MA30003/MA41003 - Linear Algebra

Problem Sheet 1*

July 20, 2017

1 Vector spaces

Problem 1.1. *Let $\mathbb{F}$ be a field and let $M_{m\times n}(\mathbb{F})$ denote the set of all $m \times n$ matrices with entries are from the field $\mathbb{F}$. Define vector addition and and scalar multiplication on $M_{m\times n}(\mathbb{F})$ over $\mathbb{F}$ as follows:*

For $A, B \in M_{m\times n}(\mathbb{F})$ and $\alpha \in \mathbb{F}$, $(A+B)_{ij} = A_{ij} + B_{ij}$ and $(\alpha A)_{ij} = \alpha A_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Prove that $M_{m\times n}(\mathbb{F})$ is a vector space over the field $\mathbb{F}$ with respect to the operations defined as above.

Problem 1.2. *Let S be any nonempty set and $\mathbb{F}$ be any field, and let $\mathcal{F}(S, \mathbb{F})$ denote the set of all functions from S to $\mathbb{F}$. Define vector addition and scalar multiplication on $\mathcal{F}(S, \mathbb{F})$ over $\mathbb{F}$ as follows:*

For $f, g \in \mathcal{F}(S, \mathbb{F})$ and $\alpha \in \mathbb{F}$, $(f+g)(s) = f(s) + g(s)$ and $(cf)(s) = cf(s)$ for each $s \in S$. Prove that $\mathcal{F}(S, \mathbb{F})$ is a vector space over the field $\mathbb{F}$ with respect to the operations defined as above.

Problem 1.3. *Let $\mathcal{F}$ be a field and $\mathbb{P}(\mathbb{F})$ denote the set of polynomials with coefficients from the field $\mathbb{F}$. With respect to the usual addition of polynomials and scaler multiplication, prove that $\mathbb{P}(\mathbb{F})$ is vector space over $\mathbb{F}$.*

Problem 1.4. *Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define*

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations?

2 Subspaces

Problem 2.1. *Consider the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$. Which of the following subsets are subspaces of $\mathbb{R}^2$?*

1. $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 0\}$,
2. $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = -1\}$,

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3. $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\}$.

Problem 2.2. *Consider the vector space $M_{n\times n}(\mathbb{R})$ over the field $\mathbb{R}$. Which of the following subsets are subspaces of $M_{n\times n}(\mathbb{R})$?*

1. *the set of all matrices whose entries are nonnegative,*
2. *the set of all matrices invertible matrices,*
3. *the set of all symmetric matrices,*
4. *the set of all skew symmetric matrices,*
5. *the set of all upper triangular matrices,*
6. *the set of all matrices with trace zero.*

Problem 2.3. *Consider the vector space $\mathcal{F}(\mathbb{C}, \mathbb{C})$ over the field $\mathbb{C}$. Which of the following subsets are subspaces of $\mathcal{F}(\mathbb{C}, \mathbb{C})$?*

1. *the set of all even functions, (a function $f \in \mathcal{F}(\mathbb{C}, \mathbb{C})$ is called even, if $f(-s) = f(s)$ for all $s \in \mathbb{C}$)*
2. *the set of all odd functions, (a function $f \in \mathcal{F}(\mathbb{C}, \mathbb{C})$ is called odd , if $f(-s) = -f(s)$ for all $s \in \mathbb{C}$)*
3. *the set of all functions f such that $f(0) = 0$,*
4. *the set of all real valued functions,*
5. *the set of all continuous functions.*

Problem 2.4. *Consider the vector space $\mathbb{P}(\mathbb{R})$ over the field $\mathbb{R}$. Which of the following subsets are subspaces of $\mathbb{P}(\mathbb{R})$?*

1. *the set of all polynomials of degree n,*
2. *the set of all polynomials of degree less than or equal to n,*
3. *the set of all polynomials of degree greater than or equal to n,*
4. $\{p(x) \in \mathbb{P}(\mathbb{R}) : p(0) = 2017\}$,
5. $\{p(x) \in \mathbb{P}(\mathbb{R}) : p(0) = 0\}$,
6. $\{p(x) \in \mathbb{P}(\mathbb{R}) : p(1729) = p(1887)\}$.