Linear Systems I

Generally a problem of computing solution to a BVP is converted to the problem of solving a system of linear equations.

→ discretization of domain

→ collocation

Linear Algebra

$$Ax = b$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$

To find $x \in \mathbb{R}^{n \times 1}$ such that

$$Ax = b$$

For us today, $m = n$. (A is square)

Given $Ax = b$, compute x.

A is invertible $\Rightarrow$ x is unique.

$$x = A^{-1} b$$

Question: If A and b are NOT EXACTLY known, what is the effect on the solution we obtain?!

Plan:

- Introduce the concept of distance on the vectors and matrices.
- Condition number.
- Sensitivity analysis.

Vector norm :

$\|\cdot\| \longrightarrow \mathbb{R}^n \to \mathbb{R}$ which satisfies

i) $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$.

ii) $\|\alpha x\| = \boxed{|\alpha| \, \|x\|}$; $\alpha \in \mathbb{R}$

iii) $\|x+y\| \leq \|x\| + \|y\|$

How far x is from y , $\|x-y\|$

Examples

i) $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/n}$; $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

ii) $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$ as $p \to \infty$

iii) $\|x\|_p = \boxed{\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}}$ $p > 1$

Matrix - norms:

$\|\cdot\| : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$

i) $\|A\| > 0$, and $\|A\| = 0$ iff $A = 0$.

ii) $\|\alpha A\| = |\alpha| \, \|A\|$ for $\alpha \in \mathbb{R}$
$A \in \mathbb{R}^{n\times n}$

iii) $\|A+B\| \leq \|A\| + \|B\|$

iv) $\|AB\| \leq \|A\| \, \|B\|$

Example:

$$\|A\|_F = \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

where $A = [a_{ij}]_{i=1, j=1}^{n \quad n}$

Frobenius - norm of A.

Induced - matrix norms:

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

[Matrix as a linear function from $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$x \longmapsto Ax$$ under the action of $A \in \mathbb{R}^{n \times n}$.

$$\frac{\|Ax\|_2}{\|x\|_2}$$

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$$

$$x \qquad\qquad\qquad Ax$$]

