

## Partial Differential Equations (MA20103)

### Assignment -1

#### I- Power Series Solution

1. Find power series solution of  $y'' + xy' + x^2y = 0$  about  $x = 0$

$$\left[ \text{Ans: } y = c_0 \left( 1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 - \dots \right) + c_1 \left( x - \frac{1}{6}x^3 + \frac{1}{40}x^5 - \dots \right) \right]$$

2. Find the power series solution of  $y'' + (x-3)y' + y = 0$  in powers of  $(x-2)$

$$\left[ \text{Ans: } y = c_0 \left( 1 - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 \dots \right) \right. \\ \left. + c_1 \left( (x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 \dots \right) \right]$$

3. Solve  $y'' - 2x^2y' + 4xy = x^2 + 2x + 4$  in powers of  $x$

$$\left[ \text{Ans: } y = c_0 \left( 1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 - \dots \right) + c_1 \left( x - \frac{1}{6}x^4 - \frac{1}{63}x^7 - \dots \right) + 2x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7 + \dots \right]$$

4. Apply Frobenius method to solve the following differential equation in series about  $x = 0$

$$2x^2y'' - xy' + (1-x^2)y = 0$$

$$\left[ \text{Ans: } y = ax \left( 1 + \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} + \dots \right) + bx^{\frac{1}{2}} \left( 1 + \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 7} + \dots \right) \right]$$

5. Show that  $x = 0$  is a regular singular point of  $(2x + x^3)y'' - y' - 6xy = 0$  and find its series solution about  $x = 0$ .

$$\left[ \text{Ans: } y = a \left( 1 + 3x^2 + \frac{3}{5}x^4 + \dots \right) + bx^{\frac{3}{2}} \left( 1 + \frac{3}{8}x^2 - \frac{3}{128}x^4 + \dots \right) \right]$$

6. Solve  $x^2y'' + xy' + (x^2 - 1)y = 0$  in series near  $x = 0$ .

$$\left[ \text{Ans: } y = ax^{-1} \left( -\frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + \dots \right) + bx^{-1} \left( 1 + \frac{x^2}{2^2} - \frac{1}{2^2 \cdot 4} \left( \frac{2}{2} + \frac{1}{4} \right) x^4 + \dots \right) + \right. \\ \left. bx^{-1} \ln x \left( -\frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + \dots \right) \right]$$

## II Legendre equation and Legendre polynomial

7. Show that

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

where  $P_n(x)$  is the Legendre function of 1<sup>st</sup> kind.

8. Express  $P(x) = x^4 + 2x^3 + 2x^2 - x - 3$  in terms of Legendre's polynomials.

9. Show that

(i)  $P_n(1) = 1$

(ii)  $P_n(-x) = (-1)^n P_n(x)$

10. Prove that when  $n$  is odd  $P_n(0) = 0$

11. Prove that  $C + \int P_n dx = \frac{P_{n+1} - P_{n-1}}{2n+1}$  where  $C$  is a constant.

## III Bessel equation and Bessel function

12. Show that  $J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \cos x$  and  $J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x$  where  $J_n(x)$  is the Bessel function of 1<sup>st</sup> kind.

13. Prove that

$$J_0' = -J_1$$

$$J_2 - J_0 = 2J_0''$$

14. Establish the differential formula

$$x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x), \quad n = 0, 1, 2, \dots$$

15. Prove that

$$\frac{d}{dx}(x J_n J_{n+1}) = x(J_n^2 - J_{n+1}^2)$$

