Module 1

Existence and Uniqueness of Solutions

Lecture 1

1.1 Preliminaries

There are many instances where a physical problem is represented by differential equations may be with initial or boundary conditions. The existence of solutions for mathematical models is vital as otherwise it may not be relevant to the physical problem. This tells us that existence of solutions is a fundamental problem. The Module 1 describes a few method for establishing the existence, naturally under certain premises. We first look into a few preliminaries for the ensuing discussions. In this lecture, we consider a class of functions satisfying Lipschitz condition, which plays an important role in the qualitative theory of differential equations. Its applications in showing the existence of a unique solution and continuous dependence on initial conditions are dealt with in this module.

Definition 1.1.1. A real valued function $f : D \to \mathbb{R}$ defined in a region $D \subset \mathbb{R}^2$ is said to satisfy *Lipschitz condition* in the variable x with a Lipschitz constant K, if the inequality

$$|f(t, x_1) - f(t, x_2)| \le K|x_1 - x_2|, \tag{1.1}$$

holds whenever $(t, x_1), (t, x_2)$ are in D. In such a case, we say that f is a member of the class $\mathrm{Lip}(D, K)$.

As a consequence of Definition 1.1.1, a function f satisfies Lipschitz condition if and only if there exists a constant $K > 0$ such that

$$\frac{|f(t, x_1) - f(t, x_2)|}{|x_1 - x_2|} \le K, \quad x_1 \ne x_2,$$

whenever $(t, x_1), (t, x_2)$ belong to D. Now we wish to find a general criteria which would ensure the Lipschitz condition on f. The following is a result in this direction. For simplicity, we assume the region D to be a closed rectangle.

Theorem 1.1.2. *Define a rectangle R by*

$$R = \{(t, x) : |t - t_0| \le p,\ |x - x_0| \le q\},$$

where p, q are some positive real numbers. Let $f : R \to \mathbb{R}$ be a real valued continuous function. Let $\frac{\partial f}{\partial x}$ be defined and continuous on R. Then, f satisfies the Lipschitz condition on R.

Proof. Since $\frac{\partial f}{\partial x}$ is continuous on R, we have a positive constant A such that

$$\left|\frac{\partial f}{\partial x}(t, x)\right| \leq A, \tag{1.2}$$

for all $(t, x) \in R$. Let $(t, x_1), (t, x_2)$ be any two points in R. By the mean value theorem of differential calculus, there exists a number s which lies between x_1 and x_2 such that

$$f(t, x_1) - f(t, x_2) = \frac{\partial f}{\partial x}(t, s)(x_1 - x_2).$$

Since the point $(t, x) \in R$ and by the inequality (1.2), we have

$$\left|\frac{\partial f}{\partial x}(t, s))\right| \leq A,$$

or else, we have

$$|f(t, x_1) - f(t, x_2)| \leq A|x_1 - x_2|,$$

whenever $(t, x_1), (t, x_2)$ are in R. The proof is complete. □

The following example illustrates that the existence of partial derivative of f is not necessary for f to be a Lipschitz function.

Example 1.1.3. Let $R = \{(t, x) : |t| \leq 1,\ |x| \leq 1\}$ and let $f(t, x) = |x|$ for $(t, x) \in R$. Then, the partial derivative of f at $(t, 0)$ fails to exist but f satisfies Lipschitz condition in x on R with Lipschitz constant $K = 1$.

The example below shows that there exists functions which do not satisfy the Lipschitz condition.

Example 1.1.4. Let $f(t, x) = x^{1/2}$ be defined on the rectangle $R = \{(t, x) : |t| \leq 2,\ |x| \leq 2\}$. Then, f does not satisfy the inequality (1.1) in R. This is because

$$\frac{f(t, x) - f(t, 0)}{x - 0} = x^{-1/2}, \quad x \neq 0,$$

is unbounded in R.

If we alter the domain in Example 1.1.4, f may satisfy the Lipschitz condition, *e.g.*, if $R = \{(t, x) : |t| \leq 2,\ 2 \leq |x| \leq 4\}$.

Gronwall Inequality

The integral inequality, due to Gronwall, plays a useful part in the study of several properties of ordinary differential equations. In particular, we propose to employ it to establish the uniqueness of solutions.

Theorem 1.1.5. *(Gronwall inequality) Assume that $f, g : [t_0, \infty] \to \mathbb{R}_+$ are non-negative continuous functions. Let $k > 0$ be a constant. Then, the inequality*

$$f(t) \le k + \int_{t_0}^{t} g(s)f(s)ds, \quad t \ge t_0,$$

implies the inequality

$$f(t) \le k \ \exp\Big(\int_{t_0}^{t} g(s)ds\Big), \quad t \ge t_0.$$

Proof. By hypotheses, we have

$$\frac{f(t)g(t)}{k + \int_{t_0}^{t} g(s)f(s)ds} \le g(t), \quad t \ge t_0. \tag{1.3}$$

Since,

$$f(t)g(t) = \frac{d}{dt}\Big(k + \int_{t_0}^{t} g(s)f(s)ds\Big),$$

by integrating (1.3) between the limits t_0 and t, we have

$$\ln\Big(k + \int_{t_0}^{t} g(s)f(s)ds\Big) - \ln k \le \int_{t_0}^{t} g(s)ds.$$

In other words,

$$k + \int_{t_0}^{t} g(s)f(s)ds \le k \exp\Big(\int_{t_0}^{t} g(s)ds\Big). \tag{1.4}$$

The inequality (1.4) together with the hypotheses leads to the desired conclusion. □

An interesting and useful consequence is :

Corollary 1.1.6. *Let f and k be as in Theorem 1.1.5 If the inequality*

$$f(t) \le k \int_{t_0}^{t} f(s)ds, \quad t \ge t_0,$$

holds then,

$$f(t) \equiv 0, \ \textit{for} \ \ t \ge t_0.$$

Proof. For any $\epsilon > 0$, we have

$$f(t) < \epsilon + k \int_{t_0}^{t} f(s)ds, \quad t \ge t_0.$$

By Theorem 1.1.5, we have

$$f(t) < \epsilon \exp k(t - t_0), \quad t \ge t_0.$$

Since ϵ is arbitrary, we have $f(t) \equiv 0$ for $t \ge t_0$. □

EXERCISES

1. Prove that $f(t, x) = x^{1/2}$ as defined in Example 1.1.4 does not admit partial derivative with respect to x at $(0, 0)$.

2. Show that $f(t, x) = \frac{e^{-x}}{1+t^2}$ defined for $0 < x < p,\ 0 < t < N$ (where N is a positive integer) satisfies Lipschitz condition with Lipschitz constant $K = p$.

3. Show that the following functions satisfy the Lipschitz condition in the rectangle indicated and find the Lipschitz constant.
   (i) $f(t, x) = e^t \sin x, \qquad |x| \le 2\pi\ , \quad |t| \le 1\ ;$
   (ii) $f(t, x) = (x + x^2)\,\frac{\cos t}{t^2}, \quad |x| \le 1\ , \quad |t-1| \le \frac{1}{2}\ ;$
   (iii) $f(t, x) = \sin(xt), \qquad |x| \le 1\ , \quad |t| \le 1.$

4. Show that the following functions do not satisfy the Lipschitz condition in the region indicated.
   (i) $f(t, x) = \exp(\frac{1}{t^2})\, x,\ \ f(0, x) = 0, \quad |x| \le 1, \quad |t| \le 1.$
   (ii) $f(t, x) = \frac{\sin x}{t}, \qquad f(0, x) = 0, \qquad |x| < \infty, \quad |t| \le 1.$
   (iii) $f(t, x) = \frac{e^t}{x^2}, \qquad f(t, 0) = 0, \qquad |x| \le \frac{1}{2}, \quad |t| \le 2.$

5. Show that the IVP
$$x' + d(t)x = h(t),\ x(t_0) = x_0;\ \ t, t_0 \in I,$$
has a unique solution. Assume the continuity of d and h on I.

6. Let $I \subset \mathbb{R}$ and let $f, g, h : I \to \mathbb{R}_+$ be non-negative continuous functions. Then, prove that the inequality
$$f(t) \le h(t) + \int_{t_0}^{t} g(s)f(s)ds, \quad t \ge t_0,\, t \in I,$$
implies the inequality
$$f(t) \le h(t) + \int_{t_0}^{t} g(s)h(s)\exp\Big(\int_{t_0}^{s} g(u)du\Big)ds, \quad t \ge t_0.$$
$\big\{$Hint: Let $z(t) = \int_{t_0}^{t} g(s)f(s)ds$. Then,
$$z'(t) = g(t)f(t) \le g(t)[h(t) + z(t)].$$
Hence,
$$z'(t) - g(t)z(t) \le g(t)h(t).$$
Multiply by $\exp(-\int_{t_0}^{t} g(s)ds)$ on either side of this inequality and integrate over $[t_0, t]\big\}$.

Lecture 2

1.2 Picard's Successive Approximations

We begin with an initial value problem
$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.5}$$

where $D \subset \mathbb{R}^2$ is an open connected set and $f : D \to \mathbb{R}$ is continuous in (t, x) on D. Also let (t_0, x_0) be in D. Geometrically speaking, solving (1.5) is to find a function x whose graph passes through (t_0, x_0) and the slope of x coincides with $f(t, x)$ whenever (t, x) belongs to some neighborhood of (t_0, x_0). Such a class of problems is called a local existence problem for an initial value problem. Unfortunately, the usual elementary procedures for determining solutions may not materialize for (1.5). The need perhaps is a sequential approach to construct a solution x of (1.5). This is where the method of successive approximations finds its utility. The iterative procedure for solving (1.5) is important and needs a bit of knowledge of real analysis. The key to the general theory is an equivalent representation of (1.5) by the 'integral equation'

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{1.6}$$

Equation (1.6) is called an integral equation since the unknown function x occurs under the integral sign. The ensuing result establishes the equivalence of (1.5) and (1.6).

Lemma 1.2.1. *Let $I \subset \mathbb{R}$ be an interval. A function $x : I \to \mathbb{R}$ is a solution of* (1.5) *on I if and only if x is a solution of* (1.6) *on I.*

Proof. If x is a solution of (1.5) then, it is easy to show that x satisfies (1.6). Let x be a solution of (1.6). Obviously $x(t_0) = x_0$. Differentiating both sides of (1.6), and noting that f is continuous in (t, x), we have

$$x'(t) = f(t, x(t)), \ t \in I,$$

which completes the proof. □

We do recall that f is a continuous function on D. Now we are set to define an approximations to a solution of (1.5). First of all we start with an approximation to a solution and improve it by iteration. It is expected that these iterations converge to a solution of (1.5) in the limit. The importance of equation (1.6) now springs up. In this connection, we exploit the fact that the estimates can be easily handled with integrals rather than with derivatives.

A rough approximation to a solution of (1.5) is just the constant function $x_0(t) \equiv x_0$. We may get a better approximation by substituting $x_0(t)$ on the right hand side of (1.6), thus obtaining a new approximation $x_1(t)$ given by

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))ds,$$

as long as $(s, x_0(s)) \in D$. To get a still better approximation, we repeat the process thereby defining

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))ds,$$

as long as $(s, x_1(s)) \in D$. In general, we define x_n inductively by

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))ds, \quad n = 1, 2, \ldots, \tag{1.7}$$

as long as $(s, x_{n-1}(s)) \in D$, x_n is known as the n-th successive approximation. In the literature this procedure is known as "Picard's method of successive approximations". In the next section we show that the sequence $\{x_n\}$ does converge to a unique solution of (1.5) provided f satisfies the Lipschitz condition. We conclude this section with a few examples.

Example 1.2.2. For the illustration of the method of successive approximations consider

$$x' = -x,\ x(0) = 1,\ t \geq 0.$$

It is equivalent to the integral equation

$$x(t) = 1 - \int_0^t x(s)ds.$$

The zero-th approximation is given by $x_0(t) \equiv 1$. The first approximation is

$$x_1(t) = 1 - \int_0^t x_0(s)ds = 1 - t.$$

By the definition of the successive approximations, it follows that

$$x_2(t) = 1 - \Big[\int_0^t (1-s)ds\Big] = 1 - \Big[t - \frac{t^2}{2}\Big].$$

In general, the n-th approximation is

$$x_n(t) = 1 - t + \frac{t^2}{2} + \cdots + (-1)^n \frac{t^n}{n!}.$$

Obviously, $x_n(t)$ is the n-th partial sum of the power series for e^{-t}. It is easy to directly verify that e^{-t} is the solution of the IVP.

Example 1.2.3. Consider the IVP

$$x' = \frac{2x}{t},\ t > 0,\ x'(0) = 0,\ x(0) = 0.$$

The zero-th approximation x_0 is identically zero because $x(0) = 0$. The next approximation is $x_1 \equiv 0$. Similarly it can be shown that $x_n \equiv 0$ for all n. Thus, the sequence of functions $\{x_n\}$ converges to the identically zero function. Clearly $x \equiv 0$ is a solution of the IVP. On the other hand, it is not hard to check that $x(t) = t^2$ is also a solution of the IVP which shows that if at all the successive approximations converges, they converge to one of the solutions of the IVP.

EXERCISES

1. Calculate the successive approximations for the IVP

   $$x' = g(t),\ x(0) = 0.$$

   What is the conclusion that can be drawn from the successive approximations ?

2. Solve the IVP

   $$x' = x,\ x(0) = 1,$$

   by computing the method of successive approximations.

3. Compute the first three for the solutions of the following equations

   (i) $x' = x^2, \quad x(0) = 1;$

   (ii) $x' = e^x, \quad x(0) = 0;$

   (iii) $x' = \frac{x}{1+x^2},\ x(0) = 1.$

Lecture 3

1.3 Picard's Theorem

With all the remarks and examples, the reader may have a number of doubts about the effectiveness and utility of Picard's method in practice. It may be speculated whether the successive integrations are defined at all or whether they lead to complicated computations. However, we mention that Picard's method has made a landmark in the theory of differential equations. It gives not only a method to determine an approximate solution subject to a given error but also establishes the existence of a unique solution of initial value problems under general conditions.