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

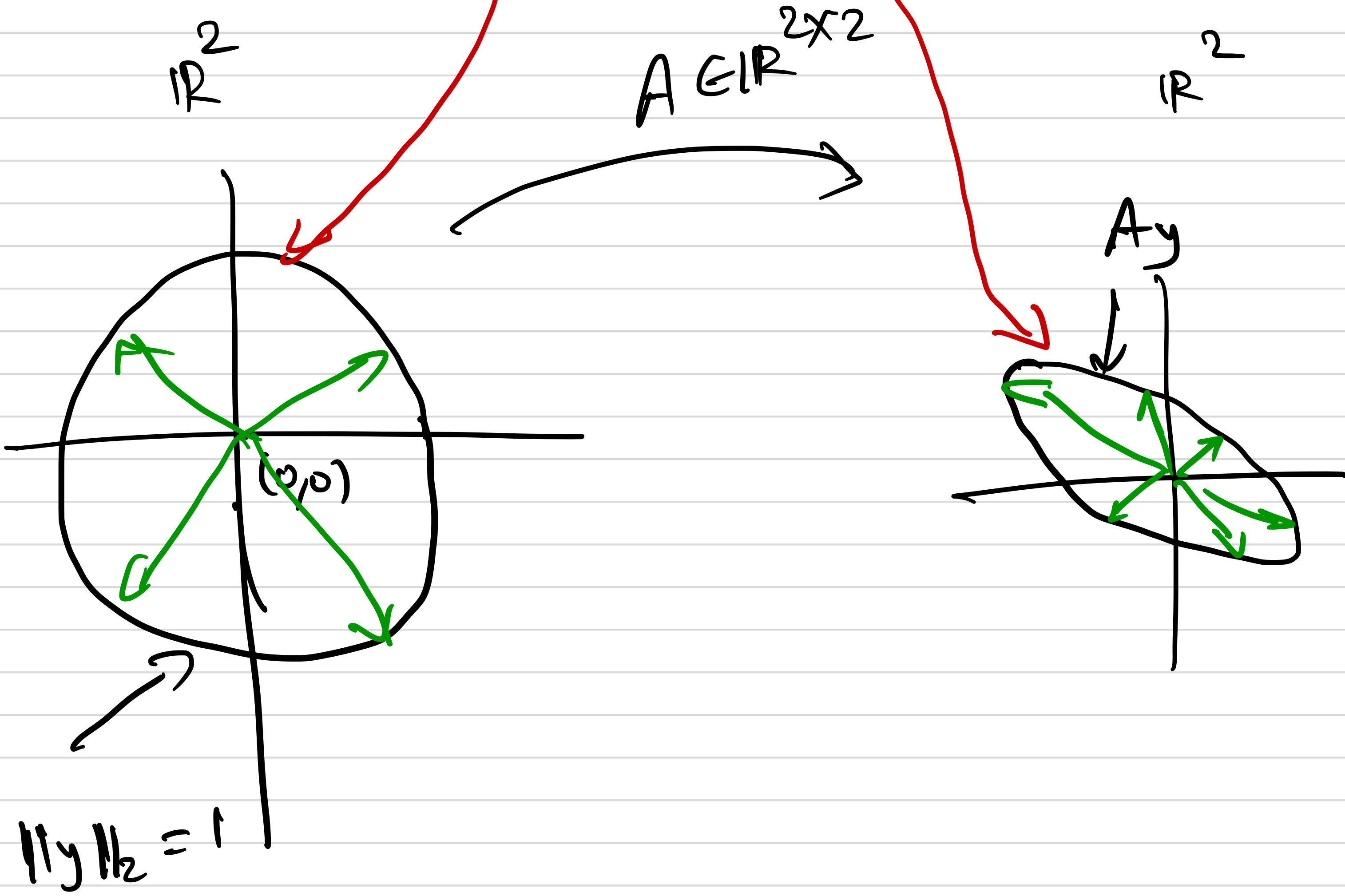
Let $\alpha = \|x\|_2$

$$= \max_{x \neq 0} \left\| \frac{Ax}{\alpha} \right\|_2$$

$$= \max_{x \neq 0} \left\| A \frac{x}{\|x\|_2} \right\|_2$$

Geometric meaning of induced norm

$$\|A\|_2 = \max_{\|y\|_2 = 1} \|Ay\|_2 \qquad \Bigg| \quad y = \frac{x}{\|x\|_2}, \quad x \neq 0$$

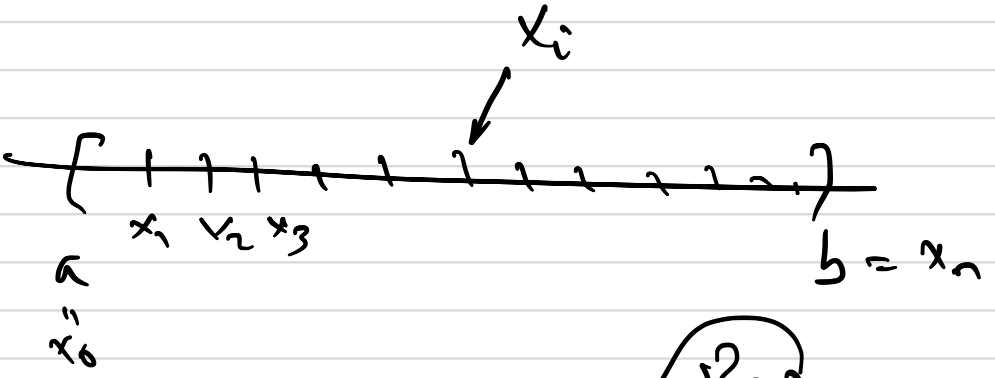


We want to solve $Ax = b$

Note that the matrices A & b may not be known exactly.

$\leftrightarrow$ Understand the effect of this on the computed solution.

