Adjacency Algebra of a Graph

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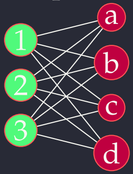
Online Short-Term Program on "Graphs, Matrices and Applications"
(Under Malaviya Mission Teacher Training Program), 29 Sep- 4 Oct, 2025

Graph $(K_{3,4})$



Adjacency matrix

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Spectrum

$$\begin{pmatrix} -\sqrt{12} & 0 & \sqrt{12} \\ 1 & 5 & 1 \end{pmatrix}.$$

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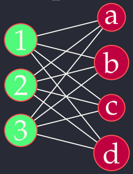

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- The adjacency matrix of $K_{m,n}$ has rank 2.

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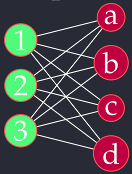

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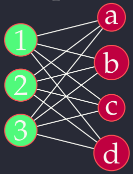

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- Let λ_1, λ_2 be the remaining two non-zero eigenvalues.

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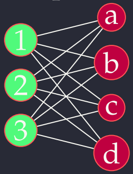

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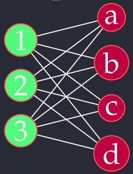	$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{pmatrix} -\sqrt{12} & 0 & \sqrt{12} \\ 1 & 5 & 1 \end{pmatrix}.$

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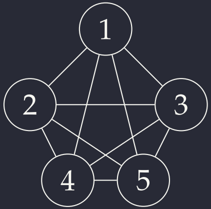
- Thus,

$$Spec(K_{m,n}) = \begin{pmatrix} -\sqrt{mn} & 0 & \sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}.$$



K_5

Eigenvectors (X^T):

$$[1,1,1,1,1]^T$$
$$[1,-1,0,0,0]^T$$
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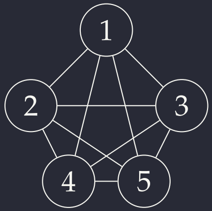
$$Spec(K_n) = \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix}.$$



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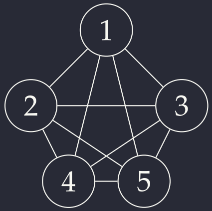
Note that the adjacency matrix of K_n equals

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix} = \mathbf{J} - I.$$



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Thus, the eigenpairs are $(\mathbf{e}, n-1)$ and $(\mathbf{e}_1 - \mathbf{e}_i, -1)$, for $2 \le i \le n$. Thus,

$$Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

$\mathbf{e} \in \mathbb{R}^n$ is the column vector of all $1's$.

Problem-1 Let $X = (V, E)$ be a simple undirected connected graph with n vertices, and let A denote its adjacency matrix. Many properties of X such as regularity or bipartiteness can be characterized from the spectrum of A. If X is large, however, investigating the spectrum of X might be cumbersome, which motivates to study "condensed" versions of A that preserve properties of its spectrum.

$\boxed{\text{How to find these condensed versions of } A\text{?}}$

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Problem-2 A graph Y is a polynomial in a graph X if the adjacency matrix of Y, $A(Y)$ is a polynomial in $A(X)$. That is $\exists p(x) \in \mathbb{C}[x]$ such that $A(Y) = p(A(X))$. Given a graph X, what are all graphs which are polynomials in X? To attack this problem, we introduce adjacency algebra of X that is exactly set of all polynomials in A. If a given graph has few nice regular properties, then adjacency algebra of X will have a basis such that whose elements are symmetric matrices with entries $0, 1$.

$\boxed{\text{How to find those graphs which have these nice basis?}}$

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- If X is a graph, then $Aut(X)$, the set of all automorphisms of X is a group which acts on the vertices of V of X hence on $V \times V$ whose orbits provides partitions to both V and $V \times V$.
- Thus knowledge on automorphism group of a graph and group actions will be very useful.

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- To define this, let $\mathcal{P} = \{V_1, \ldots, V_m\}$ $(m < n)$ be a partition of V, and, for $i, j \in [m] = \{1, \ldots, m\}$ and $u \in V_i$, let $b_{ij}(u)$ be the number of neighbors of u in V_j.

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- The partition $\mathcal{P}$ is called *equitable (or regular)* if $b_{ij}(u)$ is independent from the concrete choice of $u \in V_i$, i.e., $b_{ij}(u) = b_{ij}(v)$ for all $u, v \in V_i$.

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- In this case, the matrix $B = (b_{ij})_{i,j \in [m]}$ is called the *quotient matrix* of $\mathcal{P}$. Since it is known that for an equitable partition the eigenvalues of B are also eigenvalues of A, some spectral properties of B carry over to A. We will see them in a while, most of the content is taken from the book Godsil and Royle [3].

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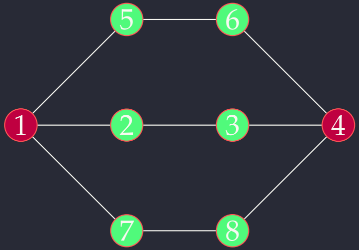
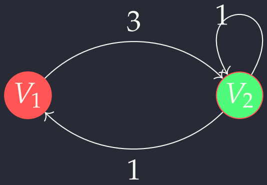
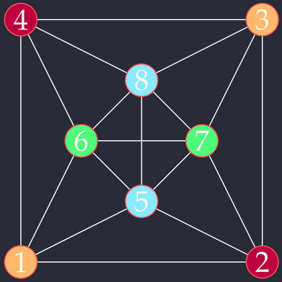
- Equitable partitions have been proven to be useful to derive, among many others, sharp eigenvalue bounds on the independence number like the celebrated ratio bound by Hoffman [5], but such results only hold when the underlying graph is regular.



$$
\begin{array}{c}
\\ 1 \\ 4 \\ 2 \\ 3 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{array}{c}
\begin{array}{cc|cccccc} 1 & 4 & 2 & 3 & 5 & 6 & 7 & 8 \end{array} \\
\left[\begin{array}{cc|cccccc}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$



	V_1	V_2
V_1	0	3
V_2	1	1



$$
\begin{array}{c}
 \\ 1 \\ 3 \\ \cdots \\ 2 \\ 4 \\ \cdots \\ 6 \\ 7 \\ \cdots \\ 5 \\ 8
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 3 & 2 & 4 & 6 & 7 & 5 & 8 \end{array} \\
\left(\begin{array}{cc|cc|cc|cc}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
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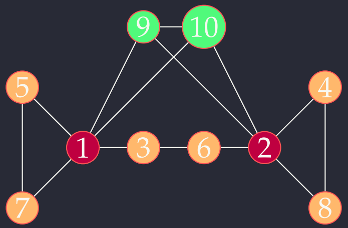
\end{array}\right)
\end{array}
$$

$$
\begin{bmatrix}
0 & 2 & 1 & 1 \\
2 & 0 & 1 & 1 \\
1 & 1 & 1 & 2 \\
1 & 1 & 2 & 1
\end{bmatrix}
$$



$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
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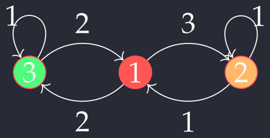
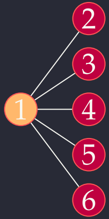


$$
\begin{bmatrix} 0 & 3 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}
$$

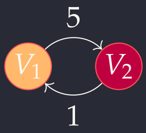
Let $X = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Then the partition

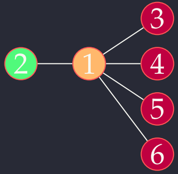
- $\pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ always an equitable partition.
- $\pi = \{\{v_1, v_2, \ldots, v_n\}\}$ is an equitable partition if and only if X is a regular graph.



$$\left[\begin{array}{c|ccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\{\{1\}, \{2, 3, 4, 5, 6\}\} \begin{bmatrix} 0 & 5 \\ 1 & 0 \end{bmatrix}$$





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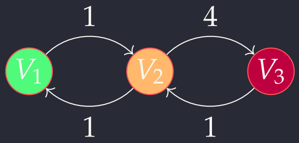
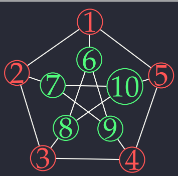
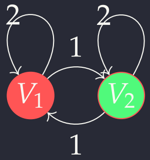
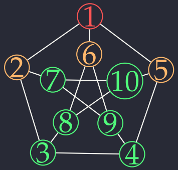
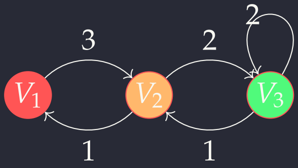
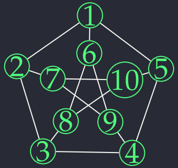


$$\{\{2\},\{1\},\{3,4,5,6\}\}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

This partition is called *distance partition* with respect to a vertex v. We denote $X_i(v) = \{w \in V | d(v,w) = i\}$.

In this example $v = 2$, $X_0(2) = \{2\}, X_1(2) = \{1\}, X_2(2) = \{3,4,5,6\}$.

Graph	Quotient graph	Quotient Matrix	Eigenvalues
		$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\{1, 3\}$
		$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$	$\{-1, 3, 1\}_{\text{distance partition}}$
		$\begin{bmatrix} 3 \end{bmatrix}$	$\{3\}$

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There is an elegant linear-algebraic way to characterize an equitable partition, that is very useful and gives insight into the structure of the eigenvectors of a graph.

Let $S = \{1,2,3,\ldots,10\}$ and $\pi = \{\{1,2,3,4,5\},\{6,7,8,9,10\}\}$ be a partition of S.

$$\text{Let } P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \text{ Then } P^T P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

$$(P^T P)^{-1} P^T = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}.$$

$$
\begin{aligned}
N = P(P^TP)^{-1}P^T \quad &= \quad P\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \\
&= \begin{bmatrix}
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}
\end{bmatrix}
\end{aligned}
$$

N is a doubly stochastic matrix.

$\boxed{\text{A partition } \pi \text{ is equitable if and only if } N \text{ commutes with } A.}$

If $V = \{1,2,3,4,5,6,7,8\}$ then $\pi = \{\{1,4,6\},\{2,5\},\{7,8\},\{3\}\}$ is a partition of V with cells $C_1 = \{1,4,6\}, C_2 = \{2,5\}, C_3 = \{7,8\}, C_4 = \{3\}$, and π is a 4-partition.

The characteristic matrix of π is $P = [c_1, c_2, c_3, c_4], P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

$$P^T P = \begin{bmatrix} c_1^T c_1 & 0 & 0 & 0 \\ 0 & c_2^T c_2 & 0 & 0 \\ 0 & 0 & c_3^T c_3 & 0 \\ 0 & 0 & 0 & c_4^T c_4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, since P has orthogonal columns then we have Notice that the diagonals are just the cardinalities of the cells, *i.e.*, $c_i^T c_i = |C_i|$.

Let $S = \{1,2,3,4,5\}$ and $\pi = \{\{3\},\{2,4\},\{1,5\}\}$ be a partition of S.

- Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $P^T P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.
- Further $(P^T P)^{-1} P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$.
- Finally $P(P^T P)^{-1} P^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$.

Is there a graph with this partition as an equitable partition? Of course every partition is equitable for K_5. We look for a nontrivial graph.

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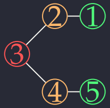
Consider the path graph P_5. ①–②–③–④–⑤
Consider the distance partition of P_5 w.r.to vertex 3.



$$(P^TP)^{-1}P^TA = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$ Now

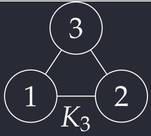
$$(P^TP)^{-1}P^TAP = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} P = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = B,$$ the quotient matrix.

$$AP = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let $u \in V(X)$. Then the uj-entry of AP is the number of neighbours of u that lie in cell C_j. If $u \in C_i$, then this number is b_{ij}. Now, the uj-entry of PB is also b_{ij}, because the only nonzero entry in the $u-$row of P is a 1 in the i-column.

Therefore $AP = PB$ and so $P^T AP = P^T PB$ or equivalently $(P^T P)^{-1} P^T AP = B$.

Graph	Partition of $\{1,2,3\}$	Ch Matrix	Quotient Matrix	Eigenvalues
	$\{\{1\},\{2\},\{3\}\}$	$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\{-1,-1,2\}$
K_3	$\{\{1,2,3\}\}$	$P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 2 \end{bmatrix}$	$\{2\}$
K_3	$\{\{1\},\{2,3\}\}$ $\{\{2\},\{1,3\}\}$ $\{\{3\},\{1,2\}\}$	$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$	$\{-1,2\}$

Every partition of $\{1,2,3\}$ is an equitable partition for K_3.

First case:

$$(P^TP)^{-1}P^TAP = A = B.$$

Second case:

$$(P^TP)^{-1}P^TAP = \frac{1}{3}\begin{bmatrix}1 & 1 & 1\end{bmatrix} A \begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}[6] = [2] = B.$$

Third case:

$$\begin{aligned}(P^TP)^{-1}P^TAP &= \begin{bmatrix}1 & 0\\0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\0 & 1\end{bmatrix}\\ &= \begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 2\\1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix} = B.\end{aligned}$$

- Let $A \in M_n(\mathbb{R})$.
- Suppose that W is A-invariant subspace of $\mathbb{R}^n$ and let $\beta = \{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}\}$ be a basis for W and consider the matrix $P = \begin{bmatrix} \mathbf{y_1} & \mathbf{y_2} & \cdots & \mathbf{y_k} \end{bmatrix}$.
- Now, since W is A-invariant then $A\mathbf{y_i} \in \mathbf{W}$ and therefore $A\mathbf{y_i}$ can be written as a linear combination of the basis vectors β. Therefore, there is some vector $\mathbf{b_i} \in \mathbb{R}^\mathbf{k}$ such that $A\mathbf{y_i} = \mathbf{P}\mathbf{b_i}$. This holds for each $i = 1, 2, \ldots, k$ and therefore if we set $\mathbf{B} = \begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \cdots & \mathbf{b_k} \end{bmatrix}$ then

$$AP = PB.$$

Lemma

Let X be a graph with adjacency matrix A and let π be a partition of $V(X)$ with characteristic matrix P. Then π is equitable if and only if the column space of P is A-invariant.