In all of what follows we assume that the function $f : R \to \mathbb{R}$ is bounded by L and satisfies the Lipschitz condition with the Lipschitz constant K on the closed rectangle

$$R = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \leq a,\ |x - x_0| \leq b,\ a > 0,\ b > 0\}.$$

Before proceeding further, we need to show that the successive approximations defined by (1.7) are well defined on an interval I. That is, to define x_{j+1} on I, it is necessary to show that $(s, x_j(s))$ lies in R, for each s in I and $j \geq 1$.

Lemma 1.3.1. *Let* $h = \min\left(a, \dfrac{b}{L}\right)$*. Then, the successive approximations given by* (1.7) *are defined on* $I = |t - t_0| \leq h$*. Further,*

$$|x_j(t) - x_0| \leq L\,|t - t_0| \leq b, \quad j = 1, 2, \ldots, t \in I. \tag{1.8}$$

Proof. The method of induction is used to prove the lemma. Since $(t_0, x_0) \in R$, obviously $x_0(t) \equiv x_0$ satisfies (1.8). By the induction hypothesis, let us assume that, for any $0 < j \leq n$, x_n is defined on I and satisfies (1.8). Consequently $(s, x_n(s)) \in R$, for all s in I. So, x_{n+1} is defined on I. By definition, we have

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds, \quad t \in I.$$

Using the induction hypothesis, it now follows that

$$|x_{n+1}(t) - x_0| = \Big|\int_{t_0}^{t} f(s, x_n(s))ds\Big| \leq \int_{t_0}^{t} |f(s, x_n(s))|ds \leq\ L\,|t - t_0| \leq\ Lh \leq b.$$

Thus, x_{n+1} satisfies (1.8). This completes the proof. $\square$

We now state and prove the Picard's theorem, a fundamental result dealing with the problem of existence of a unique solution for a class of nonlinear initial value problems. Recall that the closed rectangle is defined in Lemma 1.3.1.

Theorem 1.3.2. *(Picard's Theorem) Let* $f : R \to \mathbb{R}$ *be continuous and be bounded by* L *and satisfy Lipschitz condition with Lipschitz constant* K *on the closed rectangle* R*. Then, the successive approximations* $n = 1, 2, \ldots,$ *given by* (1.7) *converge uniformly on an interval*

$$I : |t - t_0| \leq h,\ h = \min\left(a, \frac{b}{L}\right),$$

to a solution x *of the IVP* (1.5)*. In addition, this solution is unique.*

Proof. We know that the IVP (1.5) is equivalent to the integral equation (1.6) and it is sufficient to show that the successive approximations x_n converge to a unique solution of (1.6) and hence, to the unique solution of the IVP (1.5). First, note that

$$x_n(t) = x_0(t) + \sum_{i=1}^{n} \big[x_i(t) - x_{i-1}(t)\big]$$

is the n-th partial sum of the series

$$x_0(t) + \sum_{i=1}^{\infty} \big[x_i(t) - x_{i-1}(t)\big] \tag{1.9}$$

The convergence of the sequence $\{x_n\}$ is equivalent to the convergence of the series (1.9). We complete the proof by showing that:

(a) the series (1.9) converges uniformly to a continuous function $x(t)$;

(b) x satisfies the integral equation (1.6);

(c) x is the unique solution of (1.5).

To start with we fix a positive number $h = \min(a, \frac{b}{L})$. By Lemma 1.2.1 the successive approximations $x_n, n = 1, 2, \ldots,$ in (1.7) are well defined on $I : |t - t_0| \leq h$. Henceforth, we stick to the interval $I^+ = [t_0, t_0 + h]$. The proof on the interval $I^- = [t_0 - h, t_0]$ is similar except for minor modifications.

We estimate $x_{j+1}(t) - x_j(t)$ on the interval $[t_0, t_0 + h]$. Let us denote

$$m_j(t) = |x_{j+1}(t) - x_j(t)|;\ j = 0, 1, 2, \ldots.$$

Since f satisfies Lipschitz condition and by definition, we have

$$\begin{aligned} m_j(t) &= \Big|\int_{t_0}^{t} \big[f(s, x_j(s)) - f(s, x_{j-1}(s))\big]ds\Big| \\ &\leq K \int_{t_0}^{t} \big|x_j(s) - x_{j-1}(s)\big|ds, \end{aligned}$$

or, in other words,

$$m_j(t) \leq K \int_{t_0}^{t} m_{j-1}(s)ds. \tag{1.10}$$

By direct computation,

$$\begin{aligned} m_0(t) &= |x_1(t) - x_0(t)| = \Big|\int_{t_0}^{t} f(s, x_0(s))ds\Big| \\ &\leq \int_{t_0}^{t} |f(s, x_0(s))|ds \\ &\leq L(t - t_0). \end{aligned} \tag{1.11}$$

We claim that

$$m_j(t) \leq LK^j \frac{(t - t_0)^{j+1}}{(j + 1)!}, \tag{1.12}$$

for $j = 0, 1, 2, \ldots,$ and $t_0 \le t \le t_0 + h$. The proof of the claim is by induction. For $j = 0$, (1.12) is, in fact, (1.11). Assume that for an integer $1 \le p \le j$ the assertion (1.12) holds. That is,

$$\begin{aligned} m_{p+1}(t) \le K \int_{t_0}^{t} m_p(s) ds \le K \int_{t_0}^{t} LK^p \, \frac{(s-t_0)^{p+1}}{(p+1)!} \, ds \\ \le L \; K^{p+1} \, \frac{(t-t_0)^{p+2}}{(p+2)!}, \quad t_0 \le t \le t_0 + h, \end{aligned}$$

which shows that (1.12) holds for $j = p+1$ or equivalently, (1.12) holds for all $j \ge 0$. So, the series $\sum_{j=0}^{\infty} m_j(t)$ is dominated by the series

$$\frac{L}{K} \sum_{j=0}^{\infty} \frac{K^{j+1} h^{j+1}}{(j+1)!},$$

which converges to $\frac{L}{K}(e^{Kh} - 1)$ or else, the series (1.9) converges uniformly and absolutely on the $I^+ = [t_0, t_0 + h]$. Let

$$x(t) = x_0(t) + \sum_{n=1}^{\infty} \big[x_n(t) - x_{n-1}(t)\big]; \quad t_0 \le t \le t_0 + h. \tag{1.13}$$

Since the convergence is uniform, the limit function x is continuous on $I^+ = [t_0, t_0 + h]$. Also, the points $(t, x(t)) \in R$ for all $t \in I$ and thereby completing the proof of (a).

We now show that x satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds, \;\; t \in I. \tag{1.14}$$

By the definition of successive approximations

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s)) ds, \tag{1.15}$$

from which, we have

$$\begin{aligned} \big|x(t) - x_0 - \int_{t_0}^{t} f(s, x(s)) ds\big| &= \big|x(t) - x_n(t) + \int_{t_0}^{t} f(s, x_{n-1}(s)) ds - \int_{t_0}^{t} f(s, x(s)) ds\big| \\ &\le |x(t) - x_n(t)| + \int_{t_0}^{t} \big|f(s, x_{n-1}(s)) - f(s, x(s))\big| ds. \end{aligned} \tag{1.16}$$

Since $x_n \to x$ uniformly on I, and $|x_n(t)| \le b$ for all n and for $t \in I^+$, it follows that $|x(t)| \le b$ for all $t \in I^+$. Now the Lipschitz condition on f implies

$$\begin{aligned} |x(t) - x(0) - \int_{t_0}^{t} f(s, x(s)) ds| &\le |x(t) - x_n(t)| + K \int_{t_0}^{t} |x(s) - x_{n-1}(s)| ds \\ &\le |x(t) - x_n(t)| + Kh \max_{t_0 \le s \le t_0 + h} |x(s) - x_{n-1}(s)|. \end{aligned} \tag{1.17}$$

The uniform convergence of x_n to x on I^+ now implies that the right hand side of (1.17) tends to zero as $n \to \infty$. But the left side of (1.17) is independent of n. Thus, x satisfies the integral equation (1.6) on I^+ which proves (b).

Uniqueness : Let us now prove that, if $\bar{x}(t)$ and $x(t)$ are any two solutions of the IVP (1.5), then they coincide on $[t_0, t_0 + h]$. Let $\bar{x}(t)$ and $x(t)$ satisfy (1.6) which yields

$$|\bar{x}(t) - x(t)| \leq \int_{t_0}^{t} |f(s, \bar{x}(s)) - f(s, x(s))| ds. \tag{1.18}$$

Both $\bar{x}(s))$ and $x(s)$ lie in R for all s in $[t_0, t_0 + h]$ and hence, it follows that

$$|\bar{x}(t) - x(t)| \leq K \int_{t_0}^{t} |\bar{x}(s)) - x(s)| ds.$$

By an application of the Gronwall inequality, we arrive at

$$|\bar{x}(t) - x(t)| \equiv 0 \quad \text{on} \quad [t_0, t_0 + h],$$

which means $\bar{x}(t) \equiv x(t)$. This proves (c), completing the proof of the theorem. □

Another important feature of Picard's theorem is that a bound for the error in the case of truncated computation at the n-th iteration can also be obtained. Indeed, we have a result dealing with such a bound on the error.

Corollary 1.3.3. *The error $x(t) - x_n(t)$ satisfies the estimate*

$$|x(t) - x_n(t)| \leq \frac{L}{K} \frac{(Kh)^{n+1}}{(n+1)!} e^{Kh}; \quad t \in [t_0, t_0 + h]. \tag{1.19}$$

Proof. We know

$$x(t) = x_0(t) + \sum_{j=0}^{\infty} \big[x_{j+1}(t) - x_j(t)\big]$$

and

$$x(t) - x_n(t) = \sum_{j=n}^{\infty} \big[x_{j+1}(t) - x_j(t)\big].$$

Consequently, by (1.12) we have

$$\begin{aligned} |x(t) - x_n(t)| \leq \sum_{j=n}^{\infty} \big|x_{j+1}(t) - x_j(t)\big| \leq \sum_{j=n}^{\infty} m_j(t) &\leq \sum_{j=n}^{\infty} \frac{L}{K} \frac{(Kh)^{j+1}}{(j+1)!} \\ &= \frac{L}{K} \frac{(Kh)^{n+1}}{(n+1)!} \Big[1 + \sum_{j=1}^{\infty} \frac{(Kh)^j}{(n+2)...(n+j+1)}\Big] \\ &\leq \frac{L}{K} \frac{(Kh)^{n+1}}{(n+1)!} e^{Kh}. \end{aligned}$$

. □

Example 1.3.4. Consider the IVP in Example 1.2.2. Note that all the conditions of the Picard's theorem are satisfied. To find a bound on the error $x(t) - x_n(t)$, we determine K and L. It is quite clear that $K = 1$. Let R be the closed rectangle around $(0, 1)$ *i.e.,*

$$R = \{(t, x) : |t| \le 1, |x - 1| \le 1\}.$$

Then, $L = 1$. Suppose the error is not to exceed ϵ. The question is to find a number n such that $|x - x_n| \le \epsilon$. To achieve this, a sufficient condition is

$$\frac{L}{K}\frac{(Kh)^{n+1}}{(n+1)!}e^{Kh} < \epsilon.$$

We have to find an n such that $\frac{1}{(n+1)!} < \epsilon e^{-1}$ or, in other words, $(n+1)! > \epsilon^{-1}e$ which holds since $\epsilon^{-1}e$ is finite and $(n+1)! \to \infty$. For instance, when $\epsilon = 1$, we may choose any $n \ge 2$, so that the error is less than 1.

A doubt may arise whether the Lipschitz condition can be dropped from the hypotheses in Picard's theorem. The answer is the negative and the following example makes it clear.

Example 1.3.5. Consider the IVP

$$x' = 4x^{3/4},\ x(0) = 0.$$

Obviously $x_0(t) \equiv 0$. But this fact implies that $x_1(t) \equiv 0$, a result which follows by the definition of successive approximations. In fact, in this case $x_n(t) \equiv 0$ for all $n \ge 0$. So, $x(t) \equiv 0$ is a solution to the IVP. But $x(t) = t^4$ is yet another solution of the IVP which contradicts the conclusion of Picard's theorem which shows that the Picard's theorem may not hold in case the Lipschitz condition on $f(t, x)$ is altogether dropped. Also $f(t, x) = 4x^{3/4}$ does not satisfy the Lipschitz condition in any closed rectangle R containing the point $(0, 0)$.

EXERCISES

1. Show that the error due to the truncation at the n-th approximation tends to zero as $n \to \infty$.

2. Consider an IVP $x' = f(x)$, $x(0) = 0$, where $f(x)$ satisfies all the conditions of Picard's theorem. Guess the unique local solution if it is given that $f(0) = 0$. Does the conclusion so reached still holds in case $f(x)$ is replaced by $g(t, x)$ and $g(t, .) \equiv 0$ along with the Lipschitz property of $g(t, x)$ in x?

3. Determine the constant L, K and h for the IVP.

   (i) $x' = x^2,\ x(0) = 1, \quad R = \{(t, x) : |t| \le 2, |x - 1| \le 2\}$,

   (ii) $x' = \sin x,\ x(\frac{\pi}{2}) = 1, \quad R = \{(t, x) : |t - \frac{\pi}{2}| \le \frac{\pi}{2}, |x - 1| \le 1\}$,

   (iii) $x' = e^x,\ x(0) = 0, \quad R = \{(t, x) : |t| \le 3, |x| \le 4\}$.