<u>Case (1)</u> Assume A is known exactly. However b is not known exactly.



$$\frac{d^2y}{dx^2} = f$$

$y(a) = \ldots$

$y(b) = \ldots$

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{(\underbrace{x_{i+1} - x_i}_{h})^2} = f(x_i)$$

$$Ax = b \quad \text{--- (1)}$$

b not known exactly.

$$A\hat{x} = b + \delta b$$

where $\hat{x}$ is an exact solution. & $A\hat{x} = b + \delta b$

Denote by $\hat{x} = x + \delta x$

$$A(x + \delta x) = b + \delta b \quad \text{--- (2)}$$

Subtracting (1) from (2)

$$A\delta x = \delta b$$

$$\Rightarrow \quad \delta x = A^{-1} \delta b$$

$$\Rightarrow \quad \|\delta x\|_2 = \|A^{-1} \delta b\|_2$$

$$\leq \|A^{-1}\|_2 \, \|\delta b\|_2 \quad \text{--- } (*)$$

From (1), $\|b\|_2 \leq \|A\|_2 \|x\|_2$

$$\Rightarrow \quad \frac{1}{\|x\|_2} \leq \|A\|_2 \frac{1}{\|b\|_2} \quad \text{--- } (*')$$

From $(*)$ & $(*')$ $\boxed{\dfrac{\|\delta x\|_2}{\|x\|_2} \leq \|A\|_2 \|A^{-1}\|_2 \dfrac{\|\delta b\|_2}{\|b\|_2}}$

<u>Definition</u>: Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then condition number of A w.r.t. induced 2-norm is defined as

$$k_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

<u>Exercise</u>:

$$\|I\|_2 = 1.$$

For any invertible matrix A

$$I = AA^{-1}$$

$$\underbrace{1 = \|I\|_2} = \|AA^{-1}\|_2 \leq \|A\|_2 \|A^{-1}\|_2$$

$$= k_2(A)$$

$$\Rightarrow \quad k_2(A) \geq 1$$

Take $A = I$

$$1 = \|I\|_2 = \|II^{-1}\|_2 \leq \|I\|_2 \|I^{-1}\|_2$$

$$= 1$$

$$\Rightarrow k_2(I) = 1$$

<u>Example:</u> b is not exactly known.

$$\frac{\|\delta b\|_2}{\|b\|_2} \approx 10^{-4}$$

$$k_2(A) \approx 10^2$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq k_2(A) \frac{\|\delta b\|_2}{\|b\|_2}$$

$$10^2 \cdot 10^{-4}$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \lesssim 10^{-2}$$

<u>Ex:</u> $A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$

$b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$$b + \delta b = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix} \quad ; \delta b = \begin{bmatrix} -0.01 \\ 0.01 \end{bmatrix}$$

$$A\hat{x} = b + \delta b$$

$$\hat{x} = x + \delta x = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$$

<u>Check:</u>

$$A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$$

$$k_2(A) = 3.992 \times 10^{6}$$

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq k_2(A) \frac{\|\delta b\|_2}{\|b\|_2}$$

10^{6}

10^{-5}

$$b + \delta b = \begin{bmatrix} 1999.01 \\ 1997.01 \end{bmatrix}$$

$$A\hat{x} = \begin{bmatrix} 1999.01 \\ 1997.01 \end{bmatrix}$$

$$\hat{x} = \begin{pmatrix} 1.01 \\ 0.99 \end{pmatrix}$$

magnification:

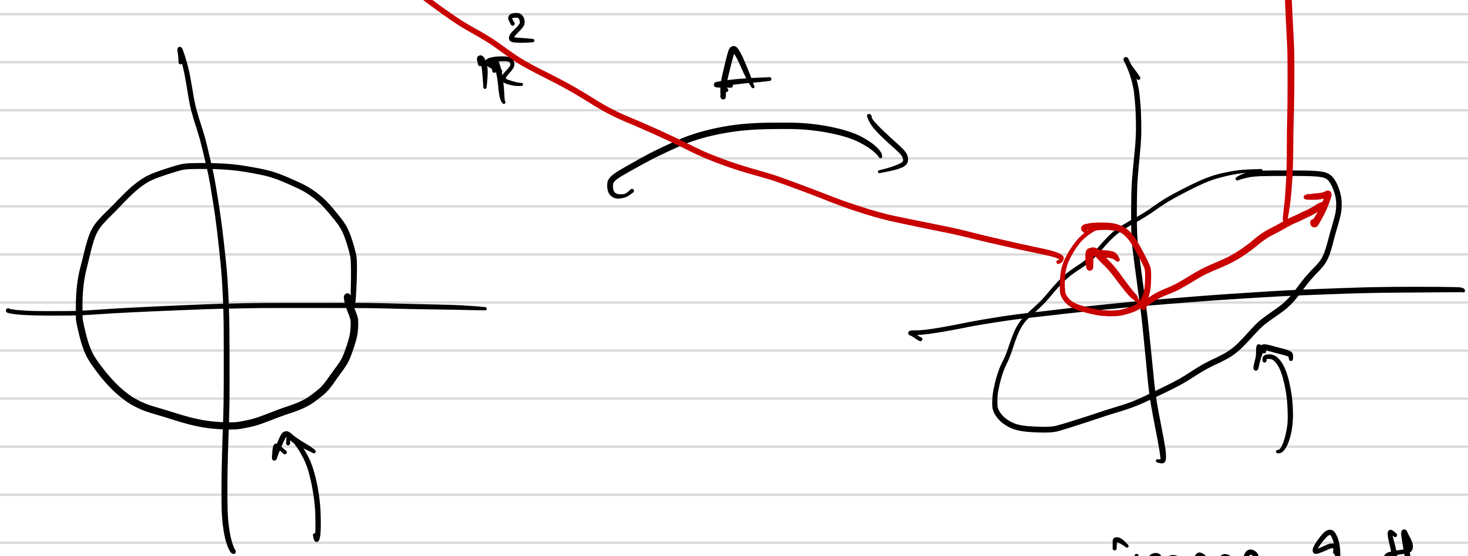
maximum magnification of A:

$$\text{magmax}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2$$

minimum magnification of A:

$$\text{min mag}(A) = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$= \min_{\|x\|_2 = 1} \|Ax\|_2$$



unit circle
$\|x\|_2 = 1$

image of the unit circle
Ax

<u>Exercise:</u> For an invertible matrix $A \in \mathbb{R}^{n \times n}$, prove that

$$\text{maxmag}(A) = \frac{1}{\text{minmag}(A^{-1})}.$$

$$\text{maxmag}(A^{-1}) = \frac{1}{\text{minmag}(A)}.$$

Consequence of the exercise:

$$k_2(A) = \|A\|_2 \, \|A^{-1}\|_2$$

$$= \text{maxmag}(A) \cdot \text{maxmag}(A^{-1})$$

$$\boxed{k_2(A) = \frac{\text{max.mag}(A)}{\text{min mag}(A)}}$$

Exercise:

Let $A = \begin{pmatrix} 1/\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$

where ε is such that $|\varepsilon|$ is very close to zero. but $\varepsilon \neq 0$.

Fundamentals of matrix computations
  - David Watkins

Numerical linear algebra and applications
  - Biswa Nath Datta

Matrix Computations
  - Golub, Van Loan

Proof, refer the books.

Sensitivity result.

Let $Ax = b$ and $(A+\delta A)(x+\delta x) = (b+\delta b)$.

A is non-singular and $\frac{\|\delta A\|_2}{\|A\|_2} < \frac{1}{k_2(A)}$.

then

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \frac{k_2(A)\left[\frac{\|\delta A\|_2}{\|A\|_2} + \frac{\|\delta b\|_2}{\|b\|_2}\right]}{1 - k_2(A)\left(\frac{\|\delta A\|_2}{\|A\|_2}\right)}$$

Understanding ill-conditioned matrices.

$A \in \mathbb{R}^{n\times n}$ with a <u>very high</u> condition number.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$$

$$Ax = b \Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$1 \le k_2(A) = \frac{\max \operatorname{mag}(A)}{\min \operatorname{mag}(A)} = \frac{1}{\min \operatorname{mag}(A)}$$

$\Rightarrow$ $k_2(A) >> 1$

$\Rightarrow$ $\min \operatorname{mag}(A) << 1$

$\exists$ a unit vector $c \in \mathbb{R}^n$ such that

$$\|Ac\| << 1$$

$$\boxed{\min_{\|c\|=1} \|Ac\| = \min \operatorname{mag}(A)}$$

$$Ac = \sum_{i=1}^{n} a_i c_i \qquad A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\Rightarrow$ Columns of A are "almost" linearly dependent.