Proof.

The column space of P is A-invariant if and only if there is a matrix B such that $AP = PB$. If π is equitable, we can choose B as its quotient matrix. □

- Let B be the quotient matrix of an equitable partition. Then B is similar to a symmetric matrix. We have $(P^TP)^{-1}P^TAP = B$. Let $P^TP = K = D^2$ and $M = P^TAP$. Then it is clear that M is a symmetric matrix. We have $B = K^{-1}M = D^{-2}M$. Then $DBD^{-1} = D^{-1}MD^{-1}$. That is B is similar to a symmetric matrix $D^{-1}MD^{-1}$. Hence all eigenvalues of B real.
- Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be the characteristic matrix corresponding to partition of $\{\{3\},\{2,4\},\{1,5\}$ of $\{1,2,3,4,5\}$. Then

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

If $AP = PB$, then it is easy to see by induction that $A^k P = PB^k$ for $k \in \mathbb{N}$ and more generally, if $f(x)$ is a polynomial, then $f(A)P = Pf(B)$. If f is a polynomial such that $f(A) = 0$, then $Pf(B) = 0$. Since the columns of P are linearly independent, this implies that $f(B) = 0$. *This shows that the minimal polynomial of B divides the minimal polynomial of A, and therefore every eigenvalue of B is an eigenvalue of A.*

Theorem
If π is an equitable partition of a graph X, then the characteristic polynomial of B divides the characteristic polynomial of $A(X)$.

Proof.
Let P be the characteristic matrix of π. If X has n vertices, then let Q be an $n \times (n - |\pi|)$ matrix whose columns, together with those of P, form a basis for $\mathbb{R}^n$. Then there are matrices C and D such that $AQ = PC + QD$. That is we have

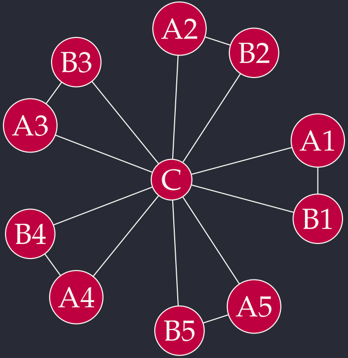
$$AP = PB, AQ = PC + QD.$$

$$A[P \;\; Q] = [AP \;\; AQ] = [PB \;\; PC + QD = [P \;\; Q] \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

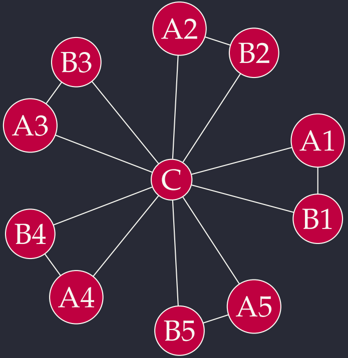
$$[P \;\; Q]^{-1} A [P \;\; Q] = \begin{bmatrix} B & C \\ O & D \end{bmatrix}.$$

Thus A is similar to $\begin{bmatrix} B & C \\ O & D \end{bmatrix}$. $\square$

- We can also get information about few eigenvectors of A from eigenvectors of B. Let v is an eigenvector of B with eigenvalue θ. That is $Bv = \theta v$. But we have $AP = PB$ hence $A(Pv) = (AP)v = (PB)v = P(Bv) = \theta Pv$.
- Thus Pv is eigenvector of A with same eigen value θ.
- We say that the eigenvector v of B "lifts" to an eigenvector of A.
- Alternatively, we may argue that if the column space of P is A-invariant, then it must have a basis consisting of eigenvectors of A. Each of these eigenvectors is constant on the cells of P, and hence has the form Pv, where $v \neq 0$. If $APv = \theta Pv$, then it follows that $Bv = \theta v$.
- If the column space of P is A-invariant, then so is its orthogonal complement; from this it follows that we may divide the eigenvectors of A into two classes;
  - those that are constant on the cells of π, which have the form Pv for some eigenvector of B, and
  - those that sum to zero on each cell of π.



- It is Friendship graph F_5 with 11 vertices. Any two vertices have exactly one common neighbor.



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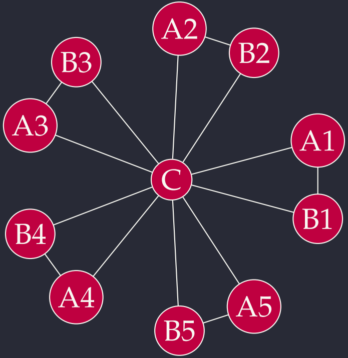
- Friendship graph (or Dutch windmill) F_k with $2k + 1$ vertices and $3k$ edges. Easy to see F_k is connected with diameter 2, as any of two vertices have a common neighbor.



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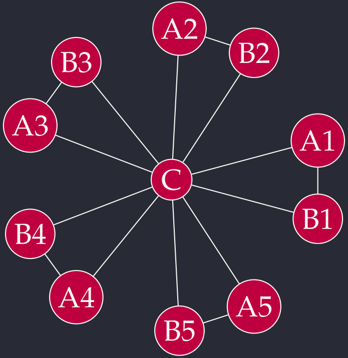
- There is a vertex which is adjacent to all the vertices (Friendship Theorem). Note this is not a part of definition. This is a property of F_k. Once we prove this theorem star graph is a subgraph of F_k.



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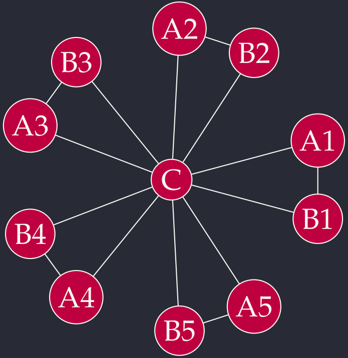
- Let $\{\{C\}, \{A_1, B_1, \ldots, A_5, B_5\}\}$ be a partition of vertex set.



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- Let $\{\{C\}, \{A_1, B_1, \ldots, A_5, B_5\}\}$ be a partition of vertex set.
- Then it is an equitable partition and its quotient matrix $B = \begin{bmatrix} 0 & 10 \\ 1 & 1 \end{bmatrix}$. In general $B = \begin{bmatrix} 0 & 2k \\ 1 & 1 \end{bmatrix}$.

- The characteristic polynomial of B of F_k is $x^2 - x - 2k$. Hence the eigenvalues of B are $\frac{1 \pm \sqrt{1+8k}}{2}$.

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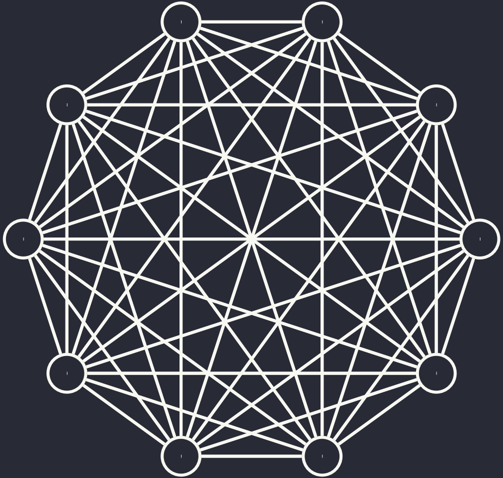
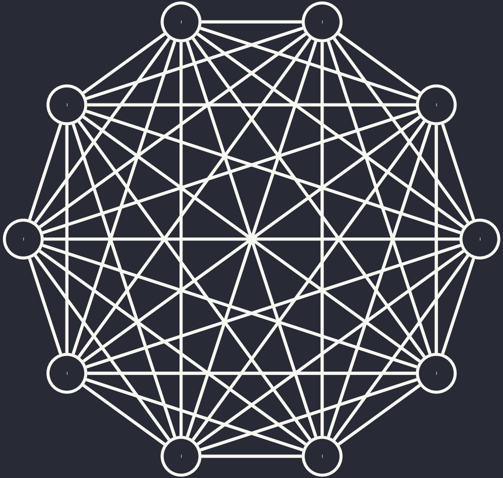
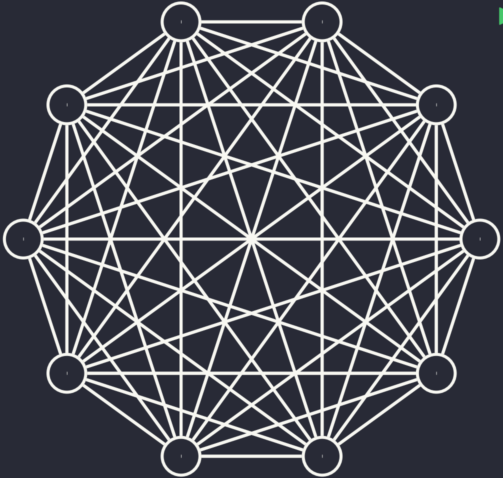
- The characteristic polynomial of F_k is $(x+1)(x^2-1)^{k-1}(x^2-x-2k)$.
- For a complete graph K_n, V with a single cell is an equitable partition, with characteristic matrix $P = [1,1,\ldots,1]^T$ and quotient matrix $B = [n-1]$. The eigenvalue of B is $n-1$ with eigenvector $[1]$. Then $P[1] = \mathbf{e} = P$. The remaining eigenvectors are $\{x \in \mathbb{R}^n | x \perp P\} = \{(x_1, x_2, \ldots, x_n)^T | x_1 + x_2 + \cdots + x_n = 0\}$.

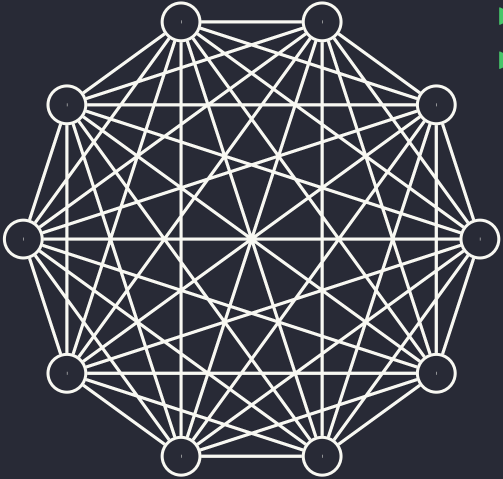




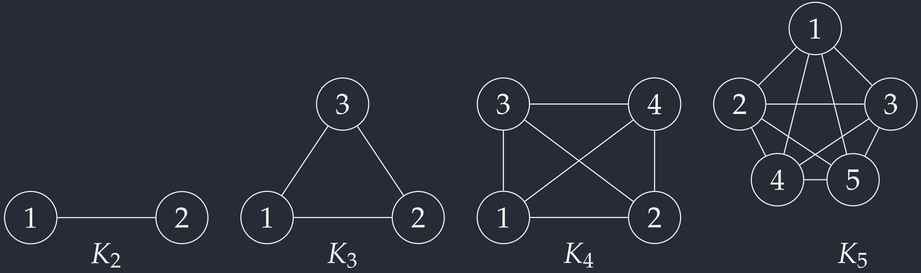
- Do you know this graph?



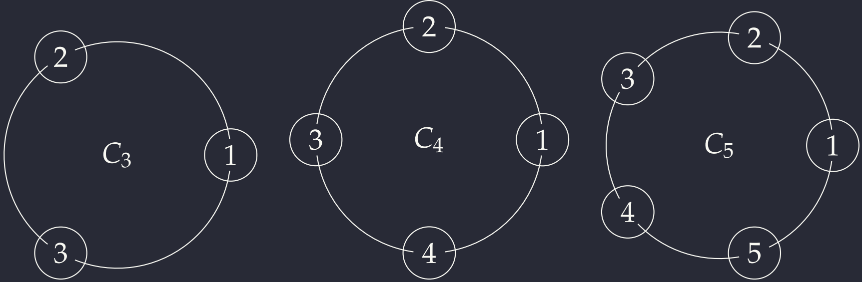
- Do you know this graph?
- Can you share few of its properties?



- Do you know this graph?
- Can you share few of its properties?
- What are entries of its adjacency matrix?



- There is an edge between any two vertices (any two vertices are adjacent).
- Total number of edges are $\frac{n(n-1)}{2}$.
- The degree of each vertex is $n-1$.
- Every simple graph with n vertices is obtained from K_n by deleting some edges.



- All the vertices form a cycle.
- The degree of each vertex is 2.
- There are n edges.

- Let G be a group and let S be a non-empty subset of G that does not contain the identity element of G. Then the **Cayley digraph/graph** associated with the pair (G, S), denoted $\mathrm{Cay}(G, S)$, has the set G as its vertex set and for any two vertices $x, y \in G$, (x, y) is an edge if $xy^{-1} \in S$.