Is Picard's theorem applicable in the above three problems? If so find the least n such that the error left over does not exceed $2, 1$ and 0.5 respectively for the three problems.

Lecture 4

1.4 Continuation And Dependence On Initial Conditions

As usual we assume that the function $f(t,x)$ in (1.5) is defined and continuous on an open connected set D and let $(t_0, x_0) \in D$. By Picard's theorem, we have an interval

$$I : t_0 - h \leq t \leq t_0 + h,$$

where $h > 0$ such that the closed rectangle $R \subset D$. Since the point $(t_0+h, x(t_0+h))$ lies in D there is a rectangle around $(t_0+h, x(t_0+h))$ and lying entirely in D. By applying Theorem 1.3.2, we have the existence of a unique solution $\hat{x}$ passing through the point $(t_0+h, x(t_0+h))$ and whose graph lies in D (for $t \in [t_0+h, t_0+h+\hat{h}], \hat{h} > 0$). If the solution $\hat{x}$ coincides with x on I, then $\hat{x}$ satisfies the IVP (1.5) on the interval $[t_0+h, t_0+h+\hat{h}] \supset I$. In that case the process may be repeated till the graph of the extended solution reaches the boundary of D. Naturally such a procedure is known as the continuation of solutions of the IVP (1.5). The continuation method just described can also be extended to the left of t_0.

Now we formalize the above discussion. Let us suppose that a unique solution x of (1.5) exists, on the interval I^* say $h_1 < t < h_2$ with $(t, x(t)) \in D$ for $t \in I^*$ and let

$$|f(t,x)| \leq L \text{ on } D, \ (t, x(t)) \in D \text{ and } h_1 < t_0 < h_2.$$

Consider the sequence

$$\big\{x\big(h_2 - \frac{1}{n}\big)\big\}, n = 1, 2, 3, \ldots.$$

By (1.6), for sufficiently large n, we have

$$\begin{aligned} |x(h_2 - \frac{1}{m}) - x(h_2 - \frac{1}{n})| &\leq \int_{h_2-(1/n)}^{h_2-(1/m)} |f(s, x(s))|ds, \quad (m > n) \\ &\leq L\Big|\frac{1}{m} - \frac{1}{n}\Big|. \end{aligned}$$

So, the sequence $\big\{x(h_2 - \frac{1}{n})\big\}$ is Cauchy and

$$\lim_{n\to\infty} x\big(h_2 - \frac{1}{n}\big) = \lim_{t\to h_2-0} x(t) = x(h_2 - 0),$$

exists. Suppose $\big(h_2, x(h_2-0)\big)$ is in D. Define $\hat{x}$ as follows

$$\begin{aligned} \hat{x}(t) &= x(t), \quad h_1 < t < h_2 \\ \hat{x}(h_2) &= x(h_2 - 0). \end{aligned}$$

By noting

$$\hat{x}(t) = x_0 + \int_{t_0}^{t} f(s, \hat{x}(s))ds, \quad h_1 < t \leq h_2,$$

it is easy to show that $\hat{x}$ is a solution of (1.5) existing on $h_1 < t \leq h_2$.

Exercise : Prove that $\hat{x}$ is a solution of (1.5) existing on $h_1 < t \leq h_2$.

Now consider a rectangle around $P : (h_2, x(h_2 - 0))$ lying inside D. Consider a solution of (1.5) through P. As before, by Picard's theorem there exists a solution y through the point P on an interval

$$h_2 - \alpha \le t \le h_2 + \alpha,\ \alpha > 0 \ \text{ and with } \ h_2 - \alpha \ge h_1.$$

Now define z by

$$\begin{aligned} z(t) &= \hat{x}(t), \quad h_1 < t \le h_2 \\ z(t) &= y(t), \quad h_2 \le t \le h_2 + \alpha. \end{aligned}$$

Claim: z is a solution of (1.5) on $h_1 < t \le h_2 + \alpha$. Since y is a unique solution of (1.5) on $h_2 - \alpha \le t \le h_2 + \alpha$, we have

$$\hat{x}(t) = y(t), \quad h_2 - \alpha \le t \le h_2.$$

We note that z is a solution of (1.5) on $h_2 \le t \le h_2 + \alpha$ and so it only remains to verify that z' is continuous at the point $t = h_2$. Clearly,

$$z(t) = \hat{x}(h_2) + \int_{h_2}^{t} f(s, z(s))ds, \quad h_2 \le t \le h_2 + \alpha. \tag{1.20}$$

Further,

$$\hat{x}(h_2) = x_0 + \int_{t_0}^{h_2} f(s, z(s))ds. \tag{1.21}$$

Thus, the relation (1.20) and (1.21) together yield

$$\begin{aligned} z(t) &= x_0 + \int_{t_0}^{h_2} f(s, z(s))ds + \int_{h_2}^{t} f(s, z(s))ds \\ &= x_0 + \int_{t_0}^{t} f(s, z(s))ds, \qquad h_1 \le t \le h_2 + \alpha. \end{aligned}$$

Obviously, the derivatives at the end points h_1 and $h_2 + \alpha$ are one-sided.
We summarize :

Theorem 1.4.1. *Let*

(i) $D \subset \mathbb{R}^{n+1}$ *be an open connected set and let* $f : D \to \mathbb{R}$ *be continuous and satisfy the Lipschitz condition in* x *on* D*;*

(ii) f *be bounded on* D *and*

(iii) x *be a unique solution of the IVP* (1.5) *existing on* $h_1 < t < h_2$.

Then,

$$\lim_{t \to h_2 - 0} x(t)$$

exists. If $(h_2, x(h_2 - 0)) \in D$*, then* x *can be continued to the right of* h_2.

We now study the continuous dependence of solutions on initial conditions. Consider

$$x' = f(t,x), x(t_0) = x_0. \tag{1.22}$$

Let $x(t; t_0, x_0)$ be a solution of (1.22). Then, $x(t; t_0, x_0)$ is a function of time t, the initial time t_0 and the initial state x_0. The dependence on initial conditions is to know about the behavior of $x(t; t_0, x_0)$ as a function of t_0 and x_0. Under certain conditions, indeed $x(t; t_0, x_0)$ is a continuous function of t_0 and x_0. This amounts to saying that the solution $x(t; t_0, x_0)$ of (1.22) stays in a neighborhood of solutions $x^*(t; t_0^*, x_0^*)$ of

$$x' = f(t,x),\ x(t_0^*) = x_0^*. \tag{1.23}$$

provided $|t_0 - t_0^*|$ and $|x_0 - x_0^*|$ are sufficiently small.

Theorem 1.4.2. *Let $I = [a,b]$ and let $x(t) = x(t; t_0, x_0)$ and $x^*(t) = x(t; t_0^*, x_0^*)$ be solutions of the IVPs* (1.22) *and* (1.23) *respectively on I. Suppose that $(t, x(t)), (t, x^*(t)) \in D$ for $t \in I$. Further, let $f \in Lip(D,K)$ be bounded by L in D. Then, for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that*

$$|x(t) - x^*(t)| < \epsilon,\ t \in I, \tag{1.24}$$

whenever $|t_0 - t_0^*| < \delta$ and $|x_0 - x_0^*| < \delta$.

Proof. It is first of all clear that the solutions $x(t)$ and $x^*(t)$ with $x(t_0) = x_0$ and $x^*(t_0^*) = x_0^*$ exists uniquely. Without loss of generality let $t_0^* \geq t_0$. From Lemma 1.2.1, we have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \tag{1.25}$$

$$x^*(t) = x_0^* + \int_{t_0^*}^{t} f(s, x^*(s))ds. \tag{1.26}$$

From (1.25) and (1.26) we obtain

$$x(t) - x^*(t) = x_0 - x_0^* + \int_{t_0^*}^{t} \big[f(s,x(s)) - f(s,x^*(s))\big]ds + \int_{t_0}^{t_0^*} f(s,x(s))ds. \tag{1.27}$$

With absolute values on both sides of (1.27) and by the hypotheses, we have

$$\begin{aligned}|x(t) - x^*(t)| &\leq |x_0 - x_0^*| + \int_{t_0^*}^{t} |f(s,x(s)) - f(s,x^*(s))|ds + \int_{t_0}^{t_0^*} |f(s,x(s))|ds \\ &\leq |x_0 - x_0^*| + \int_{t_0^*}^{t} K|x(s)) - x^*(s)|ds + L|t_0 - t_0^*|.\end{aligned}$$

Now by the Gronwall inequality, it follows that

$$|x(t) - x^*(t)| \leq \big[|x_0 - x_0^*| + L|t_0 - t_0^*|\big] \exp[K(b-a)] \tag{1.28}$$

for all $t \in I$. Given any $\epsilon > 0$, choose

$$\delta(\epsilon) = \frac{\epsilon}{2\exp[K(b-a)]} \min\Big[1, \frac{1}{L}\Big].$$

From (1.28), we obtain

$$|x(t) - x^*(t)| \leq \Big[\frac{\epsilon}{2\exp\{K(b-a)\}} + \frac{L\epsilon}{2L\exp\{K(b-a)\}}\Big] \exp K[(b-a)] = \epsilon.$$

if $|t_0 - t_0^*| < \delta(\epsilon)$ and $|x_0 - x_0^*| < \delta(\epsilon)$, which completes the proof. □

Remark on Theorems 1.4.1 and 1.4.2 :
These theorems clearly exhibit the crucial role played by the Gronwall inequality. Indeed the Gronwall inequality has many more applications in the qualitative theory of differential equations which we shall see later.

EXERCISES

1. Consider a linear equation $x' = a(t)x$ with initial conditions $x(t_0) = x_0$, where $a(t)$ is a continuous function on an interval I containing t_0. Solve the IVP and show that the solution $x(t; t_0, x_0)$ is a continuous function of (t_0, x_0) for each fixed $t \in I$.

2. Consider the IVPs

   (i) $x' = f(t, x),\ x(t_0) = x_0^*$,

   (ii) $y' = g(t, y),\ y(t_0) = y_0^*$,

   where $f(t, x)$ and $g(t, x)$ are continuous functions in (t, x) defined on the rectangle
   $$R = \{(t, x) : |t - t_0| \le a,\ |x - x_0| \le b\},$$
   where (t_0, x_0^*) and (t_0, y_0^*) are in R. In addition, let
   $$f \in \mathrm{Lip}(R, K) \text{ and } |f(t, x) - g(t, x)| \le \epsilon \text{ for all } (t, x) \in R,$$
   for some positive number ϵ. Let $x(t; t_0, x_0^*)$ and $y(t; t_0, y_0^*)$ be two solutions of (i) and (ii) respectively on $I : |t - t_0| \le a$. If $|x_0^* - y_0^*| \le \delta$, then show that
   $$|x(t) - y(t)| \le \delta \exp\big(K|t - t_0|\big) + (\epsilon/K)\big(\exp(K|t - t_0|) - 1\big),\ t \in I.$$

3. Let the conditions (i) to (iii) of Theorem 1.4.1 hold. Show that $\lim x(t)$ as $t \to h_1 + 0$ exists. Further, if the point $(h_1, x(h_1 + 0))$ is in D, then show that x can be continued to the left of h_1.

Lecture 5

1.5 Existence of Solutions in the Large

We have seen earlier that the Theorem 1.3.2 is about the existence of solutions in a local sense. In this section, we consider the problem of existence of solutions not in the local sense. Existence of solutions in the large is also known as non-local existence. Before embarking on technical results let us have look at an example.
Example : By Picard's theorem the IVP
$$x' = x^2,\ x(0) = 1,\ \ -1 \le t, x \le 1,$$
has a solution existing on
$$-\frac{1}{2} \le t \le \frac{1}{2},$$
where as its solution is
$$x(t) = \frac{1}{1 - t},\ -\infty < t < 1.$$

Actually, by direct computation, we have an interval of existence larger than the one which we obtain by an application of Picard's theorem. In other words, we need to strengthen the Picard's theorem in order to recover the larger interval of existence.

Now we take up the problem of existence in the large. Under certain restrictions on f, we prove the existence of solutions of IVP

$$x' = f(t,x),\ x(t_0) = x_0, \tag{1.29}$$

on the whole (of a given finite) interval $|t - t_0| \leq T$, and secondly on $-\infty < t < \infty$. We say that x exists "non-locally" on I if x a solution of (1.29) exists on I. The importance of such problems needs little emphasis due to its necessity in the study of oscillations, stability and boundedness of solutions of IVPs. The non-local existence of solutions of IVP(1.29) is dealt in the ensuing result.

Theorem 1.5.1. *We define a strip S by*

$$S = \{(t,x) : |t - t_0| \leq T \text{ and } |x| < \infty\},$$

where T is some finite positive real number. Assume that $f : S \to \mathbb{R}$ is continuous and $f \in Lip(S, K)$. Then, the successive approximations defined by (1.7) *for the IVP*(1.29) *exist on $|t - t_0| \leq T$ and converge to a solution x of* (1.29).

