I + \alpha_2 A + \dots + \alpha_{k+1} A^k = 0$$

Claim: $\alpha_i = 0 \quad \forall 1 \le i \le k+1$

$i \to j$: $A^k_{ij} \neq 0$, $A^{k-1}_{ij} = 0$, $A^s_{ij} = 0$ $\forall 1 \le s \le k-1$

$\Rightarrow \alpha_{k+1} = 0$

$\Rightarrow \{I, A, \dots, A^k\}$ – L.I.

$A^s_{ij'} = 0 \quad \forall 1 \le s \le k-2$

$diam(G) = d$

$\{I, A, \cdots, A^d\}$ — L.I.

$\alpha_1 I + \cdots + \alpha_{d+1} A^d \neq 0 \quad \forall \alpha_1, \cdots \alpha_{d+1} \in \mathbb{R}$

$\Rightarrow$ any d^{th} degree polynomial will not be minimal polynomial for A.

ie) deg. of min. poly. $\geq d+1$

$\Rightarrow$ $A(G)$ has atleast $(d+1)$ distinct eigenvalues.

ie) $d < k \rightarrow$ # of distinct is $d+1 \leq k$