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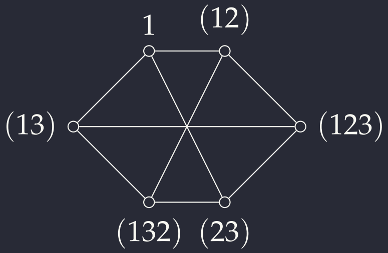

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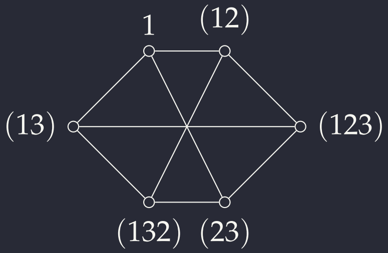
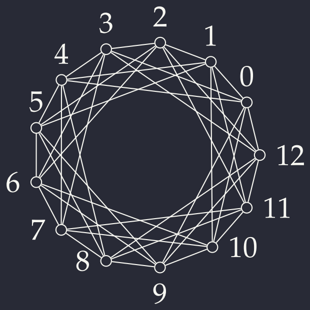


For example,

$$G = S_3 = \{1, (12), (13), (23), (123), (132)\},$$

be symmetric group on 3 symbols and $S = \{(12), (13), (23)\}$ then S is generating set of G as every permutation can be expressed as product of transpositions. The corresponding Cayley graph $Cay(G, S)$ is $K_{3,3}$ as shown .

Cayley graphs on cyclic groups are called **circulant graphs.**



An important class of circulant graphs are **Paley graphs**. Let q be a prime power such that $q \equiv 1 \pmod 4$, and $\mathbb{F}_q$ denote a finite field of order q, The Paley graph $P(q)$ of order q is a graph whose vertices are elements of the finite field $\mathbb{F}_q$ in which two vertices are adjacent if and only if their difference is a non-zero square in $\mathbb{F}_q$. Since in $\mathbb{F}_q$ there are $\frac{q-1}{2}$ non-zero squares (quadratic residues) hence $P(q)$ is $\frac{q-1}{2}$-regular graph. As we aware -1 is a square in modulo q if and only if $q \equiv 1 \pmod 4$. Consequently $a - b$ is a square if and only if $b - a$ is a square. On the other hand if $q \equiv 3 \pmod 4$, then $P(q)$ is a digraph.The following graph is $P(13)$.

The **Kneser graph** $K(n,k)$ is an undirected graph whose vertices correspond to the k-element subsets of the set $\{1,2,\ldots,n\}$, and where two vertices are adjacent if and only if the corresponding subsets are disjoint.

$$V(K(n,k)) = \{A \subseteq \{1,\ldots,n\} \mid |A| = k\}$$

$$E(K(n,k)) = \{\{A,B\} \mid A \cap B = \emptyset\}$$

Note: The graph is well-defined only when $n \geq 2k$, because otherwise no two k-subsets can be disjoint.

- **Number of vertices:**
$$|V(K(n,k))| = \binom{n}{k}$$
- **Edges:** Two k-subsets are adjacent if they are disjoint.
- **Degree:** The degree of each vertex is
$$\deg = \binom{n-k}{k}$$

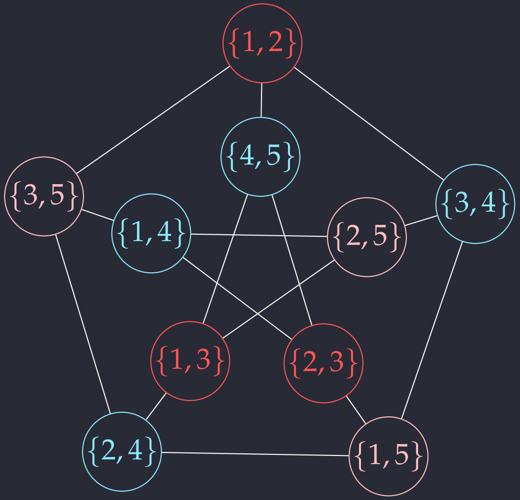
- **Automorphism group:** The automorphism group of $K(n,k)$ is isomorphic to the symmetric group S_n acting on the k-subsets of $\{1,\ldots,n\}$.
- **Vertex-transitivity:** Kneser graphs are vertex-transitive.

Important Theorem (Lovász, 1978):
Lovász used the Borsuk–Ulam theorem to prove the chromatic number of Kneser graphs:

$$\chi(K(n,k)) = n - 2k + 2$$

This was a solution to the famous **Kneser conjecture**.

- Can you see $K(n,1)$ is the complete graph K_n?
- For $n \geq 5$, show that diameter of $K(n,2)$ is 2.
- Let $n \geq 5$, $K(n,2)$ is $\binom{n}{2}$-regular. Let $u, v \in V(K(n,2))$. Then the number of common neighbours of u and v is $\binom{n-4}{2}$ if they are adjacent, else $\binom{n-3}{2}$ and is independent of vertices chosen. That is the graphs $K(n,2)$ are **Strongly regular graphs.**


$\{1,2\}$
$\{4,5\}$
$\{3,5\}$
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$\{1,4\}$
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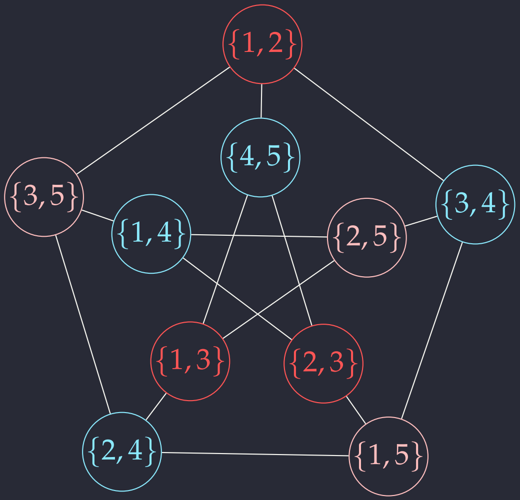

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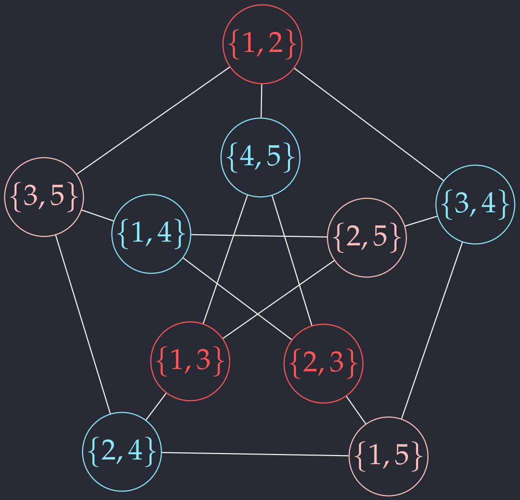

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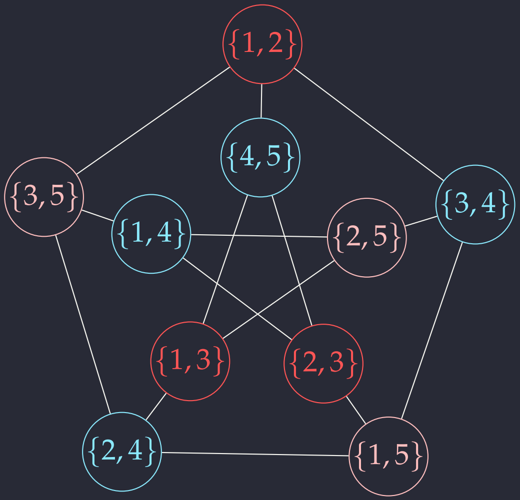

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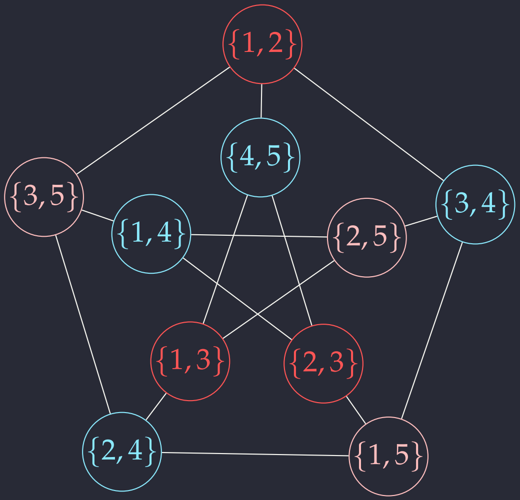

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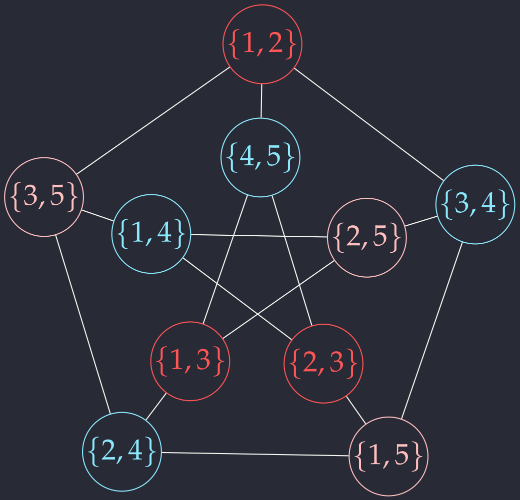

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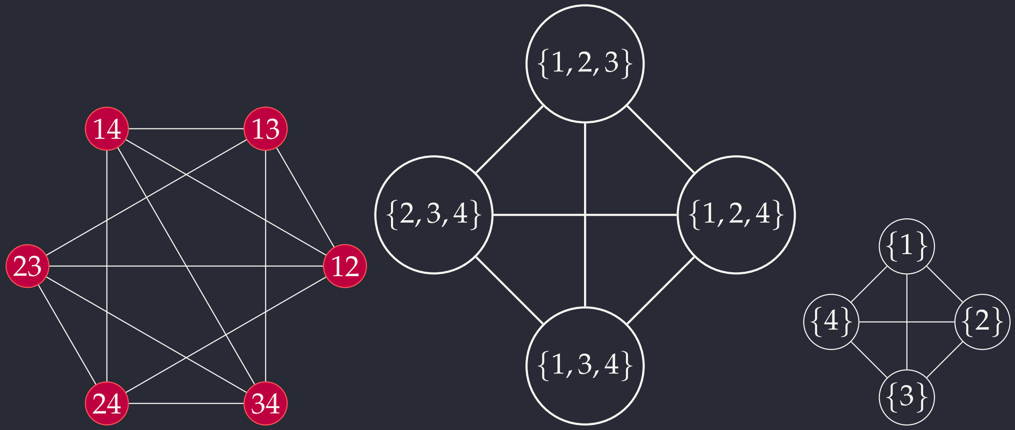
- The diameter of the graph is 2.

Johnson graph (J(n,k))

Property	**Value / Description**
Vertex set	All k-element subsets of an n-element set
Number of vertices	$\binom{n}{k}$
Adjacency rule	Two vertices are adjacent if their subsets intersect in $k-1$ elements
Degree of each vertex	$k(n-k)$
Number of edges	$\frac{1}{2} \cdot \binom{n}{k} \cdot k(n-k)$
Regular	Yes, $k(n-k)$-regular
Connected	Yes, for $n > k \geq 1$
Diameter	$\min(k, n-k)$
Girth	3 if $k \geq 2$
Vertex-transitive	Yes
Edge-transitive	Yes
Distance-transitive	Yes
Strongly regular	Only in some special cases (e.g., $J(5,2)$)


14
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{2}
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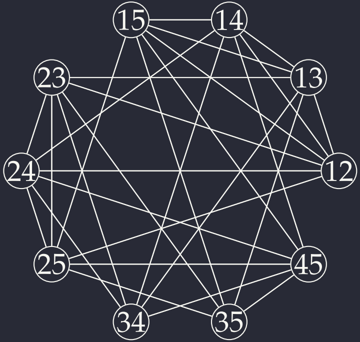

Property	Value / Description
Number of vertices	$\binom{5}{2} = 10$
Number of edges	$\frac{10 \cdot 6}{2} = 30$
Vertex degree	6 (regular)
Connectivity	Connected
Diameter	2
Girth	3 (contains triangles)
Chromatic number	4
Clique number	4
Is strongly regular	Yes
Strongly regular parameters	$(10, 6, 3, 4)$
Line graph of	K_5
Complement to	Petersen graph

Recall a permutation group S_3.
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$$S_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

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Number of bijective maps of a set with n elements to itself
= Number of ways placing n rooks on $n \times n$ chess board
=Number of permutation matrices of order n.

Is a permutation matrix doubly stochastic?

Automorphism group of a graph X

- The collection of all automorphisms of a graph X, denoted $\mathrm{Aut}(X)$, forms a group under composition of maps.
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- Also, for each $g \in \mathrm{Aut}(X)$ the corresponding permutation matrix is be denoted by P_g. If $g = (1\ 2\ 3) \in S_3$, then $P_g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

What is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =??$ What is $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =??$

Is $JP_g = P_gJ$?

Is $\forall g \in S_n\ A(K_n)P_g = P_gA(K_n)$?

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Lemma

Let A be the adjacency matrix of a graph X. Then $g \in Aut(X)$ if and only if $P_g A = AP_g$.

Proof.

Let g be a permutation of $V(X) = \{v_1, v_2, \ldots, v_n\}$, and $g(v_i) = v_h, g(v_j) = v_k$.

