MA20103 - Partial differential equations

Problem Sheet IV *

November 12, 2017

1 Wave, Laplace and Heat equations

Problem 1.1. *(Use d'Alembert's method) The ends of a stretched string of length $L = 1$ are fixed at $x = 0$ and $x = 1$. The string is set to vibrate from the rest by releasing it from an initial triangular shape modeled by the function*

$$f(x) = \begin{cases} \frac{3}{10}x, & \text{if } 0 \leq x \leq \frac{1}{3} \\ \frac{3}{20}(1-x), & \text{otherwise} \end{cases} \tag{1}$$

Determine subsequent motion of the string, given that $c = \pi$.

Problem 1.2. *Solve the motion of a string of length $L = \frac{\pi}{2}$ if $c = 1$ and the initial displacement and velocity are given by $f(x) = 0$ and $g(x) = x \cos x$.*

Problem 1.3. *Solve the wave equation for a string of unit length, subject to the given conditions.*

1. $f(x) = \frac{1}{2}\sin \pi x, g(x) = 0$ *and* $c = \pi$,
2. $f(x) = \sin \pi x \cos \pi x, g(x) = 0$ *and* $c = \pi$,
3. $f(x) = x \sin \pi x, g(x) = 0$ *and* $c = \pi$
4. $f(x) = x(1-x), g(x) = sin \pi x$ *and* $c = 1$

Problem 1.4. *Solve the heat equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2}, \quad 0 < x < \pi, \quad t > 0,$$

$$u(0,t) = 0 \quad \textit{and} \quad u(\pi, t) = 0, \ t > 0,$$

$$u(x, 0) = 100, \quad 0 < x < \pi$$

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Problem 1.5. *Solve the heat equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = 0 \quad \textit{and} \quad u(\pi,t) = 0, \ t > 0,$$
$$u(x,0) = 30 \sin x, \quad 0 < x < \pi$$

Problem 1.6. *Solve the heat equation:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = 0 \quad \textit{and} \quad u(1,t) = 0, \ t > 0,$$
$$u(x,0) = e^{-x}, \quad 0 < x < 1$$

Problem 1.7. *Consider the Laplace equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} = 0, \qquad 0 < x < a, \ 0 < y < b,$$

with the boundary conditions :

$$u(x,0) = f_1(x), \qquad u(x,b) = f_2(x), \quad 0 < x < a,$$
$$u(0,y) = g_1(y), \quad u(a,y) = g_2(y), \quad 0 < y < b.$$

Solve the problem for the following data:

1. $a = 1, b = 2, f_2(x) = x, f_1(x) = g_1(y) = g_2(y) = 0.$
2. $a = 1, b = 1, f_1(x) = 0, f_2(x) = 100, g_1(y) = 0, g_2(y) = 100.$
3. $a = 2, b = 1, f_1(x) = 100, f_2(x) = g_1(y) = 0, g_2(y) = 100(1 - y).$
4. $a = b = 1, f_1(x) = \sin 7\pi x, f_2(x) = \sin \pi x, g_1(y) = \sin 3\pi y, g_2(y) = \sin 6\pi y.$