# INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
## MA21201/MA31005 - Real Analysis
## Problem Sheet 2(To be updated)
## Autumn 2021

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**Problem 1** Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Show that the convergence of $(|x_n|)$ need not imply the convegence of $(x_n)$. What about the converse?

**Problem 2** Find the limit of the following sequences:

1. $\lim((2n)^{\frac{1}{n}})$.
2. $\lim(\frac{n^2}{n!})$.
3. $\lim(\sin \frac{n\pi}{2})$.
4. $\lim(\sin n!c\pi)$, where $c \in \mathbb{Q}$.

**Problem 3** 1. Construct a sequence of rational numbers having limit in the irrationals.

2. Construct a sequence of irrational numbers having limit in the rationals.

**Problem 4** True or False:

1. sum of two divergent series is divergent.
2. product (termwise) of divergent series is divergent.

**Problem 5** Show that if $z_n := (a^n + b^n)^{\frac{1}{n}}$ where $0 < a < b$, then $\lim(z_n) = b$

**Problem 6** Find the following limits, if exist:

1. $\lim(\frac{1}{n} + \frac{1}{2n} + \cdots + \frac{1}{n^2})$.
2. $\{(\frac{2}{1})(\frac{3}{2})^2 \ldots (\frac{n+1}{n})^n\}^{\frac{1}{n}} = e$.

**Problem 7** Let $A$ be an infinite subset of $\mathbb{R}$ that is bounded above and let $u = \sup A$. Show that there exists an increasing sequence $(x_n)$ with $x_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(x_n)$.

**Problem 8** A real number is called algebraic if it is a root of an algebraic equation $a_0 + a_1x + \cdots + a_nx^n = 0$ where the coefficients $a_0, a_1, \ldots, a_n$ are integers. Prove that the set of algebraic numbers is countable.

**Problem 9** Suppose that every subsequence of $(x_n)$ has a convergent subsequence that converges to 0. Then show that $\lim x_n = 0$.

**Problem 10** Show that, if $a \geq 0, b \geq 0$ and $a < b$, then $\sqrt{a} < \sqrt{b}$.

**Problem 11** Let $(x_n)$ be a Cauchy sequence with non-zero values. Show that $(\frac{1}{x_n})$ is Cauchy if and only if $(\frac{1}{x_n})$ is bounded.

**Problem 12** Ture or False:

1. If $\sum a_n$ with $a_n > 0$ is convergent, then $\sum a_n^2$ is convergent.
2. If $\sum a_n$ and $b_n$ are convergent, then $\sum a_nb_n$ is convergent.
3. If $\sum a_n$ with $a_n > 0$ is convergent, then $\sum \sqrt{a_n}$ is convergent.
4. If $\sum a_n$ with $a_n > 0$ is convergent, then $\sum \sqrt{a_na_{n+1}}$ is convergent.

**Problem 13** Show that if a series is condionally convergent, then the series obtained from its positive terms is divergent, and the series obtained from its negative terms is divergent.

**Problem 14** Discuss the convergence or the divergence of the series with $n$-th terms:

1. $\frac{n}{(n+1)(n+2)}$.
2. $(n^2(n+1))^{-\frac{1}{2}}$.
3. $(\ln n)^{-\ln \ln n}$.
4. $n!e^{-n^2}$.
5. $\frac{2.4...(2n)}{3.5...(2n+1)}$.

**Problem 15** Suppose $(a_n)$ is a sequence of strictly positive terms such that $a_n \to 0$. Show that $\sum a_n$ converges if and only if the series $\sum \sin a_n$ is absolutely convergent.