Similarly, I can prove, rows of A are "almost" linearly dependent.

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix} \quad ; \quad k_2(A) \simeq 10^6$$

$$Ax = b + \delta b \qquad b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$$

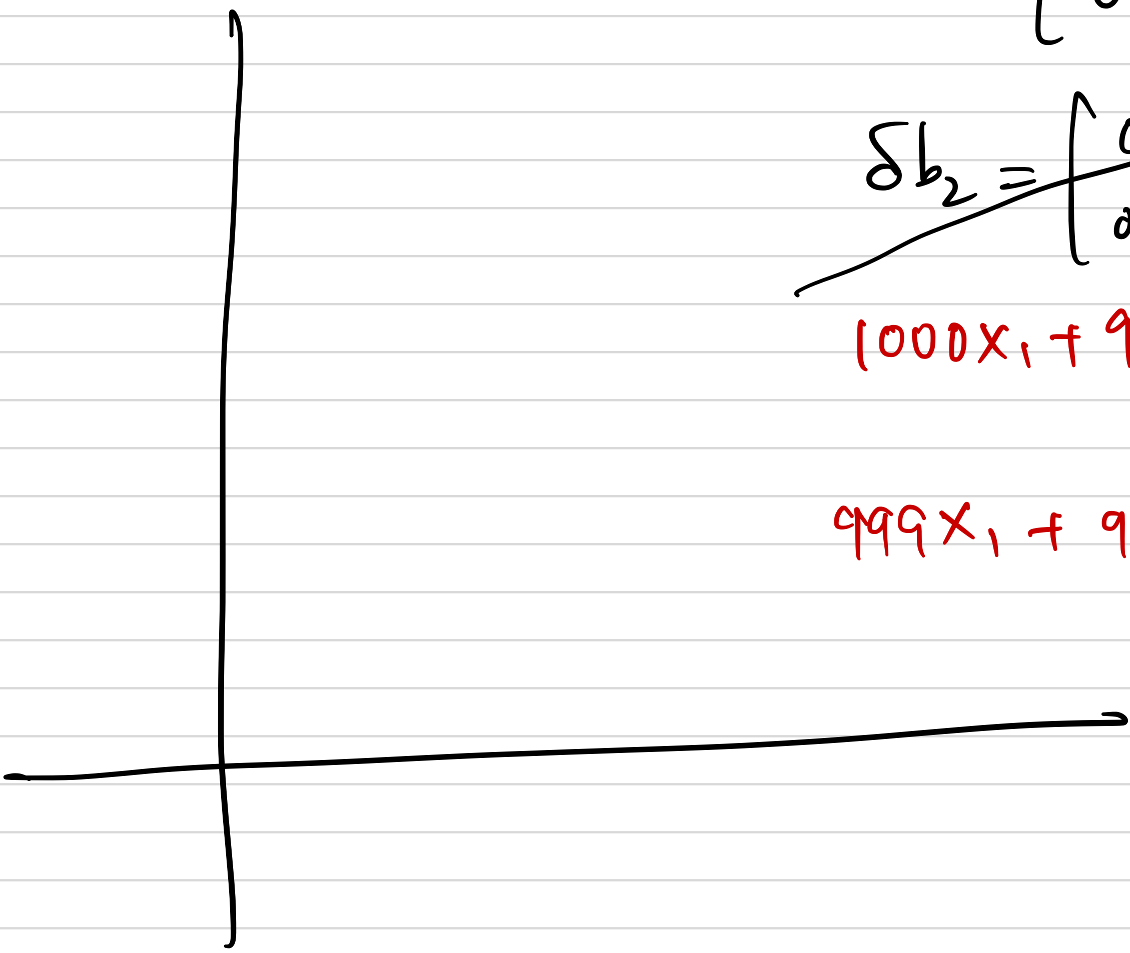
$$\boxed{\begin{aligned} 1000x_1 + 999x_2 &= 1999 \\ 999x_1 + 998x_2 &= 1997 \end{aligned}}$$

$$\delta b_1 = \begin{bmatrix} -0.01 \\ 0.01 \end{bmatrix}$$

$$\cancel{\delta b_2 = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}}$$

$$1000x_1 + 999x_2 = 1998.01$$

$$999x_1 + 998x_2 = 1997.01$$



Hibert matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ & & & \\ & & & \end{bmatrix}$$