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$$\begin{aligned} P_g A = AP_g \quad &\Leftrightarrow \quad A_{hk} = A_{ij} \Leftrightarrow \{v_h, v_k\} \in E \text{ if and only if } \{v_i, v_j\} \in E \\ &\Leftrightarrow \quad g \text{ is an automorphism of } X. \end{aligned}$$

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$$P_g A = A P_g \Leftrightarrow P_g(\mathbf{J} - I - A) = (\mathbf{J} - I - A) P_g.$$

□

Theorem

Let $X = (V, E)$ be a graph and $g \in Aut(X)$. Then

1. *If $v \in V$, then $\deg(v) = \deg(g(v))$.*
2. *If $u, v \in V$, then $d(u, v) = d(g(u), g(v))$.*

Definition

Let X be a graph. If $H \leq Aut(X)$ is a group of automorphisms of X, we say that u and v are similar under H if there is an automorphism in H which maps u to v. The equivalence classes defined by this similarity are called the orbits of the graph by H. The partition of X consisting of the set of orbits by H is called an orbit partition of X.

Proposition

An orbit partition is an equitable partition.

Proof.

Let $O_1, O_2, \ldots, O_r$ be an orbit partition of X. Suppose $u, v \in O_i$. Then there is an automorphism $\phi \in Aut(X)$ such that $\phi(u) = v$. Since ϕ maps O_j to O_j and preserves valency, u and v must have same number of neighbors in C_j. $\square$

Let $DS(A)$ denotes the set of all doubly stochastic matrices that commute with A.

> ### Definition
> *Let A be the adjacency matrix of a graph X. If every extreme point of $DS(A)$ is a permutation matrix, then the graph X is called compact.*

Note that $DS(A)$ is a convex set and contains all of the permutation matrices that commute with A, *i.e.*, all automorphisms of X. Hence, if X is compact, then the automorphisms of X are precisely the extreme points of $DS(A)$.

Proposition

If X is compact, then every equitable partition is an orbit partition.

- A graph $X = (V, E)$ is said to be a **vertex transitive** (**edge transitive**) graph if $\mathrm{Aut}(X)$ acts transitively on V (E).
- That is X is vertex transitive if for any two vertices $x, y \in V, x \neq y$ there exists $g \in \mathrm{Aut}(X)$ such that $g(x) = y$.
- For example, $\mathrm{Aut}(K_n) \cong \mathcal{S}_n$ and $\mathrm{Aut}(C_n) \cong \mathcal{D}_n$, hence the graphs K_n and C_n are vertex transitive.

Cayley graph is vertex transitive

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Let $X = \mathrm{Cay}(G, S)$ be a Cayley graph. Then, for every $g \in G$

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Thus, $G \subseteq \mathrm{Aut}(X)$. Hence, if $a, b \in V = G$ then, the group element $a^{-1}b$ takes a to b. □

Theorem

Let X be a vertex transitive graph and π the orbit partition of some subgroup G of $Aut(X)$. If π has a singleton cell $\{u\}$, then every eigenvalue of X is an eigenvalue of B.

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- Let $A = \{1,2\}$. Then $A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$. Let $R_1 = \{(1,1)\}, R_2 = \{(1,2)\}, R_3 = \{(2,1)\}, R_4 = \{(2,2)\}$ find $\cup R_i$.

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  - What is the vector space spanned by the set $\{M_{R_1}, M_{R_2}, M_{R_3}, M_{R_4}\}$. Is it closed with respect to multiplication?

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- Let $A = \{1, 2\}$. Then $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Let $R_1 = \{(1, 1)\}, R_2 = \{(1, 2)\}, R_3 = \{(2, 1)\}, R_4 = \{(2, 2)\}$ find $\cup R_i$.
  - What is the vector space spanned by the set $\{M_{R_1}, M_{R_2}, M_{R_3}, M_{R_4}\}$. Is it closed with respect to multiplication?
  - What is the vector space spanned by the set $\{M_{R_1}, M_{R_2} + M_{R_3}, M_{R_4}\}$. Is it closed with respect to multiplication?

- Let A be a nonempty finite set. What you mean by a relation on A?
- Let R be a relation on A, then M_R be the corresponding 0,1-matrix representation of R. How do you check given relation R is symmetric or not from the matrix M_R.?
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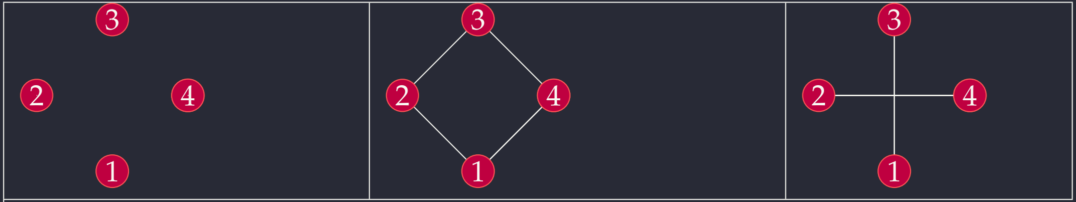
- Let $A = \{1,2,3\}$. and $R_0 = \{(1,1),(2,2),(3,3)\}$ and $R_1 = A \times A \setminus R_0$. Find M_{R_1} and $M_{R_0} + M_{R_1}$. Find the vector space spanned by $\{M_{R_0}, M_{R_1}\}$. Is it closed with respect to multiplication.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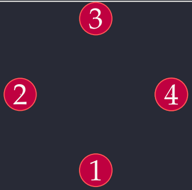
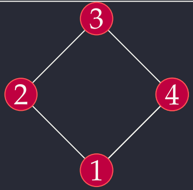
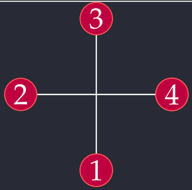
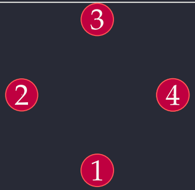
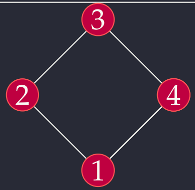
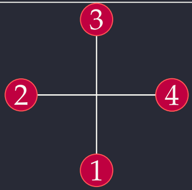






$R_0 =$ $\{(1,1),(2,2),(3,3),(4,4)\}$ $\lvert R_0\rvert = 4$	$R_1 =$ $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$ Here $\{a,b\}$ means $(a,b),(b,a)$ $\lvert R_1\rvert = 8$	$R_2 =$ $\{\{1,3\},\{2,4\}\}$ $\lvert R_3\rvert = 4$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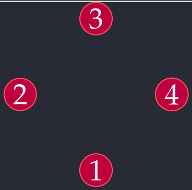	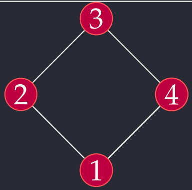	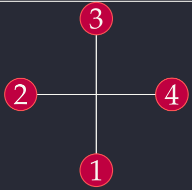
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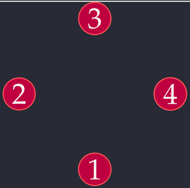
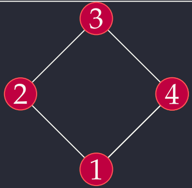
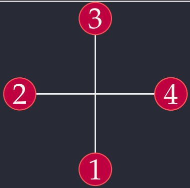
- Is $R_0 \cup R_1 \cup R_2 = V \times V$?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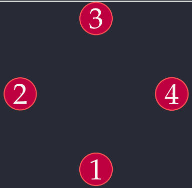	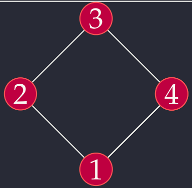	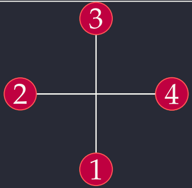
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- Is $R_0 \cup R_1 \cup R_2 = V \times V$? Find $A = A_1 + A_2$.

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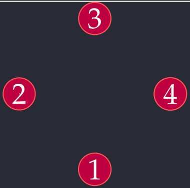
<td>
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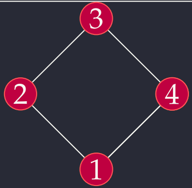
<td>
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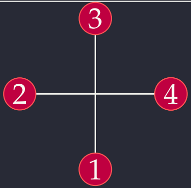
<td>
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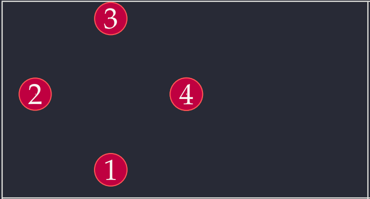
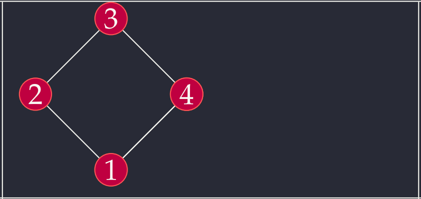
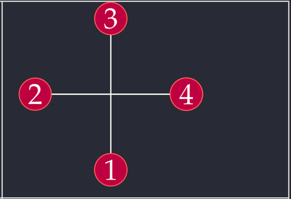
- Is $R_0 \cup R_1 \cup R_2 = V \times V$? Find $A = A_1 + A_2$. Is A a adjacency matrix of some graph?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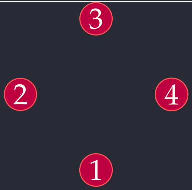	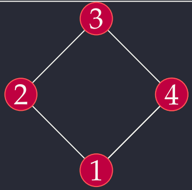	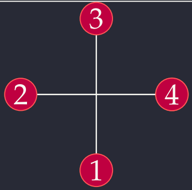
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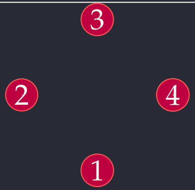	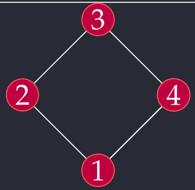	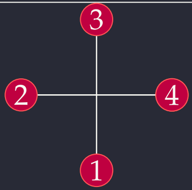
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1. Chris D. Godsil & Gordon Royle, Algebraic Graph Theory, Springer-Verlag,(2001).

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- The eigenvalues, eigenvectors, the minimal polynomial and the characteristic polynomial of a graph X are defined to be that of its adjacency matrix.

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- Let $\alpha \in \mathbb{C}$ be an algebraic element. Let $\phi_\alpha : \mathbb{Q}[x] \to \mathbb{C}$ be a map defined as $\phi_\alpha(f(x)) = f(\alpha)$. Then it is easy to check that ϕ_α is onto homomorphism and its range is $\mathbb{Q}[\alpha]$ further $\mathbb{Q}[x]/\langle m_\alpha(x)\rangle \cong \mathbb{Q}[\alpha]$, where $m_\alpha(x)$ is the minimal polynomial of α.

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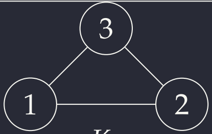
- If $A(X)$ is the adjacency matrix of a graph (or digraph) X, then $\mathbb{C}[A]$ is called *Adjacency algebra* of X. Denoted as $\mathcal{A}(X)$.

Graph	Adjacency matrix (A)	characteristic polynomial	minimal polynomial	$\mathcal{A}(X)$
 K_3	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$(x+1)^2(x-2)$	$(x+1)(x-2)$	$\{\alpha I + \beta A \mid \alpha, \beta \in \mathbb{C}\}$

$$\mathcal{A}(X) = \mathbb{C}[A] \cong \mathbb{C}[x]/\langle m_A(x)\rangle.$$

Hence $\dim \mathcal{A}(X) = \dim\left(\mathbb{C}[x]/\langle m_A(x)\rangle\right) =$ number of distinct eigenvalues of A.
If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$, $A^4 = \begin{bmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{bmatrix}$, ...
$\dim \mathcal{A}(K_n) = 2$. A basis of $\mathcal{A}(K_n)$ is $\{I, A = J - I\}$.
$\dim \mathcal{A}(K_{n,n}) = 3$. A basis of $\mathcal{A}(K_{n,n})$ is $\{I, A, A^2\}$, another useful basis is $\{I, A, A^c\}$, where A^c is the adjacency matrix of complement of $K_{n,n}$.

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- Let $A \in M_n(\mathbb{R})$. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be distinct eigenvalues of A with multiplicities $m_1, m_2, \ldots, m_r$. Then $\dim \mathcal{B}[A] = m_1^2 + m_2^2 + \cdots + m_r^2$.

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- The matrices in $\mathbb{C}[W_n]$ are the circulant matrices. The dimention of set of circulant matrices is n. They are the Cayley graphs of $\mathbb{Z}_n$. All the matrices have same eigenvectors and eigenvalues are easy to compute. Construct a polynomial with coefficients as first row of that matrix. Evaluate polynomial at the nth roots of unity to get eigenvalues.

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A matrix $A \in \mathbb{M}_n(\mathbb{F})$ is said to be a **circulant matrix** if $a_{ij} = a_{1j-i+1 \pmod n)}$. That is, for each $i \geq 2$, the elements of the i-th row of A are obtained by cyclically shifting the elements of the $(i-1)$-th row of A, one position to the right. So, it is sufficient to specify its first row.

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Lemma

Let $A \in M_n(\mathbb{F})$. Then A is a circulant matrix if and only if it is a polynomial in W_n. That is, the set of circulant matrices in $M_n(\mathbb{F})$ forms a commutative algebra. Note that as a vector space, its basis is $\{I = W_n^0, W_n^1, W_n^2, \ldots, W_n^{(n-1)}\}$.

Representer polynomial

- Let $A \in M_n(\mathbb{Z})$ be a circulant matrix. Then, from Lemma 3, there exists a unique polynomial $\gamma_A(x) \in \mathbb{Z}[x]$ of degree $\leq n-1$, called the **representer polynomial** of A such that $A = \gamma_A(W_n)$.

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- Further, one can see that if $A \in M_n(\mathbb{Z})$ is a circulant matrix, then $[a_0\ a_1 \ldots a_{n-1}]$ is the first row of A if and only if $\gamma_A(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$.