Proof. Recall that the definition of successive approximations (1.7) is

$$\left.\begin{aligned} &x_0(t) \equiv x_0, \\ &x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))ds,\ |t - t_0| \leq T. \end{aligned}\right\} \tag{1.30}$$

We prove the theorem for the interval $[t_0, t_0+T]$. The proof for the interval $[t_0-T, t_0]$ is similar with suitable modifications. First note that (1.30) defines the successive approximations on $t_0 \leq t \leq t_0 + T$. Also,

$$|x_1(t) - x_0(t)| = \Big|\int_{t_0}^{t} f(s, x_0(s))ds\Big|. \tag{1.31}$$

Since f is continuous, $f(t, x_0)$ is continuous on $[t_0, t_0 + T]$ which implies that there exists a real constant $L > 0$ such that

$$|f(t, x_0)| \leq L, \quad \text{for all } t \in [t_0, t_0 + T].$$

With this bound on $f(t, x_0)$ in (1.31), we get

$$|x_1(t) - x_0(t)| \leq L(t - t_0) \leq LT, \quad t \in [t_0, t_0 + T]. \tag{1.32}$$

The estimate (1.32) implies (by using induction)

$$|x_n(t) - x_{n-1}(t)| \leq \frac{LK^{n-1}T^n}{n!}, \quad t \in [t_0, t_0 + T]. \tag{1.33}$$

Now (1.33), as in the proof of Theorem 1.3.2, yields the uniform convergence of the series

$$x_0(t) + \sum_{n=0}^{\infty} \big[x_{n+1}(t) - x_n(t)\big],$$

and hence, the uniform convergence of the sequence $\{x_n\}$ on $[t_0, t_0+T]$ easily follows. Let x denote the limit function, namely,

$$x(t) = x_0(t) + \sum_{n=0}^{\infty} \big[x_{n+1}(t) - x_n(t)\big], \quad t \in [t_0, t_0+T]. \tag{1.34}$$

In fact, (1.33) shows that

$$\begin{aligned} |x_n(t) - x_0(t)| &= \Big| \sum_{p=1}^{n} \big[x_p(t) - x_{p-1}(t)\big] \Big| \\ &\leq \sum_{p=1}^{n} \big|x_p(t) - x_{p-1}(t)\big| \\ &\leq \frac{L}{K} \sum_{p=1}^{n} \frac{K^p T^p}{n!} \\ &\leq \frac{L}{K} \sum_{p=1}^{\infty} \frac{K^p T^p}{n!} = \frac{L}{K}(e^{KT} - 1). \end{aligned}$$

Since x_n converges to x on $t_0 \leq t \leq t_0 + T$, we have

$$|x(t) - x_0| \leq \frac{L}{K}(e^{KT} - 1).$$

Since the function f is continuous on the rectangle

$$R = \big\{(t,x) : |t - t_0| \leq T,\ |x - x_0| \leq \frac{L}{K}(e^{KT} - 1)\big\},$$

there exists a real number $L_1 > 0$ such that

$$|f(t,x)| \leq L_1,\ (t,x) \in R.$$

Moreover, the convergence of the sequence $\{x_n\}$ is uniform implies that the limit x is continuous. From the corollary (1.14), it follows that

$$|x(t) - x_n(t)| \leq \frac{L_1}{K} \frac{(KT)^{n+1}}{(n+1)!} e^{KT}.$$

Finally, we show that x is a solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t_0 \leq t \leq t_0 + T. \tag{1.35}$$

Also

$$\begin{aligned} |x(t) - x_0 - \int_{t_0}^{t} f(s,x(s))ds| &= \big|x(t) - x_n(t) + \int_{t_0}^{t} \big[f(s,x_n(s)) - f(s,x(s))\big]ds\big| \\ &\leq |x(t) - x_n(t)| + \int_{t_0}^{t} \big|f(s,x(t)) - f(s,x_n(s))ds\big| \end{aligned} \tag{1.36}$$

Since $x_n \to x$ uniformly on $[t_0, t_0+T]$, the right side of (1.36) tends to zero as $n \to \infty$. By letting $n \to \infty$, from (1.36) we indeed have

$$\big|x(t) - x_0 - \int_{t_0}^{t} f(s, x(s))ds\big| \leq 0, \quad t \in [t_0, t_0+T].$$

or else

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t \in [t_0, t_0+T].$$

The uniqueness of x follows similarly as shown in the proof of Theorem 1.3.2. □

Remark : The example cited at the beginning of this section does not contradict the Theorem 1.5.1 as $f(t,x) = x^2$ does not satisfy the strip condtion $f \in \mathrm{Lip}(S, K)$.

A consequence of the Theorem 1.5.1 is :

Theorem 1.5.2. *Assume that $f(t,x)$ is a continuous function on $|t| < \infty$, $|x| < \infty$. Further, let f satisfies Lipschitz condition on the the strip S_a for all $a > 0$, where*

$$S_a = \{(t,x) : |t| \leq a, |x| < \infty\}.$$

Then, the initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0, \tag{1.37}$$

has a unique solution existing for all t.

Proof. The proof is very much based on the fact that for any real number t there exists T such that $|t - t_0| \leq T$. Notice here that all the hypotheses of Theorem 1.5.1 are satisfied, for this choice of T, on the strip $|t - t_0| \leq T$, $|x| < \infty$. Thus, by Theorem 1.5.1, the successive approximations $\{x_n\}$ converge to a function x which is a unique solution of (1.37). □

EXERCISES

1. Supply a proof of the Theorem 1.5.1 on the interval $[t_0 - T, t_0]$.

2. Let a be a continuous function defined on $I : |t - t_0| \leq \alpha$. Prove the uniform convergence of the series for x defined by (1.34).

3. let $I \subset \mathbb{R}$ be an interval. By solving the linear equation

$$x' = a(t)x, \ x(t_0) = x_0,$$

   show that it has a unique solution x on the whole of I. Use the Theorem 1.5.1 to arrive the same conclusion.

4. By solving the IVP

$$x' = -x^2, \ x(0) = -1, \ 0 \leq t \leq T,$$

   show that the solution does not exist for $t \geq 1$. Does this example contradict Theorem 1.5.1, when $T \geq 1$?

Lecture 6

1.6 Existence and Uniqueness of Solutions of Systems

The methodology developed till now concerns existence and uniqueness of a single equation or a scalar equations which is a natural extension for the study of a system of equations or to higher order equations. In the sequel, we glance of these extensions. Let $I \subseteq \mathbb{R}$ be an interval, $E \subseteq \mathbb{R}^n$. Consider a system of nonlinear equations

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, ..., x_n), \\ x_2' &= f_2(t, x_1, x_2, ..., x_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' &= f_n(t, x_1, x_2, ..., x_n), \end{aligned} \tag{1.38}$$

where $f_1, f_2, ..., f_n : I \times E \to \mathbb{R}$ are given continuous functions. Denoting (column) vector x with components $x_1, x_2, ..., x_n$ and vector f with components $f_1, f_2, ..., f_n$, the system of equations (1.38) assumes the form

$$x' = f(t, x). \tag{1.39}$$

A general n-th order equation is representable in the form (1.38) which means the study of n-th order nonlinear equation is naturally embedded in the study of (1.39). It speaks of the importance of the study of systems of nonlinear equations, leaving apart numerous difficulties that one has to face. Consider an IVP

$$x' = f(t, x), \quad x(t_0) = x_0. \tag{1.40}$$

The proofs of local and non-local existence theorems for systems of equations stated below have a remarkable resemblance to those of scalar equations. The detailed proofs are to be supplied by readers with suitable modifications to handle the presence of vectors and their norms. Below the symbol $|.|$ is used to denote both the norms of a vector and the absolute value. There is no possibility of confusion since the context clarifies the situation.

In all of what follows we are concerned with the region D, a rectangle in $\mathbb{R}^{n+1}$ space, defined by

$$D = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\},$$

where $x, x_0 \in \mathbb{R}^n$ and $t, t_0 \in \mathbb{R}$.

Definition 1.6.1. A function $f : D \to \mathbb{R}^n$ is said to satisfy the Lipschitz condition in the variable x, with Lipschitz constant K on D if

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2| \tag{1.41}$$

uniformly in t for all $(t, x_1), (t, x_2)$ in D.

The continuity of $f(t, x)$ in x for each fixed t is a consequence, when $f(t, x)$ is Lipschitzian in x. If $f(t, x)$ is Lipschitzian on D then, there exists a non-negative, real-valued function $L(t)$ such that

$$|f(t, x)| \leq L(t), \text{ for all } (t, x) \in D.$$

In addition, there exists a constant $L > 0$ such that $L(t) \leq L$, when L is continuous on $|t - t_0| \leq a$.

Lemma 1.6.2. *Let $f : D \to \mathbb{R}^n$ be a continuous function. $x(t; t_0, x_0)$ (denoted by $x(t)$) is a solution of* (1.40) *on some interval I contained in $|t - t_0| \leq a(t_0 \in I)$ if and only if x is a solution of the integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \ t \in I. \tag{1.42}$$

Proof. First of all, we prove that the components x_i of x satisfy

$$x_i(t) = x_{0i} + \int_{t_0}^{t} f_i(s, x(s))ds, \quad t \in I, \ i = 1, 2, \ldots, n,$$

if and only if

$$x_i'(t) = f_i(t, x(t)), \ x_{0i} = x_i(t_0), \quad i = 1, 2, \ldots, n,$$

holds. The proof is exactly the same as that of Lemma 1.2.1. □

As expected, the integral equation (1.42) is now exploited to define (inductively) the successive approximations by

$$\begin{cases} x_0(t) = x_0 \\ x_n(t) = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s))ds, \qquad t \in I \end{cases} \tag{1.43}$$

for $n = 1, 2, \ldots,$. The ensuing lemma establishes that, under the stated conditions, the successive approximations are indeed well defined.

Lemma 1.6.3. *Let $f : D \to \mathbb{R}^n$ be a continuous function and be bounded by $L > 0$ on D. Define $h = \min\left(a, \dfrac{b}{L}\right)$. Then, the successive approximations are well defined by* (1.43) *on the interval $I = |t - t_0| \leq h$. Further,*

$$|x_j(t) - x_0| \leq L\,|t - t_0| < b, \quad j = 1, 2, ...$$

The proof is very similar to the proof of Lemma 1.3.1.

Theorem 1.6.4. *(Picard's theorem for system of equations). Let all the conditions of Lemma 1.6.3 hold and let f satisfy the Lipschitz condition with Lipschitz constant K on D. Then, the successive approximations defined by* (1.43) *converge uniformly on $I = |t - t_0| \leq h$ to a unique solution of the IVP* (1.40).

Corollary 1.6.5. *A bound error left due to the truncation at the n-th approximation for x is*

$$|x(t) - x_n(t)| \leq \frac{L}{K}\frac{(Kh)^{n+1}}{(n+1)!}e^{Kh}, \quad t \in [t_0, t_0 + h]. \tag{1.44}$$

Corollary 1.6.6. *Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ real matrices. Let $I \subset \mathbb{R}$ be an interval. Let $A : I \to \mathbb{R}$ be continuous on I. Then, the IVP*

$$x' = A(t)x,$$
$$x(a) = x_0, \ a \in I,$$

has a unique solution x existing on I. As a consequence the set of all solutions of

$$x' = Ax,$$

is a linear vector space of dimension n.

The proofs of Theorem 1.6.4 and Corollary 1.6.6 are exercises.

As noted earlier the Lipschitz property of f in Theorem 1.6.4 cannot be altogether dropped as shown by the following example.

Example 1.6.7. The nonlinear IVP

$$x_1' = 2x_2^{1/3}, \quad x_1(0) = 0,$$
$$x_2' = 3x_1, \qquad x_2(0) = 0,$$

in the vector form is

$$x' = f(t, x), \quad x(0) = \mathbf{0},$$

where $x = (x_1, x_2)$, $f(t, x) = (2x_2^{1/3}, 3x_1)$ and $\mathbf{0}$ is the zero vector. Obviously, $x(t) \equiv 0$ is a solution. It is easy to verify that $x(t) = (t^2, t^3)$ is yet another solution of the IVP which violates the uniqueness of the solutions of IVP .

Lecture 7

1.7 Cauchy-Peano Theorem

Let us recall that the IVP stated in Example 1.6.7 admits solutions. It is not difficult to verify, in this case, that f is continuous in (t, x) in the neighborhood of $(0, 0)$. In fact, the continuity of f is sufficient to prove the existence of a solution. The proofs in this section is based on Ascoli-Arzela theorem which in turn needs the concept of equicontinuity of a family of functions. We need the following ground work before embarking on the proof of such results. Let $I = [a, b] \subset \mathbb{R}$ be an interval. Let $F(I, \mathbb{R})$ denote the set of all real valued functions defined on I.

Definition 1.7.1. A set $E \subset F(I, \mathbb{R})$ is called equicontinuous on I if for any $\epsilon > 0$, there is a $\delta > 0$ such that for all $f \in E$,

$$|f(x) - f(y)| < \epsilon, \text{ whenever } |x - y| < \delta.$$

Definition 1.7.2. A set $E \subset F(I, \mathbb{R})$ is called uniformly bounded on I if there is a $M > 0$, such that

$$|f(x)| < M \text{ for all } f \in E \text{ and for all } x \in I.$$

Theorem 1.7.3. *(Ascoli-Arzela Theorem) Let $B \subset F(I, \mathbb{R})$ be any uniformly bounded and equicontinuous set on I. Then, every sequence of functions $\{f_n\}$ in B contains a subsequence $\{f_{n_k}\}, k = 1, 2 \ldots,$ which converges uniformly on every compact sub-interval of I.*

Theorem 1.7.4. *(Peano's existence theorem) Let $a > 0, t_0 \in \mathbb{R}$. Let $S \subset \mathbb{R}^2$ be a strip defined by*

$$S = \{(t, x) : |t - t_0| \le a, \ |x| \le \infty\}.$$

Let $I : [t_0, t_0 + a]$. Let $f : S \to \mathbb{R}$ be a bounded continuous function. Then, the IVP

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.45}$$

has at least one solution existing on $[x_0 - a, x_0 + a]$.