Hilb(n)

Singular Value Decomposition (1930)

Let $A \in \mathbb{R}^{m \times n}$. Then there exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ and numbers $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2 > 0$ where $r \leq \min\{m, n\}$ such that

$$A = U \Sigma V^T$$

where $\Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots \; \sigma_r^2 & \\ 0 & & 0 \end{bmatrix}_{m \times n}$

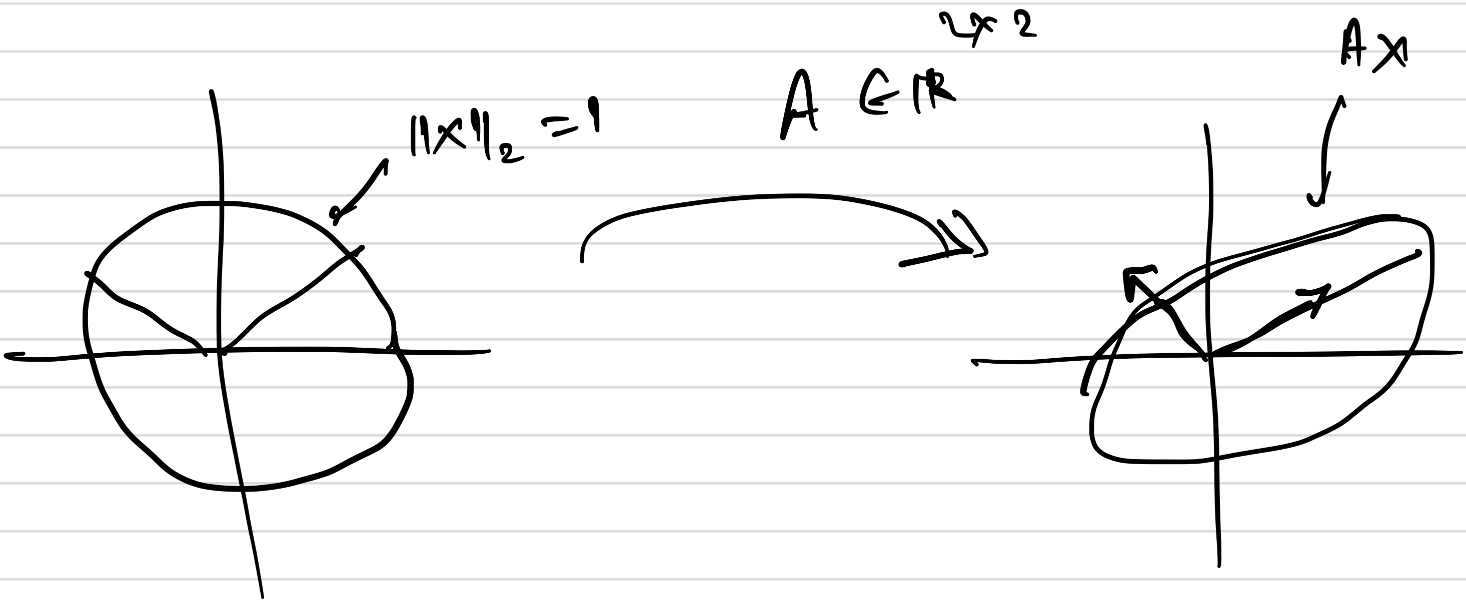
$r = \text{rank}(A)$

$U = [u_1 \; \cdots \; u_m]$

$V = [v_1, \cdots, v_m]$

$A = \sigma_1^2 u_1 v_1^T + \sigma_2^2 u_2 v_2^T + \cdots + \sigma_r^2 u_r v_r^T$

maxmag(A) minmag(A)



$[U, S, V] = \text{svd}(A)$

$A = USV^T$