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- In particular, there is a one-to-one correspondence between the set of $0,1$ circulant matrices and the set of $0,1$-polynomials of degree $\leq n-1$.
- If X is a circulant graph/digraph, with n vertices, then $\mathcal{A}(X) \subseteq \mathbb{C}[W_n]$.

Eigenvalues and Eigenvectors of circulant graphs

Let ζ_n be the primitive nth root of unity, *i.e.*, $\zeta_n^n = 1$ but $\zeta_n^k \neq 1$, for $1 \leq k \leq n-1$. Then Then the **Fourier matrix** $F_n \in \mathbb{C}^{n\times n}$ is defined as:

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_n & \zeta_n^2 & \cdots & \zeta_n^{n-1} \\ 1 & \zeta_n^2 & \zeta_n^4 & \cdots & \zeta_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_n^{n-1} & \zeta_n^{2(n-1)} & \cdots & \zeta_n^{(n-1)^2} \end{bmatrix}$$

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Definition

A matrix $A \in M_n(\mathbb{C})$ is diagonalizable, if it is similar to a diagonal matrix.

That is A is diagonalizable if there exists a nonsingular matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

Theorem

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Part 1: If A has n linearly independent eigenvectors, then A is diagonalizable.
Assume A has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Note that the eigenvalues are not necessarily distinct.
Let P be the $n \times n$ matrix whose columns are these eigenvectors:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

Since the eigenvectors are linearly independent, the matrix P is invertible.
Now, consider the product AP. We can compute this column by column:

$$AP = A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix}.$$

Because each $\mathbf{v}_i$ is an eigenvector ($A\mathbf{v}_i = \lambda_i \mathbf{v}_i$), this becomes:

$$AP = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \end{bmatrix}.$$

This result can be factored as the product of P and a diagonal matrix D:

$$\begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD,$$

where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.
We have therefore shown that:

$$AP = PD.$$

Since P is invertible, we can multiply both sides on the left by P^{-1} to obtain:

$$P^{-1}AP = D.$$

This is precisely the condition for A to be diagonalizable. Thus, A is diagonalizable.

Part 2: If A is diagonalizable, then it has n linearly independent eigenvectors.
Assume A is diagonalizable. Then, by definition, there exists an invertible matrix P and a diagonal matrix D such that:

$$P^{-1}AP = D \quad \text{or, equivalently,} \quad A = PDP^{-1}.$$

Let the columns of P be $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and let the diagonal entries of D be $\lambda_1, \lambda_2, \ldots, \lambda_n$, so that $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.
Starting from $A = PDP^{-1}$, we multiply both sides on the right by P:

$$AP = PD.$$

Now, let's examine this equation column by column.
On the left-hand side:

$$AP = A\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \end{bmatrix}.$$

On the right-hand side:

$$PD = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_2 \mathbf{v}_2 & \cdots & \lambda_n \mathbf{v}_n \end{bmatrix}.$$

Equating the corresponding columns of AP and PD, we find that for each $i = 1, 2, \ldots, n$:

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

This means that each column $\mathbf{v}_i$ of P is an eigenvector of A with corresponding eigenvalue λ_i.
Finally, since P is an invertible matrix, its columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are, by definition, linearly independent.
Therefore, A has n linearly independent eigenvectors.

Conclusion: We have shown that if A has n linearly independent eigenvectors, then it is diagonalizable, and conversely, if A is diagonalizable, then it has n linearly independent eigenvectors. This completes the proof.

If A is diagonalizable, then the rank of A is equal to the number of nonzero eigenvalues of A.

Proof.

- First note that eigenvalues of a diagonal matrix are diagonal elements of matrix. Rank of a diagonal matrix is equal to number of nonzero diagonal elements. Thus rank of a diagonal matrix is equal to number of nonzero eigenvalues. Extra info: Hence the minimal poynomial of A is the product of distinct linear factors.
- If A is similar to diagonal matrix, then there exists a nonsingular matrix P such that $PAP^{-1} = D$, where D is the diagonal matrix. Rank of A= Rank of PAP^{-1} = Rank of D. Also eigenvalues of A is same as eigenvalues of D. Hence result follows.

$\square$

Lemma

Let A be a real symmetric matrix. If u and v are eigenvectors of A with different eigenvalues, then u and v are orthogonal.

Proof.

Suppose $Au = \lambda u$ and $Av = \tau v$, where $\lambda \neq \tau$. Consider $\lambda u^T v$. Then $\lambda u^T v = (\lambda u)^T v = (Au)^T v = u^T A v = \tau u^T v$. Hence $(\lambda - \tau) u^T v = 0$. Thus $u^T v = 0$, follows from the fact that $\lambda \neq \tau$. Note $u^T v$ and $(\lambda - \tau)$ are real numbers. $\square$

Lemma

The eigenvalues of a real symmetric matrix A are real numbers.

Proof.

Let u be an eigenvector of A with eigenvalue λ. Then by taking the complex conjugate of the equation $Au = \lambda u$ we get $A\bar{u} = \bar{\lambda}\bar{u}$, and so $\bar{u}$ is also an eigenvector of A. Now, by definition an eigenvector is not zero, so $u^T \bar{u} > 0$. By the previous lemma, u and $\bar{u}$ cannot have different eigenvalues, so $\lambda = \bar{\lambda}$, and the claim is proved. $\square$

Lemma

Let A be a real symmetric $n \times n$ matrix. If U is an A-invariant subspace of $\mathbb{R}^n$, then $U^\perp$ is also A-invariant.

Proof.

For any two vectors u and v, we have

$$v^T(Au) = (Av)^T u.$$

If $u \in U$, then $Au \in U$; hence if $v \in U^\perp$, then $v^T Au = 0$. Consequently, $(Av)^T u = 0$ whenever $u \in U$ and $v \in U^\perp$. This implies that $Av \in U^\perp$ whenever $v \in U^\perp$, and therefore $U^\perp$ is A-invariant. $\square$

Lemma

Let A be an $n \times n$ real symmetric matrix. If U is a nonzero A-invariant subspace of $\mathbb{R}^n$, then U contains a real eigenvector of A.

Proof.

Let R be a matrix whose columns form an orthonormal basis for U. Then, because U is A-invariant, $AR = RB$ for some square matrix B. Since $R^T R = I$, we have

$$R^T AR = R^T RB = B.$$

Which implies that B is symmetric, as well as real. Since every symmetric matrix has at least one eigenvalue, we may choose a real eigenvector u of B with eigenvalue λ. Then $ARu = RBu = \lambda Ru$, and since $u \neq 0$ and the columns of R are linearly independent, $Ru \neq 0$. Therefore, Ru is an eigenvector of A contained in U. $\square$

Theorem

Let A be a real symmetric $n \times n$ matrix. Then $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.

Proof.

Let $\{u_1, \ldots, u_m\}$ be an orthonormal (and hence linearly independent) set of $m < n$ eigenvectors of A, and let M be the subspace that they span. Since A has at least one eigenvector, $m \geq 1$. The subspace M is A-invariant, and hence $M^\perp$ is A-invariant, and so $M^\perp$ contains a (normalized) eigenvector u_{m+1} Then $\{u_1, \ldots, u_m, u_{m+1}\}$ is an orthonormal set of $m + 1$ eigenvectors of A. Therefore, by induction, a set consisting of one normalized eigenvector can be extended to an orthonormal basis consisting of eigenvectors of A. $\square$

Corollary

If A is an $n \times n$ real symmetric matrix, then there are matrices L and D such that $L^T L = LL^T = I$ and $LAL^T = D$, where D is the diagonal matrix of eigenvalues of A.

Proof.
Let L be the matrix whose rows are an orthonormal basis of eigenvectors of A. So $L^T = \begin{bmatrix} L_1 & L_2 & \dots & L_n \end{bmatrix}$, where L_i is the eigenvector of A corresponding to the eigenvalue λ_i for all $i = 1, 2, \dots, n$. Consider $(L^T L)_{ij} =< L_i, L_j^T >= \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$ (since the rows of L form an orthonormal basis). Hence, $L^T L = I$. Similarly, since L is orthogonal, let $M = L^T$ then $LL^T = M^T M = I$. To prove the second part of the theorem, consider,

$$\begin{aligned} LAL^T &= LA[L_1, L_2, \cdots, L_n] = L[AL_1, AL_2, \cdots, AL_n] \\ &= L[\lambda_1 L_1, \lambda_2 L_2, \cdots, \lambda_n L_n = [\lambda_1 LL_1, \lambda_2 LL_2, \cdots, \lambda_n LL_n] \\ &= [\lambda_1 e_1, \lambda_2 e_2, \cdots, \lambda_n e_n] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D. \end{aligned}$$

□

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Let X be a graph with adjacency matrix A. Then, for every positive integer k, $(A^k)_{ij}$ equals the number of walks of length k from the vertex v_i to the vertex v_j.

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Assume the result is true for $k = L$ and let us consider the matrix A^{L+1}. Then,

$$(A^{L+1})_{ij} = \sum_{h=1}^{n} (A^L)_{ih}.(A)_{hj}.$$

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Let X be a graph with adjacency matrix A. Then, for every positive integer k, $(A^k)_{ij}$ equals the number of walks of length k from the vertex v_i to the vertex v_j.

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Proof by induction on k.

Base Step: If $k = 1$, by definition $A_{ij} = \begin{cases} 1, & \text{if } v_i, v_j \, areadjacent \\ 0, & otherwise. \end{cases}$

Assume the result is true for $k = L$ and let us consider the matrix A^{L+1}. Then,

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Therefore, $(A^{L+1})_{ij}$ equals the number of walks of length L from v_i to v_h and then a walk of length one (adjacency) from v_h to v_j, for all vertices $v_h \in V(X)$.

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Proof.

Since the adjacency matrix is a real symmetric matrix, its minimal polynomial is the product of distinct linear polynomials. Hence, $\dim(\mathcal{A}(X))$ also equals the number of distinct eigenvalues of A. Thus, if the graph X has diameter d, then it has at least $d + 1$ distinct eigenvalues. $\square$

If all eigenvalues of a simple graph are equal, then its diameter is zero. Thus, a simple graph has only one distinct eigenvalue if and only if it is a null graph.
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$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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Proof.

Let X be a graph with two distinct eigenvalues, then $\dim \mathcal{A}(X) = 2$. Hence, I and A form a basis of $\mathcal{A}(X)$. Consequently $A^2 = aI + bA$, for some $a, b \in \mathbb{N}$. Thus, $(A^2)_{ii} = a$ for all i. □

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Are are able to see? A graph X has two distinct eigenvalues if and only if X is disjoint union of complete graphs with same number of vertices.

How to check a given graph is connected or not?

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What about the converse? If every entry of $(I+A)^{n-1}$ is positive, then X is a connected graph.

k-th distance matrix of a graph

Definition

Let $X = (V, E)$ be a connected graph with diameter d. Then, for $0 \leq k \leq d$, the k-th distance matrix of X, denoted A_k, is defined as

$$(A_k)_{rs} = \begin{cases} 1, & \text{if } d(v_r, v_s) = k \\ 0, & \text{otherwise.} \end{cases}$$

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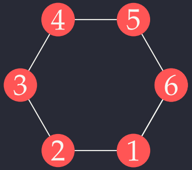
If $X = C_4$, the cycle graph on four vertices, then

$$A_0 = I_4, A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

1. Are able to see A_2 is the adjacency matrix of X^c in the above example?
2. If X is connected graph with a diameter $d \geq 2$, then $A_2 + A_3 + \cdots + A_d =$??

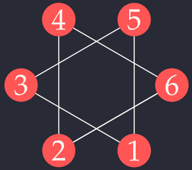
$$A_0 = I_6 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$R_0 = \{(x,y) | d(x,y) = 0\}$



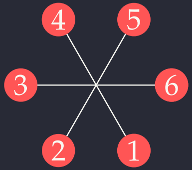
$$A_1 = A = \begin{bmatrix} & 1 & & & & 1 \\ 1 & & 1 & & & \\ & 1 & & 1 & & \\ & & 1 & & 1 & \\ & & & 1 & & 1 \\ 1 & & 1 & & & \end{bmatrix}$$

$R_1 = \{(x,y) | d(x,y) = 1\}$



$$A_2 = \begin{bmatrix} & & 1 & & 1 & \\ & & & 1 & & 1 \\ 1 & & & & 1 & \\ & 1 & & & & 1 \\ 1 & & 1 & & & \\ & 1 & & 1 & & \end{bmatrix}$$

$R_2 = \{(x,y) | d(x,y) = 2\}$



$$A_3 = \begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{bmatrix}$$

$R_3 = \{(x,y) | d(x,y) = 3\}$

$$A_1^2 = A_2^2 = 2A_0 + A_2, A_3^2 = A_0, A_1 A_2 = A_1 + 2A_3, A_1 A_3 = A_2, A_1 A_3 = A_1.$$

$$A_1 = A, A_2 = A^2 - 2I, A_3 = \frac{1}{2}A^3 - \frac{3}{2}A.$$

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> Definition (Paul M. Weichsel [2])
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> *Let X be a connected graph with diameter d and let $A_k(X)$, for $0 \leq k \leq d$, be the k-th distance matrix of X. Then, X is said to be a* **distance polynomial graph** *if $A_k(X) \in \mathcal{A}(X)$, for $0 \leq k \leq d$.*

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The complete graph K_n, Cycle graph C_n, Complete bipartite graph $K_{n,n}$ and Petersen graph are few examples of distance polynomial graphs.