Proof. The proof of the theorem is first dealt on $[t_0, t_0 + a]$ and the proof on $[t_0 - a, t_0]$ is similar with suitable modifications. Let the sequence of functions $\{x_n\}$ be defined by, for $n = 1, 2 \cdots$

$$x_n(t) = x_0, \quad x_0 \le t \le x_0 + \frac{a}{n}, \quad t \in I,$$

$$x_n(t) = x_0 + \int_{t_0}^{t_0 - \frac{a}{n}} f(s, x_n(s))ds \quad \text{if} \quad t_0 + \frac{ka}{n} \le t \le t_0 + \frac{(k+1)a}{n}, \quad k = 1, 2, \ldots, n \tag{1.46}$$

We note that x_n is defined on $[t_0, t_0 + \frac{a}{n}]$ to start with and thereafter defined on

$$\Big[t_0 + \frac{ka}{n}, t_0 + \frac{(k+1)a}{n}\Big], \quad k = 1, 2, \ldots, n.$$

By hypotheses $\exists\ M > 0$, such that $|f(t, x)| \le M$, whenever $(t, x) \in S$. Let t_1, t_2 be two points in $[t_0, t_0 + a]$. Then,

$$|x(t_1) - x(t_2)| = 0 \quad \text{if} \quad t_1, t_2 \in \Big[t_0, t_0 + \frac{a}{n}\Big].$$

For any $t_1 \in [t_0, t_0 + \frac{a}{n}], \quad t_2 \in \big[t_0 + \frac{ka}{n}, t_0 + \frac{(k+1)a}{n}\big]$

$$\begin{aligned} |x_n(t_1) - x_n(t_2)| &= \Big| \int_{t_1 - (a/n)}^{t_2 - (a/n)} f(s, x_n(s))ds \Big| \\ &\le M|t_2 - t_1|, \end{aligned}$$

or else

$$|x_n(t_1) - x_n(t_2)| \le M|t_2 - t_1|, \quad \forall\ t_1, t_2 \in I. \tag{1.47}$$

Let ϵ be given with the choice of $\delta = \epsilon/M$. From equation (1.47), we have

$$|x_n(t_1) - x_n(t_2)| \le \epsilon \ \text{ if } \ |t_1 - t_2| < \delta,$$

which is same as saying that $\{x_n\}$ is uniformly continuous on I. Again by (1.47), for all $t \in I$

$$|x_n(t)| \le |t_0| + M|x - \frac{a}{n} - t_0| \le |t_0| + Ma,$$

or else $\{x_n\}$ is uniformly bounded on I. By Ascoli-Arzela theorem (see Theorem 1.7.3) $\{x_n\}$ has a uniformly convergent subsequence $\{x_{n_k}\}$ on I. The limit of $\{x_{n_k}\}$ is continuous on I since the convergence on I is uniform. By letting $k \to \infty$ in

$$x_{n_k} = x_0 + \int_{t_0}^{t} f(s, x_{n_k}(s))ds - \int_{t - a/n_k}^{t} f(s, x_{n_k}(s))ds,$$

we have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t \in I.$$

Since

$$\Big| \int_{t - a/n_k}^{t} f(s, x_{n_k}(s))ds \Big| \to 0 \ \ as \ \ k \to \infty, \tag{1.48}$$

and consequently x is a solution of (1.45), finishing the proof. □

Remark Although f is continuous on S, f may not be bounded since S is not so. The same proof has a modification when S is replaced a rectangle R (of finite area) except that we have to ensure that $(t, x_n(t)) \in R$. In this case $(t, x_n(t)) \in S$ for all $t \in I$ is obvious. With these comments, we have

Theorem 1.7.5. *Let $\bar{R}$ be a rectangle*

$$\bar{R} = \{(t,x) : |t - t_0| \le a, |x - x_0| \le b\}, \quad a \ge 0, b \ge 0, \ t, x_0, t_0 \in \mathbb{R}$$

and $f : \bar{R} \to \mathbb{R}$ be a continuous function. Let $|f(t,x)| \le M$ for all $(t,x) \in \bar{R}$, $h = min(a, \frac{b}{M})$ and let $I_h = |t - t_0| \le h$, then the IVP (1.45) *has a solution x defined on I_h.*

Proof. The proof is exactly similar to that of Theorem 1.7.4. We note that, for all n, $(t, x_n(t)) \in \bar{R}$ if $t \in I_h$. The details of the proof is left as an exercise. □

Theorem 1.7.4 has an alternative proof, details are given beow.

Proof of Theorem 1.7.4. Define a sequence $\{x_n\}$ on I_h by, for $n \ge 1$,

$$x_n(t) = \begin{cases} x_0, & \text{if } \ t \le t_0; \\ x_0 + \int_{t_0}^t f(s, x_n(s - \frac{a}{n}))ds, & \text{if } \ t_0 \le s \le t_0 + h. \end{cases}$$

Since the sequence is well defined on $[t_0, t_0 + \frac{a}{n}]$, it is well defined on $[t_0, t_0 + h]$. It is not very difficult to show that $\{x_n\}$ is uniformly continuous and uniformly bounded on I_h. By an application of Ascoli-Arzela theorem, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging uniformly (to say x) on I_h. Uniform convergence implies that x is continuous on I_h. By definition

$$x_{n_k}(t) = x_0 + \int_{t_0}^t f\Big(s, x_{n_k}\Big(s - \frac{a}{n_k}\Big)\Big)ds, \quad t \in I_h \tag{1.49}$$

Since $x_{n_k} \to x$ uniformly on I_h, by letting $k \to \infty$ in (1.49), we get

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds, \quad t \in I_h$$

that is, x is a solution of the IVP (1.45). □

EXERCISES

1. Represent the linear n-th order IVP

$$x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = b(t),$$
$$x(t_0) = x_0, \ x'(t_0) = x_1, \ \ldots, \ x^{(n-1)}(t_0) = x_{n-1},$$

   as a system. Prove that it has a unique solution.

2. Sketch the proof of Theorem 1.7.5.

3. Give a proof of Theorem 1.7.5.

4. Sketch the proof of Theorem 1.28 on $[t_0 - h, t_0]$.

Module 2

Linear Differential Equations of Higher Order

Lecture 9

2.1 Introduction

In this chapter, we introduce a study of a particular class of differential equations, namely the linear differential equations. They occur in many branches of sciences and engineering and so a systematic study of them is indeed desirable. Linear equations with constant coefficients have more significance as far as their practical utility is concerned since closed form solutions are known by just solving algebraic equations. On the other hand linear differential equations with variable coefficients pose a formidable task while obtaining closed form solutions. In any case first we need to ascertain whether these equations do admit solutions at all. In this chapter, we show that a general nth order linear equation admits precisely n linearly independent solutions. Before embarking into the details, the uniqueness of solutions of initial value problems for linear equations has been established in Module 1 .We recall the following

Theorem 2.1.1. *Assume that $a_0, a_1, \cdots, a_n$ and b are real valued continuous functions defined on an interval $I \subseteq \mathbb{R}$ and that $a_0(t) \neq 0$, for all $t \in I$. Then the IVP*

$$\left.\begin{array}{c} a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = b(t), t \in I \\ x(t_0) = \alpha_1, x'(t_0) = \alpha_2, \cdots, x^{(n-1)}(t_0) = \alpha_n, t_0 \in I \end{array}\right\} \tag{2.1}$$

has a unique solution existing on I.

2.2 Linear Dependence and Wronskian

The concept of linear dependence and independence has a special role to play in the study of linear differential equations. It naturally leads us to the concept of the general solution of a linear differential equation. To begin with, the concept of Wronskian and its relation to linear dependence and independence of functions is established.

Consider real or complex valued functions defined on an interval I contained in $\mathbb{R}$. The interval I could be possibly the whole $\mathbb{R}$. We recall the following definition.

Definition 2.2.1. *(Linear dependence and independence)* Two functions x_1 and x_2 defined on an interval I are said to be linearly dependent on I, if and only if there exist two constants c_1 and c_2, at least one of them is non-zero, such that $c_1x_1 + c_2x_2 = 0$ on I. Functions x_1 and x_2 are said to be independent on I if they are not linearly dependent on I.

Remark : Definition 2.2.1 implies that in case two functions $x_1(t)$ and $x_2(t)$ are linearly independent and, in addition,

$$c_1x_1(t) + c_2x_2(t) \equiv 0, \quad \forall\, t \in I,$$

then c_1 and c_2 are necessarily both zero. Thus, if two functions are linearly dependent on an interval I then one of them is a constant multiple of the other. The scalars c_1 and c_2 may be real numbers.

Example 2.2.2. Consider the functions

$$x_1(t) = e^{\alpha t} \text{ and } x_2(t) = e^{\alpha(t+1)}, \quad t \in \mathbb{R},$$

where α is a constant. Since x_1 is a multiple of x_2, the two functions are linearly dependent on $\mathbb{R}$.

Example 2.2.3. $\sin t$ and $\cos t$ are linearly independent on the interval $I = [0, 2\pi]$.

The above discussion of linear dependence of two functions defined on I is readily extended for a set of n functions where $n \geq 2$. These extensions are needed in the study of linear differential equations of order $n \geq 2$. In the ensuing definition, we allow the functions which are complex valued.

Definition 2.2.4. A set of n real(complex) valued functions $x_1, x_2, \cdots, x_n$, $(n \geq 2)$ defined on I are said to be linearly dependent on I, if there exist n real (complex) constants $c_1, c_2, \cdots, c_n$, not all of them are simultaneously zero, such that

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0,\ t \in \mathbb{R}.$$

The functions $x_1, x_2, \cdots, x_n$ is said to be linearly independent on I if they are not linearly dependent on I.

Example 2.2.5. Let α is a constant.The functions

$$x_1(t) = e^{i\alpha t},\ x_2(t) = \sin \alpha t,\ x_3(t) = \cos \alpha t, t \in \mathbb{R},$$

where α is a constant. It is easy to note that x_1 can be expressed in terms of x_2 and x_3 which shows that the given functions are linearly dependent on $\mathbb{R}$.

It is a good question to enquire about the sufficient conditions for the linear independence of a given set of functions. We need the concept of Wronskian to ascertain the linear independence of two or more differentiable functions.

Definition 2.2.6. *(Wronskian)* The Wronskian of two differentiable functions x_1 and x_2 defined on I is a function W defined by the determinant

$$W[x_1(t), x_2(t)] = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}, \quad t \in I.$$

Theorem 2.2.7. *If the Wronskian of two functions x_1 and x_2 on I is non-zero for at least one point of the interval I, then the functions x_1 and x_2 are linearly independent on I.*

Proof. The proof is by method of contradiction. Let us assume on the contrary that the functions x_1 and x_2 are linearly dependent on I. Then there exist constants (at least one of them is non-zero) c_1 and c_2 such that

$$c_1x_1(t) + c_2x_2(t) = 0 \;\forall\, t \in I. \tag{2.2}$$

By differentiating, (2.2) we have

$$c_1x_1'(t) + c_2x_2'(t) = 0 \;\forall\, t \in I. \tag{2.3}$$

By assumption there exists a point, say $t_0 \in I$, such that

$$\begin{vmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{vmatrix} = x_1(t_0)x_2'(t_0) - x_2(t_0)x_1'(t_0) \neq 0. \tag{2.4}$$

From (2.2) and , we obtain

$$\begin{aligned} c_1x_1(t_0) + c_2x_2(t_0) = 0 \\ c_1x_1'(t_0) + c_2x_2'(t_0) = 0. \end{aligned} \tag{2.5}$$

Looking upon (2.5) as a system of linear equations with c_1 and c_2 as unknown quantities, from the theory of algebraic equations we know that if (2.4) holds, then the system (2.5) admits only zero solution i.e., $c_1 = 0$ and $c_2 = 0$. This is a contradiction to the assumption and hence the theorem is proved. □

As an immediate consequence, we have :

Theorem 2.2.8. *Let $I \subseteq \mathbb{R}$ be an interval. If two differentiable functions x_1 and x_2 (defined on I) are linearly dependent on I then their Wronskian*

$$W[x_1(t), x_2(t)] \equiv 0 \quad \text{on} \quad I.$$

The proof is left as an exercise. It is easy to extend Definition 2.2.4 for a set of n functions and derive results of Theorems 2.8 and 2.9 for these sets of n functions. The proofs of the corresponding theorems are omitted as the proof is essentially the same as given in Theorem 2.8.

Definition 2.2.9. The Wronskian of n ($n > 2$) functions $x_1, x_2, \cdots, x_n$ defined and $(n-1)$ times differentiable on I is defined by the nth order determinant

$$W[x_1(t, x_2(t), \cdots, x_n(t)] = \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{vmatrix}, \quad t \in I.$$

Theorem 2.2.10. *If the Wronskian of n functions $x_1, x_2, \cdots, x_n$ defined on I is non-zero for at least one point of I, then the set of n functions $x_1, x_2, \cdots, x_n$ is linearly independent on I.*

Theorem 2.2.11. *If a set of n functions $x_1, x_2, \cdots, x_n$ whose derivatives exist up to and including that of order $(n-1)$ are linearly dependent on an interval I, then their Wronskian $W[x_1(t), x_2(t), \cdots, x_n(t)] \equiv 0$ on I.*

Remark : The converse of Theorems **??** and 2.11 may not be true in general. Two or more functions can be linearly independent on an interval and yet their Wronskian may be identically zero. For example, let $x_1(t) = t^2$ and $x_2(t) = t|t|, -\infty < t < \infty$. In fact x_1 and x_2 are linearly independent but $W[x_1(t), x_2(t)] \equiv 0$.

The situation is very different when the given functions are solutions of certain linear homogeneous differential equation. Let us discuss such a case later.

