MA30003/MA41003 - Linear Algebra

Problem Sheet 4 *

October 31, 2017

1 Inner product spaces

Problem 1.1. *In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product, where $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$. Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.*

Problem 1.2. *Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.*

Problem 1.3. *Suppose that $\langle .,.\rangle_1$ and $\langle .,.\rangle_2$ are two inner products on a vector space V. Prove that $\langle .,.\rangle = \langle .,.\rangle_1 + \langle .,.\rangle_2$ is another inner product on V.*

Problem 1.4. *Prove that if V is an inner product space, then $|\langle x, y\rangle\| = \|x\|.\|y\|$ if and only if one of the vectors x or y is a multiple of the other.*

Problem 1.5. *Let V be a vector space over F, where $F = \mathbb{R}$ or $F = \mathbb{C}$, and let W be an inner product space over F with inner product $\langle .,.\rangle$. If $T : V \to W$ is linear, Prove that $\langle x, y\rangle^{'} = \langle T(x), T(y)\rangle$ defines an inner product on V if and only if T is one -to-one.*

Problem 1.6. *Let V be an inner product space, S and S_0 be subsets of V , and W be a finite-dimensional subspace of V. Prove the following results.*

(a) *$S_0 \subseteq S$ implies that $S^{\perp} \subseteq S_0^{\perp}$.*

(b) *$S \subseteq (S^{\perp})^{\perp}$; so span $(S) \subseteq (S^{\perp})^{\perp}$.*

(c) *$W = (W^{\perp})^{\perp}$.*

(d) *$V = W \oplus W^{\perp}$.*

Problem 1.7. *Let W_1 and W_2 be subspaces of finite-dimensional inner product space. Prove that $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$ and $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$.*

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Problem 1.8. *Let T be a linear operator on an inner product space V. If $\langle T(x), y\rangle = 0$ for all $x, y \in V$, prove that $T = 0$. In face, prove this result if the equality holds for all x and y in some basis for V.*

Problem 1.9. *In each of the following parts, find the orthogonal projection of the given vector on the given subspace W of the inner product space V.*

(a) $V = \mathbb{R}^2$,$u = (2, 6)$ *, and* $W = \{(x, y) : y = 4x\}$.

(b) $V = \mathbb{R}^3$,$u = (2, 1, 3)$*, and* $W = \{(x, y, z) : x + 3y - 2z = 0\}$.

(c) $V = P(\mathbb{R})$ *with the inner product* $\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt$, $h(x) = 4 + 3x - 2x^2$*, and* $W = P_1(\mathbb{R})$

Problem 1.10. *Let $V = \mathbb{C}([-1, 1])$ with the inner product $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt$, and let W be the subspace of $P_2(\mathbb{R})$, viewed as a space of functions. Use the orthonormal basis obtained from standard ordered basis for W to compute the " best " (closets) second-degree polynomial approximation of the function $h(t) = e^t$ on the interval $[-1, 1]$.*