Natural question is which graphs are distance polynomial graphs? We will prove few results to see the necessary conditions for a graph to be a distance polynomial graph.

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Proof of Part 1

Proof.

Let $\mathbf{e} = [1, 1, \ldots, 1]^T$. Then $A\mathbf{e} = k\mathbf{e}$. Consequently, k is an eigenvalue with corresponding eigenvector $\mathbf{e}$. $\square$

Proof of Part 2

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$$k\mathbf{a}_j = (A\mathbf{a})_j = \sum_{\{v_i, v_j\} \in E} \mathbf{a}_i \leq k\mathbf{a}_j$$

as is vertex of X is adjacent to exactly k vertices and $\mathbf{a}_j \geq \mathbf{a}_i$, for all $i = 1, \ldots, n$.

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Let $A\mathbf{b} = \lambda\mathbf{b}$. As above, let b_j be an entry of $\mathbf{b}$ having the largest absolute value.

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$$|\lambda|\mathbf{b}_j = |(\lambda\mathbf{b})_j| = |(A\mathbf{b})_j| = |\sum_{\{v_i,v_j\}\in E} \mathbf{b}_i| \leq \sum_{\{v_i,v_j\}\in E} |\mathbf{b}_i| \leq k|\mathbf{b}_j|.$$

Thus, $|\lambda| \leq k$

$\mathbf{J} \in \mathcal{A}(X)$

Lemma 11 implies that if X is a connected k-regular graph then the minimal polynomial of X will have the form $(x-k)q(x)$ for some polynomial $q(x)$ with integer entries and $q(k) \neq 0$, as k is an eigenvalue of multiplicity 1. We use this idea in the next result.

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Lemma (Hoffman [3])

Let X be a connected k-regular graph on n vertices. Then, the matrix $\mathbf{J}$, consisting of all 1's, equals $\dfrac{n}{q(k)}q(A)$, i.e., $\mathbf{J} \in \mathcal{A}(X)$.

Proof: As X is a k-regular graph, its adjacency matrix A satisfies $A\mathbf{e} = k\mathbf{e}$. Let $(x-k)q(x) \in \mathbb{Z}[x]$ be the minimal polynomial of X. Hence,

$$\mathbf{J}A = A\mathbf{J} = k\mathbf{J} \quad \text{and} \quad q(A)\mathbf{e} = q(k)\mathbf{e}. \tag{1}$$

Continuation of Proof

Let $\left\{ \frac{1}{\sqrt{n}}\mathbf{e}, \mathbf{x}_2, \ldots, \mathbf{x}_n \right\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A with corresponding eigenvalues $k, \lambda_2, \ldots, \lambda_n$. Thus, $\mathbf{x}_i^T \mathbf{e} = 0$, for $2 \leq i \leq n$. Hence, $\mathbf{J}\mathbf{x}_i = \mathbf{0}$.

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$$\begin{bmatrix} d_1 & d_2 & \cdots & d_n \\ d_1 & d_2 & \cdots & d_n \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \cdots & d_n \end{bmatrix} = \mathbf{J}A = A\mathbf{J} = \begin{bmatrix} d_1 & d_1 & \cdots & d_1 \\ d_2 & d_2 & \cdots & d_2 \\ \vdots & \vdots & \ddots & \vdots \\ d_n & d_n & \cdots & d_n \end{bmatrix}.$$

Thus, $d_i = d_j$, for all i and j and hence X is a regular graph. Hence the proof.

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The eigenvalues of X^c, when X is a regular graph.

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Let X be a connected k-regular graph on n vertices with eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. Then, the eigenvalues of X^c are $n-k-1, -1-\lambda_2, \ldots, -1-\lambda_n$.

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$$\begin{aligned} UA^cU^T &= U(\mathbf{J} - I - A)U^T = U\mathbf{J}U^T - UIU^T - UAU^T \\ &= diag(n, 0, 0, \ldots, 0) - diag(1, 1, \ldots, 1) - diag(k, \lambda_2, \lambda_3, \ldots, \lambda_n) \\ &= diag(n-k-1, -1-\lambda_2, \ldots, -1-\lambda_n). \end{aligned}$$

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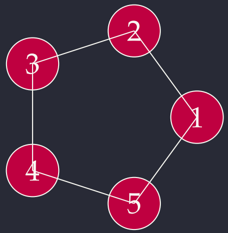
As X is a distance polynomial graph, by definition, X is already connected. If X has diameter d, then by definition, $A_k(X) \in \mathcal{A}(X)$, for $0 \leq k \leq d$. Consequently, $\mathbf{J} = \sum_{k=0}^{d} A_k(X) \in \mathcal{A}(X)$ and hence using Lemma 12, the result follows. $\square$



C_5 : connected 2 regular graph.

- Find number of common neighbors of two adjacent vertices?
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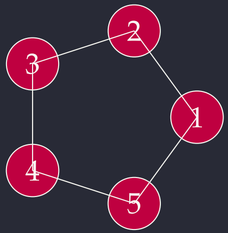
- Is the distance partition of any node is an equitable partition? Is it same for all vertices?



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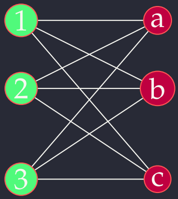

$K_{3,3}$: connected 3-regular graph.

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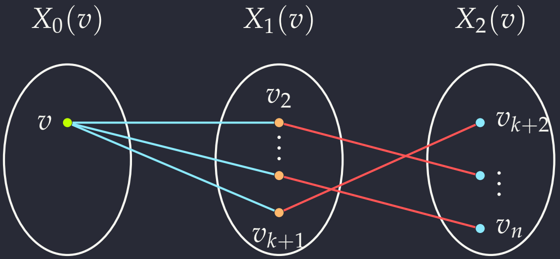

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v
v_2
v_{k+1}
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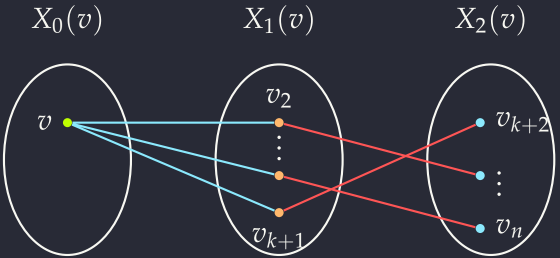
v_n



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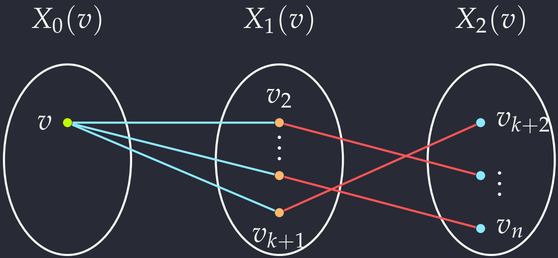

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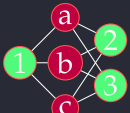
$\{X_0(v), X_1(v), X_2(v)\}$ forms an equitable partition of X called distance partition of X w.r.to v. That is independent of vertex chosen, *i.e.*, all will have same quotient graph/matrix.

The quotient matrix is

$$\begin{bmatrix} 0 & k & 0 \\ 1 & a & k-a-1 \\ 0 & c & k-c \end{bmatrix}.$$



$K_{3,3}$

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- The line graphs of the complete bipartite graphs, $lg(K_{n,n})$ are strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$. This is also a rook graph on square chess board. For example if we consider rook graph on $n \times n$ chess board, then the distance partition w.r.t. to $(1, 1)$ is an equitable partition. Here the cells are
  $C_0 = \{(1, 1)\}, C_1 = \{(1, j) \mid j \neq 1\} \cup \{(i, 1) \mid i \neq 1\}, C_2 = \{(i, j) \mid i \neq 1, j \neq 1\}$.
  $|V| = 1 + 2(n-1) + (n-1)^2 = n^2$. Distinct eigenvalues are $2n - 2, n - 2, -2$.

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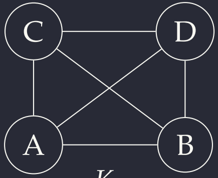
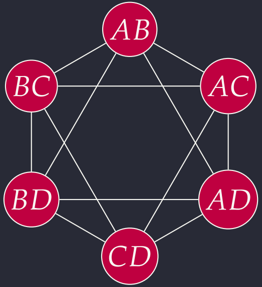
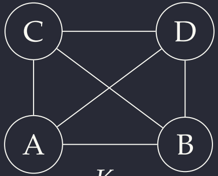
□

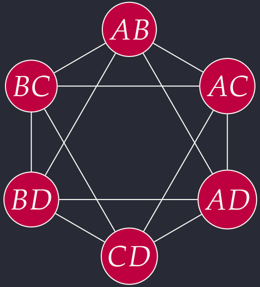




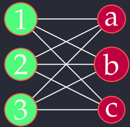
$lg(K_4), srg(6, 4, 2, 4).$



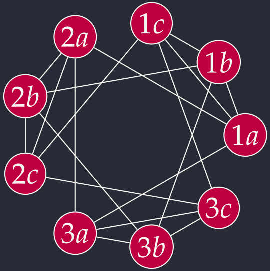
K_4



$lg(K_4), srg(6,4,2,4).$



$K_{3,3}$



$lg(K_{3,3}), srg(9,4,1,2).$

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Thus we have $\boxed{k(k-a-1) = c(n-k-1).}$

Theorem (Godsil and Royle [3])

Let A be the adjacency matrix of an (n, k, a, c)-strongly regular graph X. Then,

1. $A^2 = kI + aA + c(\mathbf{J} - I - A)$.
2. *the eigenvalues of X are k and roots of equation $x^2 - (a - c)x - (k - c) = 0$.*

Proof of first part:

To prove the first part, note that he $(i, j)^{th}$ entry of A^2 is the number of walks of length of 2 from the vertex i to the vertex j. Moreover, this number determined only by whether the vertices i and j are adjacent, non-adjacent or same.

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$$(A^2)_{ij} = \begin{cases} k & \text{whenever } i = j, \\ a & \text{if } i \neq j \text{ but } i \text{ and } j \text{ are adjacent,} \\ c & \text{if } i \neq j \text{ but } i \text{ and } j \text{ are not adjacent.} \end{cases}$$

Or equivalently, $A^2 = kI + aA + cA^c = kI + aA + c(\mathbf{J} - I - A)$.

Proof of second part

Proof.
For the second part, note that k is indeed an eigenvalue of X with eigenvector $\mathbf{e}$. Now, let λ be an eigenvalue of X with corresponding eigenvector $\mathbf{x}$. Then, $\mathbf{e}^T\mathbf{x} = 0$. Hence, using the first part

$$\begin{aligned} a(\lambda\mathbf{x}) &= a(A\mathbf{x}) = \left(A^2 - kI - c(\mathbf{J} - I - A)\right)\mathbf{x} \\ &= \lambda^2\mathbf{x} - k\mathbf{x} - c(0 - 1 - \lambda)\mathbf{x} = (\lambda^2 + c\lambda - (k-c))\mathbf{x}. \end{aligned}$$

As $\mathbf{x} \neq \mathbf{0}$, we must have $\lambda^2 - (a-c)\lambda - (k-c) = 0$. That is, λ satisfies the required equation. $\square$

The eigenvalues are k, $\frac{(a-c)\pm\sqrt{\Delta}}{2}$, where $\Delta = (a-c)^2 + 4(k-c)$.

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1. *X is a strongly regular if and only if A^2 is linear combination of the matrices I, J and A.*
2. *X is a strongly regular if and only if it has exactly three distinct eigenvalues.*

Euler conjectured that no orthogonal Latin squares existed for oddly even numbers (even numbers not divisible by 4.).

This conjecture by Euler was in 1782. In 1901, a French mathematician named Gaston Tarry (1843 – 1913) proved that $n = 6$ was indeed impossible by laboriously checking all possible cases. But Euler's conjecture that orthogonality was impossible for all oddly even numbers remained to be resolved. Until 1959, when R.C. Bose, Shrikhande and E.T. Parker disproved the conjecture.

Once Shrikhande said:

"had the rare privilege of seeing our works reported on the front page of the Sunday Edition of the New York Times of April 26, 1959."

Theorem (Friendship Theorem)

Let X be a finite graph such that any two distinct vertices have exactly one common neighbor. Then, there exists a vertex adjacent to all other vertices.

Proof.

Sketch of the Proof: Suppose, for contradiction, that no vertex is adjacent to all others.

Step 1: Regularity
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Step 1: Regularity
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Step 2: Strongly Regular Graph
X is strongly regular with parameters $srg(n, k, 1, 1)$.
The eigenvalues of X are $k, \sqrt{k-1}, -\sqrt{k-1}$. And we know that multiplicity of k as an eigenvalue is 1. Let r, s be multiplicities of $\sqrt{k-1}, -\sqrt{k-1}$ respectively, then $k + r\sqrt{k-1} - s\sqrt{k-1} = 0$ That is $k + (r-s)\sqrt{k-1} = 0$. Since $k = 0$ not possible, hence $r = s$ also not possible. Then $(r-s)^2(k-1) = k^2$ or $(k-1)|k^2$, and this is possible only when $k = 2$. This is also leads to a contradiction. $\square$

Graph	**Parameters** (v, k, a, c)	**Degree**	**Vertices**
$K_{n,n}$	$(2n, n, 0, n)$	n	$2n$
C_5	$(5, 2, 0, 1)$	2	5
Petersen	$(10, 3, 0, 1)$	3	10
Clebsch	$(16, 5, 0, 2)$	5	16
Hoffman–Singleton	$(50, 7, 0, 1)$	7	50
Gewirtz	$(56, 10, 0, 2)$	10	56
M_{22}	$(77, 16, 0, 4)$	16	77
Higman–Sims	$(100, 22, 0, 6)$	22	100

Table: Strongly regular triangle-free graphs

Here $K_{n,n}$ is an infinite family of graphs. Rest are unique.
Refer: https://www.math.ru.nl/OpenGraphProblems/Tjapko/30.html

$\mathcal{A}(X)$ of SRG

If X is a connected strongly regular graph, then $\dim(\mathcal{A}(X)) = 3$ and $\{I, A, A^c\} = \{A_0, A_1, A_2\}$ forms a basis for $\mathcal{A}(X)$.