Example 2.2.12. Consider the functions

$$x_1(t) = e^{\alpha t}\cos\beta t, \quad x_2(t) = e^{\alpha t}\sin\beta t, \quad t \in I,$$

where α and β are constants and $\beta \neq 0$. We note

$$\begin{aligned} W[x_1(t), x_2(t)] &= e^{2\alpha t}\begin{vmatrix} \cos\beta t & \sin\beta t \\ \cos\beta t - \beta\sin\beta t & \sin\beta t + \beta\cos\beta t \end{vmatrix}, \quad t \in I, \\ &= \beta e^{2\alpha t} \neq 0, \quad t \in I. \end{aligned}$$

Further x_1 and x_2 are linearly independent on I and satisfies the differential equation

$$x'' - 2\alpha x' + (\alpha^2 + \beta^2)x = 0.$$

EXERCISES

1. Show that $\sin x, \sin 2x, \sin 3x$ are linearly independent on $I = [0, 2\pi]$.

2. Verify that $1, x, x^2, \cdots, x^m$ are linearly independent on any interval $I \subseteq \mathbb{R}$.

3. Define the functions f and g on $[-1, 1]$ by

$$\left.\begin{aligned} f(x) &= 0 \\ g(x) &= 1 \end{aligned}\right\} \text{ if } x \in [-1, 0]$$

$$\left.\begin{aligned} f(x) &= \sin x \\ g(x) &= 1 - x \end{aligned}\right\} \text{ if } x \in [0, 1].$$

   Then, prove that f and g are linearly independent on $[-1, 1]$. Further verify that f and g are linearly dependent on $[-1, 0]$.

4. Prove that the n functions

$$e^{r_i t}, te^{r_i t}, \cdots, t^{k_i - 1}e^{r_i t},$$

   $i = 1, 2, \cdots, s$, where $k_1 + k_2 + \cdots + k_s = n$ and $r_1, r_2, \cdots, r_s$ are distinct numbers, and linearly independent on every interval I.

5. let I_1, I_2 and I be intervals in $\mathbb{R}$ such that $I_1 \subset I_2 \subset I$. If two functions defined on I are linearly independent on I_1 then, show that they are linearly independent on I_2.

Lecture 10

2.3 Basic Theory for Linear Equations

In this section the meaning that is attached to a general solution of the differential equation and some of its properties are studied. We stick our attention to second order equations to start with and extend the study for an n-th order linear equations. The extension is not hard at all. As usual let $I \subseteq \mathbb{R}$ be an interval. Consider

$$a_0(t)x''(t) + a_1(t)x'(t) + a_2(t)x(t) = 0, \quad a_0(t) \neq 0, \quad t \in I. \tag{2.6}$$

Later we study structure of solutions of a non-homogeneous equation of second order. Let us define an operator L on the space twice differentiable functions defined on I by the following relation

$$L(y)(t) = a_0(t)y''(t) + a_1(t)y'(t) + a_2(t)y(t) \text{ and } a_0(t) \neq 0, \quad t \in I. \tag{2.7}$$

With L in hand, (2.6) is

$$L(x) = 0 \text{ on } I.$$

The linearity of the differential operator tell us that :

Lemma 2.3.1. *The operator L is linear on the space of twice differential functions on I.*

Proof. Let y_1 and y_2 be any two twice differentiable functions on I. Let c_1 and c_2 be any constants. For the linearity of L We need to show

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2) \text{ on } I$$

which is a simple consequence of the linearity of the differential operator. □

As an immediate consequence of the Lemma (2.14), we have the superposition principle:

Theorem 2.3.2. *(Super Position Principle) Suppose x_1 and x_2 satisfy the equation* (2.6) *for $t \in I$. Then,*

$$c_1x_1 + c_2x_2,$$

also satisfies (2.6)*, where c_1 and c_2 are any constants.*

The proof is easy and hence, omitted. The first of the following examples illustrates Theorem 2.3.2 while the second one shows that the linearity cannot be dropped.

Example 2.3.3. (i) Consider the differential equation for the linear harmonic oscillator, namely

$$x'' + \lambda^2 x = 0, \ \lambda \in \mathbb{R}.$$

Both $\sin\lambda x$ and $\cos\lambda x$ are two solutions of this equation and

$$c_1 \sin\lambda x + c_2 \cos\lambda x,$$

is also a solution, where c_1 and c_2 are constants.

(ii) The differential equation

$$x'' = -x'^2,$$

admits two solutions

$$x_1(t) = \log(t + a_1) + a_2 \text{ and } x_2(t) = \log(t + a_1),$$

where a_1 and a_2 are constants. With the values of $c_1 = 3$ and $c_2 = -1$,

$$x(t) = c_1 x_1(t) + c_2 x_2(t),$$

does not satisfy the given equation. We note that the given equation is nonlinear.

Lemma (2.14) and Theorem 2.3.2 which prove the principle of superposition for the linear equations of second order have a natural extension to linear equations of order $n(n > 2)$. Let

$$L(y) = a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y, \quad t \in I \tag{2.8}$$

where $a_0(t) \neq 0$ on I. The general n-th order linear differential equation may be written as

$$L(x) = 0, \tag{2.9}$$

where L is the operator defined by the relation (2.8). As a consequence of the definition, we have :

Lemma 2.3.4. *The operator L defined by* (2.8)*, is a linear operator on the space of all n times differentiable functions defined on I.*

Theorem 2.3.5. *Suppose $x_1, x_2, \cdots, x_n$ satisfy the equation* (2.9)*. Then,*

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

also satisfies (2.9)*, where $c_1, c_2, \cdots, c_n$ are arbitrary constants.*

The proofs of the Lemma 2.3.4 and Theorem 2.3.5 are easy and hence omitted.

Theorem 2.3.5 allows us to define a general solution of (2.9) given an additional hypothesis that the set set of solutions $x_1, x_2, \cdots, x_n$ is linearly independent. Under these assumptions later we actually show that any solution x of (2.9) is indeed a linear combination of $x_1, x_2, \cdots, x_n$.

Definition 2.3.6. Let $x_1, x_2, \cdots, x_n$ be n linearly independent solutions of (2.9). Then,

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

is called the general solution of (2.9), where $c_1, c_2 \cdots, c_n$ are arbitrary constants.

Example 2.3.7. Consider the equation

$$x'' - \frac{2}{t^2}x = 0, \quad 0 < t < \infty.$$

We note that $x_1(t) = t^2$ and $x_2(t) = \dfrac{1}{t}$ are 2 linearly independent solutions on $0 < t < \infty$. A general solution x is

$$x(t) = c_1t^2 + \frac{c_2}{t}, \quad 0 < t < \infty.$$

Example 2.3.8. $x_1(t) = t, x_2(t) = t^2, x_3(t) = t^3, t > 0$ are three linearly independent solutions of the equation

$$t^3x''' - 3t^2x'' + 6tx' - 6x = 0, \quad t > 0.$$

The general solution x is

$$x(t) = c_1t + c_2t^2 + c_3t^3, t > 0.$$

We again recall that Theorems 2.3.2 and 2.3.5 state that the linear combinations of solutions of a linear equation is yet another solution. The question now is whether this property can be used to generate the general solution for a given linear equation. The answer indeed is in affirmative. Here we make use of the interplay between linear independence of solutions and the Wronskian. The following preparatory result is needed for further discussion. We recall the equation (2.7) for the definition of L.

Lemma 2.3.9. *If x_1 and x_2 are linearly independent solutions of the equation $L(x) = 0$ on I, then the Wronskian of x_1 and x_2, namely, $W[x_1(t), x_2(t)]$ is never zero on I.*

Proof. Suppose on the contrary, there exist $t_0 \in I$ at which $W[x_1(t_0), x_2(t_0)] = 0$. Then, the system of linear algebraic equation for c_1 and c_2

$$\left.\begin{array}{l} c_1x_1(t_0) + c_2(t)x_2(t_0) = 0 \\ c_1x_1'(t_0) + c_2(t)x_2'(t_0) = 0 \end{array}\right\}, \tag{2.10}$$

has a non-trivial solution. For such a nontrivial solution (c_1, c_2) of (2.10), we define

$$x(t) = c_1x_1(t) + c_2x_2(t), \quad t \in I.$$

By Theorem 2.3.2, x is a solution of the equation (2.6) and

$$x(t_0) = 0 \text{ and } x'(t_0) = 0.$$

Since an initial value problem for $L(x) = 0$ admits only one solution, we therefore have $x(t) \equiv 0, \quad t \in I$, which means that

$$c_1x_1(t) + c_2x_2(t) \equiv 0, \quad t \in I,$$

with at least one of c_1 and c_2 is non-zero or else, x_1, x_2 are linearly dependent on I, which is a contradiction. So the Wronskian $W[x_1, x_2]$ cannot vanish at any point of the interval I. □

As a consequence of the above lemma an interesting corollary is :

Corollary 2.3.10. *The Wronskian of two solutions of $L(x) = 0$ is either identically zero if the solutions are linearly dependent on I or never zero if the solutions are linearly independent on I.*

Lemma 2.3.9 has an immediate generalization of to the equations of order $n(n > 2)$. The following lemma is stated without proof.

Lemma 2.3.11. *If $x_1(t), x_2(t), \cdots, x_n(t)$ are linearly independent solutions of the equation* (2.9) *which exist on I, then the Wronskian*

$$W[x_1(t), x_2(t), \cdots, x_n(t)],$$

is never zero on I. The converse also holds.

Example 2.3.12. Consider Examples 2.3.7 and 2.20. The linearly independent solutions of the differential equation in Example 2.3.7 are $x_1(t) = t^2, x_2(t) = 1/t$. The Wronskian of these solutions is

$$W[x_1(t), x_2(t)] = -3 \neq 0 \text{ for } t \in (-\infty, \infty).$$

The Wronskian of the solutions in Example 2.3.8 is given by

$$W[x_1(t), x_2(t), x_3(t)] = 2t^3 \neq 0$$

when $t > 0$.

The conclusion of the Lemma 2.3.11 holds if the equation (2.9) has n linearly independent solutions . A doubt may occur whether such a set of solutions exist or not. In fact, Example 2.3.13 removes such a doubt.

Example 2.3.13. Let

$$L(x) = a_0(t)x''' + a_1(t)x'' + a_1(t)x' + a_3(t)x = 0.$$

Now, let $x_1(t),\ t \in I$ be the unique solution of the IVP

$$L(x) = 1,\ x(a) = 0,\ x'(a) = 0,\ x''(a) = 0;$$

$x_1(t),\ t \in I$ be the unique solution of the IVP

$$L(x) = 0,\ x(a) = 0,\ x'(a) = 1,\ x''(a) = 0;$$

and $x_3(t),\ t \in I$ be the unique solution of the IVP

$$L(x) = 0,\ x(a) = 0,\ x'(a) = 0,\ x''(a) = 1$$

where $a \in I$.. Obviously $x_1(t), x_2(t), x_3(t)$ are linearly independent, since the value of the Wronskian at the point $a \in I$ is non-zero. For

$$W[x_1(a), x_2(a), x_3(a)] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

An application of the Lemma 2.3.11 justifies the assertion. Thus, a set of three linearly independent solution exists for a homogeneous linear equation of the third order.

Now we establish a major result for a homogeneous linear differential equation of order $n \geq 2$ below.

Theorem 2.3.14. *Let $x_1, x_2, \cdots, x_n$ be linearly independent solutions of* (2.9) *existing on an interval $I \subseteq \mathbb{R}$. Then any solution x of* (2.9) *existing on I is of the form*

$$x(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t),\ t \in I$$

where $c_1, c_2, \cdots, c_n$ are some constants.

Proof. Let x be any solution of $L(x) = 0$ on I, and $a \in I$. Let

$$x(a) = a_1, x'(a) = a_2, \cdots, x^{(n-1)} = a_n.$$

Consider the following system of equation:

$$\left.\begin{array}{r} c_1x_1(a) + c_2x_2(a) + \cdots + c_nx_n(a) = a_1 \\ c_1x_1'(a) + c_2x_2'(a) + \cdots + c_nx_n'(a) = a_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_1x_1^{(n-1)}(a) + c_2x_2^{(n-1)}(a) + \cdots + c_nx_n^{(n-1)}(a) = a_n \end{array}\right\}. \tag{2.11}$$

We can solve system of equations (2.11) for $c_1, c_2, \cdots, c_n$. Since the determinant of the coefficients of $c_1, c_2, \cdots, c_n$ in the above system is not zero and since the Wronskian of $x_1, x_2, \cdots, x_n$ at the point a, it is different from zero by Lemma 2.3.11. Define

$$y(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t),\ t \in I,$$

where $c_1, c_2, \cdots, c_n$ are the solutions of the system given by (2.11). Then y is a solution of $L(x) = 0$ and in addition

$$y(a) = a_1, y'(a) = a_2, \cdots, y^{(n-1)}(a) = a_n.$$

From the uniqueness theorem, there is one and only one solution with these initial conditions. Hence $y(t) = x(t)$ for $t \in I$. This completes the proof. □

Lecture 11

By this time we note that a general of (2.9) represents a n parameter family of curves. The parameters are the arbitrary constants appearing in the general solution. Such a notion motivates us define a general solution of a non-homogeneous linear equation

$$L(x(t)) = a_0(t)x''(t) + a_1(t)x'(t) + a_2(t)x(t) = d(t),\ t \in I \tag{2.12}$$

where d is continuous on I. Formally a n parameter solution x of (2.12) is called a solution of (2.12). Loosely speaking a general solution of (2.12) "contains" n arbitrary constants. With such a definition we have:

Theorem 2.3.15. *Suppose x_p is any particular solution of* (2.12) *existing on I and that x_h is the general solution of the homogeneous equation $L(x) = 0$ on I. Then $x = x_p + x_h$ is a general solution of* (2.12) *on I.*

Proof. $x_p + x_h$ is a solution of the equation (2.12), since

$$L(x) = L(x_p + x_h) = L(x_p) + L(x_h) = d(t) + 0 = d(t), \quad t \in I$$

Or else x is a solution of (2.12), which is a n parameter family of function (since x_h is one such) and so x is a general solution of (2.12). □

Thus, if a particular solution of (2.12) is known, then the general solution of (2.12) is easily obtained by using the general solution of the corresponding homogeneous equation. The Theorem 2.3.15 has a natural extension to a n-th order non-homogeneous differential equation of the form

$$L(x(t)) = a_0(t)x^n(t) + a_1(t)x^{n-1}(t) + \cdots + a_n(t)x(t) = d(t),\ t \in I.$$

Let x_p be a particular solution existing on I. Then, the general solution of $L(x) = d$ is of the form

$$x(t) = x_p(t) + c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t), \quad t \in I$$

where $\{x_1, x_2, \cdots, x_n\}$ is a linearly independent set of n solutions of (2.9) existing on I and $c_1, c_2, \cdots, c_n$ are any constants.

