MA30003/MA41003 - Linear Algebra

Problem Sheet 3 *

August 9, 2017

1 Linear Transformations

Problem 1.1. *Prove that there exits a linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *such that* $T(1,1) = (1,0,2)$ *and* $T(2,3) = (1,-1,4)$. *Find the value of* $T(4,6)$.

Problem 1.2. *Let* V *and* W *be finite dimensional vector spaces and* $T : V \to W$ *be linear.*

1. *Prove that if* $dim(V) < dim(W)$, *then* T *cannot be onto.*
2. *Prove that if* $dim(V) > dim(W)$, *then* T *cannot be one-one.*

Problem 1.3. *Give an example of distinct linear transformations* T *and* U *such that* $N(T) = N(U)$ *and* $R(T) = R(U)$.

Problem 1.4. *Let* $T : \mathbb{R}^3 \to \mathbb{R}$ *be linear. Show that there exist scalars* a, b *and* c *such that* $T(x,y,z) = ax + by + cz$ *for all* $(x,y,z) \in \mathbb{R}^3$.

Problem 1.5. *A function* $T : V \to W$ *between vector spaces* V *and* W *is called additive if* $T(x+y) = T(x) + T(y)$ *for all* $x, y \in V$. *Prove that, if* V *and* W *are vector spaces over the field of rational numbers, then any additive function from* V *into* W *is a linear transformation.*

Problem 1.6. *Prove that there is an additive function* $T : \mathbb{R} \to \mathbb{R}$ *that is not linear.*

Problem 1.7. *Define* $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ *by* $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = (a+b) + (2d)x + bx^2$. *Find the matrix of the linear transformation with respect to the standard bases.*

Problem 1.8. *Let* V *be an* n*-dimensional vector space with an ordered basis* $\mathbb{B}$. *Define* $T : V \to \mathbb{F}^n$ *by* $T(x) = [x]_{\mathbb{B}}$. *Prove that* T*is linear.*

Problem 1.9. *Let* V *and* W *be vector spaces such that* $dim(V) = dim(W)$ *and, let* $T : V \to W$ *be linear. Show that there exist ordered bases* $\mathbb{B}_1$ *and* $\mathbb{B}_2$ *for* V *and* W, *respectively, such that the matrix of the transformation* $[T]_{\mathbb{B}_1}^{\mathbb{B}_2}$ *is a diagonal matrix.*

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Problem 1.10. *Find linear transformations $U, T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $UT = 0$(the zero transformation), but $Tu \neq 0$.*

Problem 1.11. *Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = 0$ if and only if $R(T) \subseteq N(T)$.*

Problem 1.12. *Let V be a finite dimensional vector space, and let $T : V \to V$ be linear.*

1. *Prove that, if $rank(T) = rank(T^2)$, then $R(T) \cap N(T) = \{0\}$ and $V = R(T) \bigoplus N(T)$.*
2. *Prove that $V = R(T^k) \bigoplus N(T^k)$ for some positive integer k.*