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> *Let X be a distance regular graph of diameter d. Then the set of distance matrices of X, $\{A_0(X), A_1(X), \ldots, A_d(X)\}$, forms a basis of the adjacency algebra $\mathcal{A}(X)$.*

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A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance regular Graphs, Springer-Verlag*,(1989).

Let X be a graph with diameter d and any vertices u, v of X.

$$s_{hj}(u,v) = |\{w \in V | d(u,w) = h, \ \& \ d(v,w) = j\}|.$$

That is the number of vertices w whose distance from u is h and whose distance from v is j. In distance regular graph, this number is independent of u and v but it depends on distance between the vertices. That is if $d(u,v) = i$, then $s_{hij} = s_{hj}(u,v)$.

> **Definition**
> *The intersection numbers of a distance regular graph with diameter d are the* $3(d+1)$ *numbers* S_{hij}*, where* h, i *and* j *belong to the set* $\{0, 1, \ldots, d\}$.

Let $v \in V(X)$ be vertex fixed. Let $u \in V(X)$ such that $d(v,u) = i$, where $i \in \{1,2,\ldots,d\}$. If $r \in N(u)$ then $d(v,r) \in \{i-1, i, i+1\}$.

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- a_i number of neighbours of u at a distance i from v that is $a_i = |N(u) \cap X_i(v)|$

It is clear that b_d, and c_0 are undefined. We will take $c_0 = b_d = 0$.

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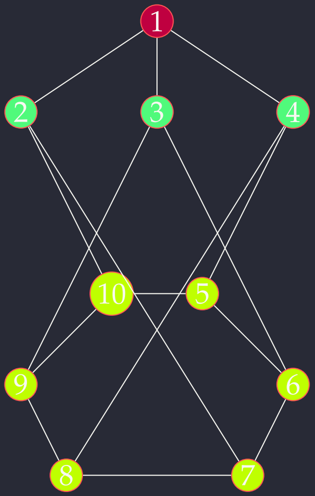
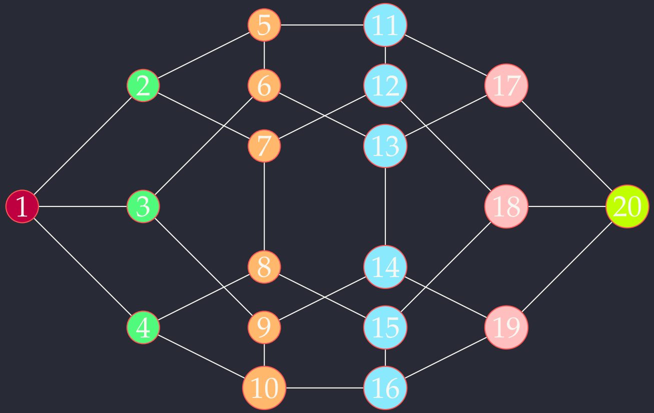
The intersection array of X denoted $i(X)$ given as $\begin{pmatrix} * & c_1 & c_2 & \ldots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \ldots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \ldots & b_{d-1} & * \end{pmatrix}$.

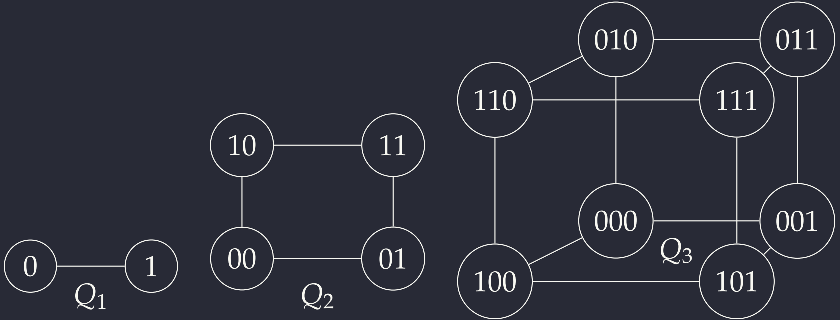
Further it is clear that $a_i + b_i + c_i = b_0$, $b_0 = k$, if X is a k- regular graph. Hence $i(X) = (k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d)$.



- Distance partition of Petersen graph.
- It is an equitable partition.
- Intersection array is $(3, 2; 1, 1)$.
- $\begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$
- Distance regular graph with diameter 2.
- Strongly regular graph.



- The Dodecahedron
- $\begin{bmatrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{bmatrix}$
- Distance transitive hence distance regular.



The k-cube, Q_k is the graph defined as follows: the vertices of Q_k are the $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$, where $\epsilon_i = 0$ *or* 1 for $1 \leq i \leq k$ and two vertices are adjacent when their symbols differ in exactly one coordinate. The graph Q_k $(k \geq 2)$ is distance-transitive, with valency k, diameter k, and intersection array $\{k, k-1, k-2, \ldots, 1; 1, 2, 3, \ldots, k\}$. For Q_3 $\{X_0(000), \ldots, X_3(000)\} = \{\{000\}, \{100, 010, 001\}, \{110, 101, 110\}, \{111\}\}$.

$$i(Q_3) = \{3, 2, 1; 1, 2, 3\}$$

The number of vertices can be obtained from the intersection array. In fact, every vertex has a constant number of vertices k_i at given distance i, that is, $k_i = |X_i(z)|$ for all $z \in V$. Indeed, this follows by induction and counting the number of edges between $X_i(z)$ and $X_{i+1}(z)$ in two ways. In particular, it follows that $k_0 = 1$ and

$$k_{i+1} = \frac{b_i k_i}{c_{i+1}}$$

for all $i = 0, 1, \ldots, d-1$. The number of vertices now follows as $v = k_0 + k_1 + \cdots + k_d$. In combinatorial arguments such as the above, it helps to draw pictures; in particular, of the so-called *distance-distribution diagram*, as depicted in Figure.

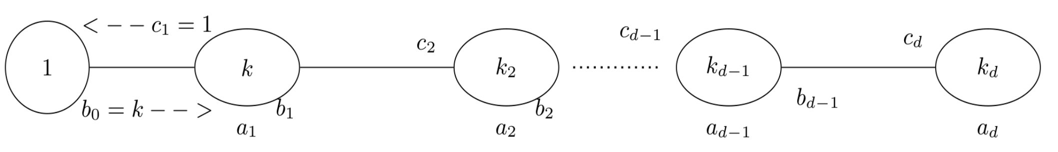


Figure: Distance-distribution diagram

Proposition

With notation as above, the following conditions hold:

(i) $k_{i+1} = \frac{b_0 b_1 \cdots b_i}{c_1 c_2 \cdots c_{i+1}}$ *is an integer for* $i = 0, 1, \ldots, d-1$,

(ii) *numbers* $nk_i, k_i a_i$ *are even for* $i = 1, 2, \ldots, d$,

(iii) nka_1 *is divisible by* 6.

Proof.

we already observed the recurrence $k_{i+1} = b_i k_i / c_{i+1}$ for all $i = 0, 1, \ldots, d-1$, and this implies that $k_{i+1} = \frac{b_0 b_1 \cdots b_i}{c_1 c_2 \cdots c_{i+1}}$ for $i = 0, 1, \ldots, d-1$. These numbers are clearly positive integers. By doubly counting all pairs (z, e), where z is an end vertex of edge e in X_i, it follows that the number of edges in X_i equals $vk_i/2$, which should be an integer. Similarly, there are $k_i a_i / 2$ edges of X within $X_i(z)$ for a fixed vertex z, and this should be an integer. Finally, the number of triangles in X equals $vka_1/6$. $\square$

Proposition

With notation as above, the following conditions hold:

(i) $1 = c_1 \leq c_2 \leq \cdots \leq c_d, \quad k = b_0 \geq b_1 \geq \cdots \geq b_{d-1},$

(ii) *If* $i + j \leq d$*, then* $c_i \leq b_j$*,*

(iii) *There is an* i *such that* $k_0 \leq k_1 \leq \cdots \leq k_i$ *and* $k_{i+1} \geq k_{i+2} \geq \cdots \geq k_d$*.*

Proof.

Let $i = 1, 2, \ldots, d$. Consider two vertices x and y at distance i, and a vertex z that is adjacent to x and at distance $i - 1$ from y. Now the c_{i-1} neighbors of y that are at distance $i - 2$ from z are all at distance $i - 1$ from x. Therefore $c_i \geq c_{i-1}$. Similarly, the b_i neighbors of y that are at distance $i + 1$ from x are at distance i from z, hence $b_{i-1} \geq b_i$. (ii) Consider two vertices x and y at distance $i + j$, and a vertex z at distance i from x and j from y. Then the c_i neighbors of z that are at distance $i - 1$ from x are at distance $j + 1$ from y. Hence $c_i \leq b_j$. (iii) It follows from (i), (ii), and Proposition 3 that $k_i^2 \geq k_{i-1}k_{i+1}$ for $i = 1, 2, \ldots, d - 1$. This implies that the k_i are unimodal: there is an i such that $k_0 \leq k_1 \leq \cdots \leq k_i$ and $k_{i+1} \geq k_{i+2} \geq \cdots \geq k_d$. $\square$

- Let A be the adjacency matrix of the cycle graph C_n.
- Then, $\gamma_A(x) = x + x^{n-1}$ is its representer polynomial and its eigenvalues are given by $\lambda_r = 2\cos(\frac{2\pi r}{n})$, for $r = 0, 1, \ldots, n-1$. It is easy to see that $\lambda_r = \lambda_{n-r}$ for $r = 1, \ldots, n-1$.
- As, the diameter of C_n is $\lfloor \frac{n}{2} \rfloor$, we see that C_n has $\lfloor \frac{n}{2} \rfloor + 1$ distinct eigenvalues and $\dim(\mathcal{A}(C_n)) = \lfloor \frac{n}{2} \rfloor + 1$. It is easy to see that the cycle graph is a distance polynomial graph, *i.e.*, its distance matrices belong to its adjacency algebra. In fact they form a basis of the adjacency algebra.
- Cycle graph is an example of distance regular graph, infact it is distance transitive graph, we will define distance transitive graphs shortly.

For example, the basis for $\mathcal{A}(C_4)$ is $\{I, A, A^2\}$ another basis is $\{I, A = A_1, A^c = A_2\}$. Same with C_5.
For C_6 and C_7 the basis are $\{I, A, A^2, A^3\}$ or $\{I, A_1, A_2, A_3\}$.

The following result shows that every symmetric circulant matrix is a polynomial in the cycle graph. Hence, the eigenvalues of every circulant graph can be computed using the eigenvalues of C_n.

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Let B be a symmetric circulant matrix with the representer polynomial $\gamma_B(x) = \sum_{i=0}^{n-1} b_i x^i$. Then $B = \sum_{i=0}^{n-1} b_i W_n^i$ and $B^T = \sum_{i=0}^{n-1} b_i W_n^{n-i}$. Consequently $b_i = b_{n-i}$, for $1 \leq i \leq n-1$. Thus, $B = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_i A_i$ and hence, the required result follows. $\square$

A.K.Lal and A.Satyanarayana Reddy, *Non-singular circulant graphs and digraphs*, Electronic Journal of Linear Algebra, Volume 26,(2013), 248–257.

Few examples of distance regular graphs and their intersection array.

Graph	Intersection array $(k, b_1, \dots, b_{d-1}; 1, c_2, \dots, c_d)$
K_n	$(n-1; 1)$
C_n	$(2, 1, 1, \dots, 1; 1, 1, \dots, 1)$ if n is odd. $(2, 1, 1, \dots, 1; 1, 1, \dots, 2)$ if n is even.

For an integer $k \geq 2$, the vertices of the Odd graph O_k are the $(k-1)$-subsets of a set of size $2k-1$, and two vertices are adjacent if the corresponding subsets are disjoint. The Odd graph O_k is distance-regular with diameter $k-1$.

For odd $k = 2l - 1$, its intersection array is
$(k, k-1, k-1, \dots, l+1, l+1, l; 1, 1, 2, 2, \dots, l-1, l-1)$.

For even $k = 2l$, the intersection array is
$(k, k-1, k-1, \dots, l+1, l+1; 1, 1, 2, 2, \dots, l-1, l-1, l)$. Consequently, the numbers a_i are zero for all $i = 0, 1, \dots, d-1$, but $a_d = l > 0$.

The Johnson graph $J(n,k)$ is a distance-regular graph with diameter $d = \min(k, n-k)$. For $0 \leq i \leq d$, the intersection numbers are given by:

$$b_i = (k-i)(n-k-i), \quad a_i = i(n-2i) + i^2, \quad c_i = i^2$$

for $i = 0, 1, 2, \ldots, d$, where $d = \min(k, n-k)$.