Example 2.3.16. Consider the equation

$$t^2x'' - 2x = 0, \quad 0 < t < \infty.$$

The two solutions $x_1(t) = t^2$ and $x_2(t) = 1/t$ are linearly independent. A particular solution x_p of

$$t^2x'' - 2x = 2t - 1, \quad 0 < t < \infty.$$

is $x_p(t) = \frac{1}{2} - t$ and so the general solution x is

$$x(t) = (\tfrac{1}{2} - t) + c_1t^2 + c_2\tfrac{1}{t}, \quad 0 < t < \infty,$$

where c_1 and c_2 are arbitrary constants.

EXERCISES

1. Suppose that z_1 is a solution of $L(y) = d_1$ and that z_2 is a solution of $L(y) = d_2$. Then show that $z_1 + z_2$ is a solution of the equation

$$L(y(t)) = d_1(t) + d_2(t).$$

2. If a complex valued function z is a solution of the equation $L(x) = 0$ then, show that the real and imaginary parts of z are also solutions of $L(x) = 0$.

3. (Reduction of the order) Consider an equation

$$L(x) = a_0(t)x'' + a_1(t)x' + a_2(t)x = 0, \quad a_0(t) \neq 0, t \in I.$$

where a_0, a_1 and a_2 are continuous functions defined on I. Let $x_1 \neq 0$ be a solution of this equation. Show that x_2 defined by

$$x_2(t) = x_1(t)\int_{t_0}^{t} \frac{1}{x_1^2(s)} \exp\Big(\int_{t_0}^{s} \frac{a_1(u)}{a_0(u)}du\Big)ds, \quad t_0 \in I,$$

is also a solution. In addition, show that x_1 and x_2 are linearly independent on I.

2.4 Method of Variation of Parameters

Recall from Theorem 2.3.15 that a general solution of the equation

$$L(x) = d(t), \tag{2.13}$$

(where $L(x)$ is given by (2.7) or (2.9)) is fully determined the moment we know x_h and x_p. It is therefore natutral to know both a particular solution x_p of (2.13) as well as the general solution x_h of the homogeneous equation $L(x) = 0$. If $L(x) = 0$ is an equation with constant coefficients, the determinatin of the general solution is not difficult. Variation of parameter is a general method gives us a particular solution. The method of variation of parameters is also effective in dealing with equations with variable coefficients . To make the matter simple let us consider a second order equation

$$L(x(t)) = a_0(t)x''(t) + a_1(t)x'(t) + a_2(t)x(t) = d(t), \quad a_0(t) \neq 0, \quad t \in I, \tag{2.14}$$

where the functions $a_0, a_1, a_2, d : I \to \mathbb{R}$ are continuous. Let x_1 and x_2 be two linearly independent solutions of the homogeneous equation

$$a_0(t)x''(t) + a_1(t)x'(t) + a_2(t)x(t) = 0, \quad a_0(t) \neq 0, \quad t \in I. \tag{2.15}$$

Then, $c_1x_1 + c_2x_2$ is the general solution of (2.15), where c_1 and c_2 are arbitrary constants. The general solution of (2.14) is determined the moment we know a particular solution x_p of (2.14). We let the constants c_1, c_2 as parameters depending on t and determine x_p . In other words, we would like to find u_1 and u_2 on I such that

$$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t), \; t \in I \tag{2.16}$$

satisfies (2.14).

In order to substitute x_p in (2.14), we need to calculate x_p' and x_p''. Now

$$x_p' = x_1'u_1 + x_2'u_2 + (x_1u_1' + x_2u_2').$$

We do not wish to end up with second order equations for u_1, u_2 and naturally we choose u_1 and u_2 to satisfy

$$x_1(t)u_1'(t) + x_2(t)u_2'(t) = 0 \tag{2.17}$$

Added to it, we already known how to solve first order equations. With (2.17) in hand we now have

$$x_p'(t) = x_1'(t)u_1(t) + x_2'(t)u_2(t). \tag{2.18}$$

Differentiation of (2.18) leads to

$$x_p'' = u_1'x_1' + u_1x_1'' + u_2'x_2' + u_2x_2''. \tag{2.19}$$

Now we substitute (2.16), (2.18) and (2.19) in (2.14) to get

$$[a_0(t)x_1''(t) + a_1(t)x_1'(t) + a_2(t)x_1(t)]u_1 + [a_0(t)x_2''(t) + a_1(t)x_2'(t) + a_2(t)x_2(t)]u_2 + \\ u_1'a_0(t)x_1' + u_2'a_0(t)x_2' = d(t),$$

and since x_1 and x_2 are solutions of (2.15), hence

$$x_1'u_1'(t) + x_2'u_2'(t) = \frac{d(t)}{a_0(t)}. \tag{2.20}$$

We solve for u_1' and u_2' from (2.17) and (2.20), to determine x_p. It is easy see

$$u_1'(t) = \tfrac{-x_2(t)d(t)}{a_0(t)W[x_1(t), x_2(t)]}$$

$$u_2'(t) = \tfrac{x_1(t)d(t)}{a_0(t)W[x_1(t), x_2(t)]}$$

where $W[x_1(t), x_2(t)]$ is the Wronskian of the solutions x_1 and x_2. Thus, u_1 and u_2 are given by

$$\left.\begin{aligned} u_1(t) &= -\int \tfrac{x_2(t)d(t)}{a_0(t)W[x_1(t),x_2(t)]}dt \\ u_2(t) &= \int \tfrac{x_1(t)d(t)}{a_0(t)W[x_1(t),x_2(t)]}dt \end{aligned}\right\} \tag{2.21}$$

Now substituting the values of u_1 and u_2 in (2.16) we get a desired particular solution of the equation (2.14). Indeed

$$x_p(t) = u_1(t)x_1(t) = u_2(t)x_2(t), \quad t \in I$$

is completely known. To conclude, we have :

Theorem 2.4.1. *Let the functions a_0, a_1, a_2 and d in* (2.14) *be continuous functions on I. Further assume that x_1 and x_2 are two linearly independent solutions of* (2.15). *Then, a particular solution $x_p(t)$ of the equation* (2.14) *is given by* (2.16).

Theorem 2.4.2. *The general solution $x(t)$ of the equation* (2.14) *on I is*

$$x(t) = x_p(t) + x_h(t),$$

where $x_p(t)$ is a particular solution given by (2.16) *and x_h is the general solution of $L(x) = 0$.*

Also, we note that we have an explicit expression for x_p which was not so while proving Theorem 2.3.15. The following example is for illustration.

Lecture 12

Example 2.4.3. Consider the equation

$$x'' - \tfrac{2}{t}x' + \tfrac{2}{t^2}x = t\sin t, \quad t \in [1, \infty).$$

Note that $x_1 = t$ and $x_2 = t^2$ are two linearly independent solutions of the homogeneous equation on $[1, \infty)$. Now

$$W[x_1(t), x_2(t)] = t^2.$$

Substituting the values of x_1, x_2, $W[x_1(t), x_2(t)]$, $d(t) = t\sin t$ and $a_0(t) \equiv 1$ in (2.21), we have

$$\begin{aligned} u_1(t) &= t\cos t - \sin t \\ u_2(t) &= \cos t \end{aligned}$$

and the particular solution is $x_p(t) = -t\sin t$. Thus, the general solution is

$$x(t) = -t\sin t + c_1 t + c_2 t^2,$$

where c_1 and c_2 are arbitrary constants.

The method of variation of parameters can be extended to the equation of order $n(n > 2)$ which we state in the form of a theorem, the proof of which has been deleted. Let us consider an equation of the n-th order

$$L(x(t)) = a_0(t)x^n(t) + a_1(t)x^{n-1}(t) + \cdots + a_n(t)x(t) = d(t), \quad t \in I. \tag{2.22}$$

Theorem 2.4.4. *Let $a_0, a_1, \cdots, a_n, d : I \to \mathbb{R}$ be continuous functions. Let*

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

be the general solution of $L(x) = 0$. Then, a particular solution x_p of (2.22) is given by

$$x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t) + \cdots + u_n(t)x_n(t),$$

where $u_1, u_, \cdots, u_n$ satisfy the equations

$$\begin{gathered}
u_1'(t)x_1(t) + u_2'(t)x_2(t) + \cdots + u_n'(t)x_n(t) = 0 \\
u_1'(t)x_1'(t) + u_2'(t)x_2'(t) + \cdots + u_n'(t)x_n'(t) = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
u_1'(t)x_1^{(n-2)}(t) + u_2'(t)x_2^{(n-2)}(t) + \cdots + u_n'(t)x_n^{(n-2)}(t) = 0 \\
a_0(t)\big[u_1'(t)x_1^{(n-1)}(t) + u_2'(t)x_2^{(n-1)}(t) + \cdots + u_n'(t)x_n^{(n-1)}(t)\big] = d(t).
\end{gathered}$$

The proof of the Theorem 2.4.4 is similar to the previous one with obvious modifications.

EXERCISES

1. Find the general solution of $x''' + x'' + x' + x = 1$ given that $\cos t, \sin t$ and e^{-t} are three linearly independent solutions of the corresponding homogeneous equation. Also find the solution when $x(0) = 0, x'(0) = 1, x''(0) = 0$.

2. Use the method of variation of parameter to find the general solution of $x''' - x' = d(t)$ where
(i) $d(t) = t$, (ii) $d(t) = e^t$, (iii) $d(t) = \cos t$, and (iv) $d(t) = e^{-t}$.
In all the above four problems assume that the general solution of $x''' - x' = 0$ is $c_1 + c_2e^{-t} + c_3e^t$.

3. Assuming that $\cos Rt$ and $\frac{\sin Rt}{R}$ form a linearly independent set of solutions of the homogeneous part of the differential equation $x'' + R^2x = f(t), R \neq 0, t \in [0, \infty)$, where $f(t)$ is continuous for $0 \le t < \infty$ show that a solution of the equation under consideration is of the form

$$x(t) = A\cos Rt + \frac{B}{R}\sin Rt + \frac{1}{R}\int_0^t \sin[R(t-s)]f(s)ds,$$

where A and B are some constants. Show that particular solution of (2.14) is not unique. (Hint : If x_p is a particular solution of (2.14) and x is any solution of (2.15) then show that $x_p + c\,x$ is also a particular solution of (2.14) for any arbitrary constant c.)

Two Useful Formulae

Two formulae proved below are interesting in themselves. They are also useful while studying boundary value problems of second order equations. Consider an equation

$$L(y) = a_0(t)y'' + a_1(t)y' + a_2(t)y = 0, \quad t \in I,$$

where $a_0, a_1, a_2 : I \to \mathbb{R}$ are continuous functions in addition $a_0(t) \neq 0$ for $t \in I$. Let u and v be any two twice differentiable functions on I. Consider

$$uL(v) - vL(u) = a_0(uv'' - vu'') + a_1(uv' - vu'). \tag{2.23}$$

The Wronskian of u and v is given by $W(u, v) = uv' - vu'$ which shows that

$$\frac{d}{dt}W(u,v) = uv'' - vu''.$$

Note that the coefficients of a_0 and a_1 in the relation (2.23) are $W'(u,v)$ and $W(u,v)$ respectively. Now we have

Theorem 2.4.5. *If u and v are twice differential functions on I, then*

$$uL(v) - vL(u) = a_0(t)\frac{d}{dt}W[u,v] + a_1(t)W[u,v], \tag{2.24}$$

where $L(x)$ is given by (2.7). In particular, if $L(u) = L(v) = 0$ then W satisfies

$$a_0\frac{dW}{dt}[u,v] + a_1 W[u,v] = 0. \tag{2.25}$$

Theorem 2.4.6. *(*Able's Formula*) If u and v are solutions of $L(x) = 0$ given by (2.7), then the Wronskian of u and v is given by*

$$W[u,v] = k\ \exp\Big[-\int \frac{a_1(t)}{a_0(t)}dt\Big],$$

where k is a constant.

Proof. Since u and v are solutions of $L(y) = 0$, the Wronskian satisfies the first order equation (2.25) and Solving we get

$$W[u,v] = k\ \exp\Big[-\int \frac{a_1(t)}{a_0(t)}dt\Big] \tag{2.26}$$

where k is a constant. □

The above two results are employed to obtain a particular solution of a non-homogeneous second order equation.