For $J(7,3)$, we have $n = 7, k = 3$, and diameter $d = \min(3,4) = 3$.
The intersection numbers are:

$$b_0 = (3-0)(7-3-0) = 12, b_1 = (3-1)(7-3-1) = 6, b_2 = (3-2)(7-3-2) = 2,$$

$$c_1 = 1^2, c_2 = 2^2, c_3 = 3^2.$$

The intersection array is: $\{12, 6, 2; 1, 4, 9\}$

Recall that the adjacency algebra $\mathcal{A}(X)$ of a distance regular graph X has a basis $\{A_0, A_1, \dots, A_d\}$ and $A_h A_i = \sum s_{hij} A_j$. This equation can be interpreted as saying that left multiplication by A_h (regarded as linear mapping of $\mathcal{A}(X)$ w.r.to given basis) is faithfully represented by $(d+1) \times (d+1)$ matrix B_h with $(B_h)_{ij} = s_{hij}$.

Proposition

The adjacency algebra $\mathcal{A}(X)$ of a distance regular graph with diameter d can be faithfully represented by an algebra of matrices with $d+1$ rows and columns. A basis of this representation is the set $\{B_0, B_1, \dots, B_d\}$ where $(B_h)_{ij}$ is the intersection number S_{hij} for $h, i, j \in \{0, 1, \dots, d\}$.

$$B_1 = \begin{bmatrix} 0 & k & 0 & 0 & \dots & 0 \\ c_1 & a_1 & b_1 & 0 & \dots & 0 \\ 0 & c_2 & a_2 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ \dots & \dots & \dots & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & 0 & 0 & \dots & c_d & a_d \end{bmatrix}$$

It is cleat that B_1 is quotient matrix (B). Hence similar to symmetric matrix. Since $c_1 c_2 \cdots c_d \neq 0$ the rank of $B - \theta I$ is $d+1$ for every $\theta \in \mathbb{R}$. Hence all eigenvalues of B are distinct.

The members of $\mathcal{A}(X)$ can now be regarded as square matrices of size $d+1$ (instead of n) a considerable simplification. What is more important is the matrix B_1 it self is sufficient. If you call B_1 as B, then

$$BB_i = b_{i-1}B_{i-1} + a_iB_i + c_{i+1}B_{i+1}.$$

Consequently each B_i $(i \geq 2)$ is a polynomial in B with coefficients depend only on the entries of B.
Since B is the image of adjacency matrix A under faithful representation, the minimal polynomial of A and B coincide.

$\boxed{AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \text{ for } i \in \{0, 1, \ldots, d\}}$

Theorem

A connected graph Γ is distance-regular if and only if:

1. *For every vertex $v \in V$, the distance partition $\{V_0(v), V_1(v), \ldots, V_d(v)\}$ is equitable*
2. *The intersection numbers p_{ij} are independent of the choice of vertex v*

Proof.

($\Rightarrow$) Suppose Γ is distance-regular with intersection numbers $\{b_k\}$ and $\{c_k\}$.
For any vertex v, consider its distance partition $\{V_0(v), V_1(v), \ldots, V_d(v)\}$. For $u \in V_k(v)$:

- Any neighbor w of u with $\partial(v,w) = k-1$ contributes to $p_{k,k-1}$, and by distance-regularity, there are exactly c_k such neighbors
- Any neighbor w of u with $\partial(v,w) = k$ contributes to $p_{k,k}$, and there are $a_k = \deg(u) - b_k - c_k$ such neighbors
- Any neighbor w of u with $\partial(v,w) = k+1$ contributes to $p_{k,k+1}$, and there are exactly b_k such neighbors

Thus the partition is equitable with intersection numbers:

$$p_{k,k-1} = c_k, \quad p_{k,k} = a_k, \quad p_{k,k+1} = b_k, \quad \text{and } p_{ij} = 0 \text{ for } |i-j| > 1$$

These numbers are independent of the choice of v by the definition of distance-regularity.
($\Leftarrow$) Suppose for every vertex v, the distance partition is equitable with intersection numbers independent of v.
Let $u, v \in V$ with $\partial(u,v) = k$. Consider the distance partition from u: $\{V_0(u), V_1(u), \ldots, V_d(u)\}$ where $v \in V_k(u)$.
By the equitable condition:

- The number of neighbors of v in $V_{k-1}(u)$ is $p_{k,k-1}$
- The number of neighbors of v in $V_{k+1}(u)$ is $p_{k,k+1}$

Since these intersection numbers are independent of the choice of u, we can define:

$$c_k = p_{k,k-1} \quad \text{and} \quad b_k = p_{k,k+1}$$

for all $0 \le k \le d$ (with appropriate boundary conditions $c_0 = b_d = 0$).
Therefore, Γ is distance-regular with intersection numbers $\{b_k\}$ and $\{c_k\}$. $\square$

Table: Distance-Regular Graphs of Valency 3

Graph	Description	Vertices
Complete graph K_4	Complete graph on 4 vertices	4
$K_{3,3}$	Complete bipartite graph	6
3-cube Q_3	Cube graph (Hamming graph $H(3,2)$)	8
Petersen graph	Famous symmetric graph	10
Heawood graph	(3,6)-cage	14
Pappus graph	Incidence graph of the Pappus configuration	18
Desargues graph	Incidence graph of Desargues config.	20
Dodecahedron graph	Vertices of a dodecahedron	20
Coxeter graph	Symmetric cubic graph	28
Tutte–Coxeter graph	Symmetric cubic distance-transitive graph	30
Foster graph	Smallest cubic symmetric graph with girth 10	90
Biggs–Smith graph	Sporadic example, girth 9	102
Tutte 12-cage	(3,12)-cage, also called Tutte's 12-cage	126

Table: Distance-regular graphs with valency 3

Intersection array	n	d	g	**Name**
$\{3;1\}$	4	1	3	K_4
$\{3,2;1,3\}$	6	2	4	$K_{3,3}$
$\{3,2,1;1,2,3\}$	8	3	4	Cube $\sim K_{3,3}^*$
$\{3,2;1,1\}$	10	2	5	Petersen $\sim O_3$
$\{3,2,2;1,1,3\}$	14	3	6	Heawood $\sim IG(7,3,1)$
$\{3,2,2,1;1,1,2,3\}$	18	4	6	Pappus $\sim IG(AG(2,3)\setminus pc)$
$\{3,2,2,1,1;1,1,2,2,3\}$	20	5	6	Desargues $\sim DO_3$
$\{3,2,1,1,1;1,1,1,2,3\}$	20	5	5	Dodecahedron
$\{3,2,2,1,1,1;1,1,1,1,2\}$	28	4	7	Coxeter
$\{3,2,2,2,1,1,1;1,1,1,1,3\}$	30	4	8	Tutte's 8-cage $\sim IG(GQ(2,2))$
$\{3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3\}$	90	8	10	Foster
$\{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$	102	7	9	Biggs–Smith
$\{3,2,2,2,2,2;1,1,1,1,1,3\}$	126	6	12	Tutte's 12-cage $\sim IG(GH(2,2))$

Distance Transitive Graphs

Definition

A graph X is said to be **distance transitive** *if for all vertices u, v, x, y of X with $d(u, v) = d(x, y)$, there is a $g \in Aut(X)$ satisfying $g(u) = x$ and $g(v) = y$.*

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Theorem

Let X be a distance transitive graph with diameter d. Then $\dim(\mathcal{A}(X)) = d + 1$.

- Suppose $G = Aut(X)$ acts distance transitively on X and $u \in V(X)$. If v and w are two vertices at distance i from u, there is an element of G that maps (u, v) to (u, w), *i.e.*, there is an element of G_u that maps v to w, and so G acts transitively on $X_i(u)$.
- Thus, the cells of the distance partition with respect to u are the orbits of G_u.
- If X has diameter d, then it follows that G acts distance transitively on X if and only if it acts transitively and, for any vertex $u \in V(X)$, the vertex stabilizer G_u has exactly $d+1$ orbits.
- In other words, the group G is transitive with rank $d+1$.

Orbital Matrices

In fact, in case of distance transitive graphs something more is true and to state it, we need the following definition.

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> **Definition**
> *Let G be a group acting on a non-empty set V. Then G also acts on $V \times V$, by $g(x, y) = (g(x), g(y))$. For each fixed element $(u, v) \in V \times V$, the set $Orb(u, v) = \{g(u, v) : g \in G\}$ is called the orbit of (u, v), under the action of G. The distinct orbits of $V \times V$ under the action of G are called* orbitals.

In the context of a graph $X = (V, E)$, the orbitals of X are the distinct orbits of $E \subset V \times V$ under the action of $\mathrm{Aut}(X)$. That is, the *orbitals* are the orbits of the arcs/non-arcs of the graph X.

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If X is a distance transitive graph then orbital matrices and the distance matrices defined earlier will coincide. Moreover, they form a basis for adjacency algebra $\mathcal{A}(X)$.

- The theory which underlies our treatment of the adjacency algebra of a distance regular graph was developed in two quite different contexts.
- First, the association schemes used by Bose in the statistical design of experiments led to an association algebra (Bose and Mesner 1959), which corresponds to our adjacency algebra.
- Concurrently, the work of Schur (1933) and Wielandt ring, of a permutation group,(1964) on the commuting algebra, or centralizer culminated in the paper of Higman (1967) which employs graph-theoretic ideas very closely related to those of this discussions.
- The connection between the theory of the commuting algebra and distance transitive graphs is also can be found.

Hadamard Product

- Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then the *Hadamard product* of $A = [a_{ij}]$ and $B = [b_{ij}]$, denoted $A \odot B$, is defined as $(A \odot B)_{ij} = a_{ij}b_{ij}$, for $1 \leq i, j \leq n$.

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Theorem (Higman [2], Brouwer, Cohen & Neumaier [4])

Let $\mathcal{M}$ be a vector subspace of symmetric $n \times n$ matrices. Then $\mathcal{M}$ has a basis of mutually disjoint $0, 1$-matrices if and only if $\mathcal{M}$ is closed under Hadamard multiplication.

Definition

A subalgebra of $\mathbb{M}_n(\mathbb{C})$ *containing the matrices* I *(Identity matrix) and* $\mathbf{J}$ *(matrix with all entries being* 1*) is called a* ***coherent algebra*** *if it is closed under conjugate-transposition and Hadamard product.*

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  - As any coherent algebra contains both I and $\mathbf{J}$, it is clear that $\mathbb{C}[\mathbf{J}]$ is the smallest coherent algebra.
- Note that $\mathbb{C}[\mathbf{J}] = \mathbb{C}[\mathbf{J} - I]$ which is same as $\mathcal{A}(K_n)$.

Let $P(\neq I)$ be a permutation matrix. Then it is easy to check that the set of all matrices which commute with P is a non-trivial example of a coherent algebra.

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$$W_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

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$\{I_n = W_n^0, W_n^1, W_n^2, \dots, W_n^{n-1}\}$ forms a basis of $\mathbb{F}[W_n]$.

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$$W_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

The minimal polynomial of W_n is $x^n - 1 = \prod_{d|n} \Phi_d(x)$ and

$\{I_n = W_n^0, W_n^1, W_n^2, \dots, W_n^{n-1}\}$ forms a basis of $\mathbb{F}[W_n]$.

We already observed that W_n is the adjacency matrix of a directed cycle.

Association Schemes

The theory offers a unifying frame work for studying combinatorial objects like codes, designs, finite geometries and graphs with high regularity. (Bannai-Ito Perspective).

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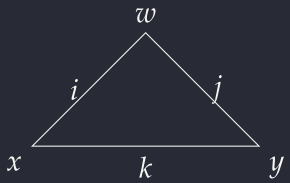
Association Scheme: $X = (\Omega, \{R_i\}_{i=0}^m)$ of class m on a set Ω. $\{R_i\}_{i=0}^m$ is partition of $\Omega \times \Omega$.

1. If $(x, y) \in R_k$, then $|\{w \in \Omega : (x, w) \in R_i, (w, y) \in R_j\}| = p_{ij}^k$.



2. $\{R_i\}_{i=0}^m = \{R_i^T\}_{i=0}^m$.
3. $R_0 = I$.

- p_{ij}^k : intersection numbers.
- $p_{ii}^0 = k_i$ valency
- X is symmetric $\Leftrightarrow R_i = R_i^T$.
- X is commutative $\Leftrightarrow p_{ij}^k = p_{ji}^k$.

- Let G be a finite group acting transitively on a finite set Ω. Then acts on $\Omega \times \Omega$ as $g(x, y) = (g(x), g(y))$. Let $R_0, R_1, \ldots, R_d$ be the G orbits of $\Omega \times \Omega$ with $R_0 = \{(x, x) | x \in \Omega\}$.
- Let G be a finite group with conjugacy classes $C_0, C_2, \ldots, C_d$. Define $(x, y) \in R_i \Leftrightarrow yx^{-1} \in C_i$. Then $(G, \{R_i\})$ is a commutative association scheme called group association scheme denoted $\chi(G)$.
  The Bose Mesner algebra of $\chi(G)$ is isomorphic to the center of the group algebra $\mathbb{C}[G]$. In this sense the theory of finite groups is "contained" in the theory of association schemes.

Primitive Schemes:Building blocks of all schemes

- Ambitious goal: Classify all primitive commutative Schemes
- Primitive Schemes serve as “building blocks” just like simple groups in the finite group theory and prime numbers for the integers.
- An association scheme is primitive if all the relation graphs (Ω, R_i) are connected graphs.
- The group-case $\chi(G$ is primitive if the 1-point stabilizer of the action is a maximal subgroup og G.
- The group association scheme $\chi(G)$ is primitive if G is finite simple group.

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