Example 2.4.7. Consider the general non-homogeneous initial value problem given by

$$L(y(t)) = d(t), \quad y(t_0) = y'(t_0) = 0, \quad t, t_0 \in I, \tag{2.27}$$

where $L(y)$ is as given in (2.14). Assume that x_1 and x_2 are two linearly independent solution of $L(y) = 0$. Let x denote a solution of $L(y) = d$. Replace u and v in (2.24) by x_1 and x to get

$$\frac{d}{dt}W[x_1,x] + \frac{a_1(t)}{a_0(t)}W[x_1,x] = x_1\frac{d(t)}{a_0(t)} \tag{2.28}$$

which is a first order equation for $W[x_1,x]$. Hence

$$W[x_1,x] = \exp\Big[-\int_{t_0}^{t}\frac{a_1(s)}{a_0(s)}ds\Big]\int_{t_0}^{t}\frac{\exp\big[\int_{t_0}^{s}\frac{a_1(u)}{a_0(u)}du\big]x_1(s)ds}{a_0(s)}ds \tag{2.29}$$

While deriving (2.29) we have used the initial conditions $x(t_0) = x'(t_0) = 0$ in view of which $W[x_1(t_0), x(t_0)] = 0$. Now using the Able's formula, we get

$$x_1x' - xx_1' = W[x_1,x_2]\int_{t_0}^{t}\frac{x_1(s)d(s)}{a_0(s)W[x_1(s),x_2(s)]}ds. \tag{2.30}$$

The equation (2.30) as well could have been derived with x_2 in place of x_1 in order to get

$$x_2 x' - x x_2' = W[x_1, x_2] \int_{t_0}^{t} \frac{x_2(s) d(s)}{a_0(s) W[x_1(s), x_2(s)]} ds. \tag{2.31}$$

From (2.30) and (2.31) one easily obtains

$$x(t) = \int_{t_0}^{t} \frac{\left[x_2(t) x_1(s) - x_2(s) x_1(t)\right] d(s)}{a_0(s) W[x_1(s), x_2(s)]} ds. \tag{2.32}$$

It is time for us to recall that a particular solution in the form of (2.32) has already been derived while discussing the method of variation of parameters.

Lecture 13

2.5 Homogeneous Linear Equations with Constant Coefficients

Homogeneous linear equations with constant coefficients is an important subclass of linear equations, the reason being that solvability of these equations reduces to he solvability algebraic equations. Now we attempt to obtain a general solution of a linear equation with constant coefficients. Let us start as usual with a simple second order equation, namely

$$L(y) = a_0 y'' + a_1 y' + a_2 y = 0, a_0 \neq 0. \tag{2.33}$$

Later we move onto a more generally equation of order $n (n > 2)$

$$L(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{2.34}$$

where $a_0, a_1, \cdots, a_n$ are real constants and $a_0 \neq 0$.

Intuitively a look at the equation (2.33) or (2.34) tells us that if the derivatives of a function which are similar in form to the function itself then such a functions might probably be a candidate to solve (2.33) or (2.34). Elementary calculus tell us that one such function is the exponential, namely e^{pt}, where p is a constant. If e^{pt} is a solution then,

$$L(e^{pt}) = a_0 (e^{pt})'' + a_1 (e^{pt})' + a_2 (e^{pt}) = (a_0 p^2 + a_1 p + a_2) e^{pt}.$$

e^{pt} is a solution of (2.34) iff

$$L(e^{pt}) = (a_0 p^2 + a_1 p + a_2) e^{pt} = 0.$$

which means that e^{pt} is a solution of (2.34) iff p satisfies

$$a_0 p^2 + a_1 p + a_2 = 0. \tag{2.35}$$

Actually we have proved the following result:

Theorem 2.5.1. *λ is a root of the quadratic equation (2.35) iff $e^{\lambda t}$ is a solution of (2.33).*

If we note

$$L(e^{pt}) = (a_0 p^n + a_1 p^{n-1} + \cdots + a_n) e^{pt}$$

then the following result is immediate.

Theorem 2.5.2. *λ is a root of the equation*

$$a_0 p^n + a_1 p^{n-1} + \cdots + a_n = 0, \qquad a_0 \neq 0 \tag{2.36}$$

iff $e^{\lambda t}$ is a solution of the equation (2.34).

Definition 2.5.3. The equations (2.35) or (2.36) are called the characteristic equations for the linear differential equations (2.33) or (2.34) respectively. The corresponding polynomials are called characteristic polynomials.

In general, the characteristic equation (2.35) has two roots, say λ_1 and λ_2. By Theorem 2.5.1, $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions of (2.33) provided $\lambda_1 \neq \lambda_2$. Let us study the characteristic equation and its relationship with the general solution of (2.33).

Case 1: Let λ_1 and λ_2 be real distinct roots of (2.35). In this case $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ are two linearly independent solutions of (2.33) and the general solution x of (2.33) is given by $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.

Case 2: When λ_1 and λ_2 are complex roots, from the theory of equations, it is well known that they are complex conjugates of each other *i.e.,* they are of the form $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. The two solutions are

$$\begin{aligned} e^{\lambda_1 t} &= e^{(a+ib)t} = e^{at}[\cos bt + i \sin bt], \\ e^{\lambda_2 t} &= e^{(a-ib)t} = e^{at}[\cos bt - i \sin bt]. \end{aligned}$$

Now, if h is a complex valued solution of the equation (2.33), then

$$L[h(t)] = L[\operatorname{Re} h(t)] + iL[\operatorname{Im} h(t)], \ t \in I,$$

since L is a linear operator. This means that the real part and the imaginary part of a solution are also solutions of the equation (2.33). Thus

$$e^{at} \cos bt, e^{at} \sin bt$$

are two linearly independent solutions of (2.33), where a and b are the real and imaginary parts of the complex root respectively. The general solution is given by

$$e^{at}[c_1 \cos bt + c_2 \sin bt], \quad t \in I.$$

Case 3: When the roots of the characteristic equation (2.35) are equal, then the root is $\lambda_1 = -a_1/2a_0$. From Theorem 2.5.1, we do have a solution of (2.33) namely $e^{\lambda_1 t}$. To find a second solution two methods are described below, one of which is based on the method of variation of parameters.

Method 1: $x_1(t) = e^{\lambda_1 t}$ is a solution and so is $ce^{\lambda_1 t}$ where c is a constant. Now let us assume that

$$x_2(t) = u(t) e^{\lambda_1 t},$$

is yet another solution of (2.33) and then determine u. Let us recall here that actually the parameter c is being varied in this method and hence method is called Variation parameters. Differentiating x_2 twice and substitution in (2.33) leads to

$$a_0 u'' + (2a_0 \lambda_1 + a_1) u' + (a_0 \lambda_1^2 + a_1 \lambda_1 + a_2) u = 0.$$

Since $\lambda_1 = -a_1/2a_0$ the coefficients of u' and u are zero. So u satisfies the equation $u'' = 0$, whose general solution is

$$u(t) = c_1 + c_2(t), \quad t \in I,$$

where c_1 and c_2 are some constants or equivalently $(c_1+c_2t)e^{\lambda_1 t}$ is another solution of (2.33). It is easy to verify that

$$x_2(t) = te^{\lambda_1 t}$$

is a solution of (2.33) and x_1, x_2 are linearly independent.

Method 2: Recall

$$L(e^{\lambda t}) = (a_0\lambda^2 + a_1\lambda + a_2)e^{\lambda t} = p(\lambda)e^{\lambda t}, \tag{2.37}$$

where $p(\lambda)$ denotes the characteristic polynomial of (2.33). From the theory of equations we know that if λ_1 is a repeated root of $p(\lambda) = 0$ then

$$p(\lambda_1) = 0 \text{ and } \left|\frac{\partial}{\partial\lambda}p(\lambda)\right|_{\lambda=\lambda_1} = 0. \tag{2.38}$$

Differentiating (2.37) partially with respect to λ, we end up with

$$\frac{\partial}{\partial\lambda}L(e^{\lambda t}) = \frac{\partial}{\partial\lambda}p(\lambda)e^{\lambda t} = \Big[\frac{\partial}{\partial\lambda}p(\lambda) + tp(\lambda)\Big]e^{\lambda t}.$$

But,

$$\frac{\partial}{\partial\lambda}L(e^{\lambda t}) = L\big(\frac{\partial}{\partial\lambda}e^{\lambda t}\big) = L(te^{\lambda t}).$$

Therefore,

$$L(te^{\lambda t}) = \Big[\frac{\partial}{\partial\lambda}p(\lambda) + tp(\lambda)\Big]e^{\lambda t}.$$

Substituting $\lambda = \lambda_1$ and using the relation in (2.38) we have $L(te^{\lambda_1 t}) = 0$ which clearly shows that $x_2(t) = te^{\lambda_1 t}$ is yet another solution of (2.34). Since x_1, x_2 are linearly independent, the general solution of (2.33) is given by

$$c_1e^{\lambda_1 t} + c_2te^{\lambda_1 t},$$

where λ_1 is the repeated root of characteristic equation (2.35).

Example 2.5.4. The characteristic equation of

$$x'' + x' - 6x = 0, \; t \in I,$$

is

$$p^2 + p - 6 = 0,$$

whose roots are $p = -3$ and $p = 2$. by case 1, e^{-3t}, e^{2t} are two linearly independent solutions and the general solution x is given by

$$x(t) = c_1e^{-3t} + c_2e^{2t}, \; t \in I.$$

Example 2.5.5. For

$$x'' - 6x' + 9x = 0, \ t \in I,$$

the characteristic equation is

$$p^2 - 6p + 9 = 0,$$

which has a repeated root $p = 3$. So (by case 2) e^{3t} and te^{3t} are two linearly independent solutions and the general solution x is

$$x(t) = c_1 e^{3t} + c_2 t e^{3t}, \quad t \in I.$$

The results which have been discussed above for a second order have an immediate generalization to a n-th order equation (2.34). The characteristic equation of (2.34) is given by

$$L(p) = a_0 p^n + a_1 p^{n-1} + \cdots + a_n = 0. \tag{2.39}$$

If p_1 is a real root of (2.39) then, $e^{p_1 t}$ is a solution of (2.34). If p_1 happens to be a complex root, the complex conjugate of p_1 *i.e.*, $\bar{p}_1$ is also a root of (2.39). In this case

$$e^{at}\cos bt \ \text{ and } e^{at}\sin bt$$

are two linearly independent solutions of (2.34), where a and b are the real and imaginary parts of p_1, respectively.

We now consider when roots of (2.39) have multiplicity(real or complex). There are two cases:

(i) when a real root has a multiplicity m_1,

(ii) when a complex root has a multiplicity m_1.

Case 1: Let q be the real root of (2.39) with the multiplicity m_1. By induction we have m_1 linearly independent solutions of (2.34), namely

$$e^{qt}, te^{qt}, t^2e^{qt}, \cdots, t^{m_1-1}e^{qt}.$$

Case 2: Let s be a complex root of (2.39) with the multiplicity m_1. Let $s = s_1 + is_2$. Then, as in Case 1, we note that

$$e^{st}, te^{st}, \cdots, t^{m_1-1}e^{st}, \tag{2.40}$$

are m_1 linearly independent complex valued solutions of (2.34). For (2.34), the real and imaginary parts of each solution given in (2.40) is also a solutions of (2.34). So in this case $2m_1$ linearly independent solutions of (2.34) are given by

$$\left.\begin{array}{c} e^{s_1 t}\cos s_2 t, \quad e^{s_1 t}\sin s_2 t \\ te^{s_1 t}\cos s_2 t, \quad te^{s_1 t}\sin s_2 t \\ t^2e^{s_1 t}\cos s_2 t, \quad t^2e^{s_1 t}\sin s_2 t \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t^{m_1-1}e^{s_1 t}\cos s_2 t, \ t^{m_1-1}e^{s_1 t}\sin s_2 t \end{array}\right\} \tag{2.41}$$

Thus, if all the roots of the characteristic equation (2.39) are known, no matter whether they are simple or multiple roots, there are n linearly independent solutions and the general solution of (2.34) is

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

where $x_1, x_2, \cdots, x_n$ are n linearly independent solutions and $c_1, c_2, \cdots, c_n$ are any constants.
To summarize :

Theorem 2.5.6. *Let $r_1, r_2, \cdots, r_s$, where $s \le n$ be the distinct roots of the characteristic equation* (2.39) *and suppose the root r_i has multiplicity m_i, $\;i = 1, 2, \cdots, s$, with*

$$m_1 + m_2 + \cdots + m_s = n.$$

Then, the n functions

$$\left.\begin{array}{l} e^{r_1t}, te^{r_1t}, \cdots, t^{m_1-1}e^{r_1t} \\ e^{r_2t}, te^{r_2t}, \cdots, t^{m_2-1}e^{r_2t} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e^{r_st}, te^{r_st}, \cdots, t^{m_s-1}e^{r_st} \end{array}\right\} \tag{2.42}$$

are the solutions of $L(x) = 0$ for $t \in I$.

EXERCISES

1. Find the general solution of
   (i) $x^{(4)} - 16 = 0$,
   (ii) $x''' + 3x'' + 3x' + x = 0$,
   (iii) $x'' + ax' + bx = 0$, for some real constants a and b,
   (iv) $x''' + 9x'' + 27x' + 27x = 0$.
2. Find the general solution of
   (i) $x''' + 3x'' + 3x' + x = e^{-t}$,
   (ii) $x'' - 9x' + 20x = t + e^{-t}$,
   (iii) $x'' + 4x = A\sin t + B\cos t$, where A and B are constants.
3. (Method of undetermined coefficients) To find the general solution of a non-homogeneous equation it is necessary to know many times a particular solution of the given equation. The method of undetermined coefficients furnishes one such solution, when the non-homogeneous term happens to be an exponential function, a trigonometric function or a polynomial. Consider an equation with constant coefficients

   $$a_0x'' + a_1x' + a_2x = d(t), \quad a_0 \neq 0, \tag{2.43}$$

   where $d(t) = Ae^{at}$, A and a are given real numbers.

   Let $x_p(t) = Be^{at}$, be a particular solution, where B is undetermined. Then, show that

   $$B = \frac{A}{P(a)}, \quad P(a) \neq 0$$

   where $P(a)$ is the characteristic polynomial. In case $P(a) = 0$, assume that the particular solution is of the form Bte^{at}. Deduce that

   $$B = A/(2a_0a + a_1) = A/P'(a), \quad P'(a) \neq 0.$$

It is also possible that $P(a) = P'(a) = 0$. Now assume the particular solution in the form $x_p(t) = Bt^2e^{at}$. Show that $B = A/2a_0 = A/P''(a)$.

4. Using the method described in Example 2.5.5, find the general solution of

   (i) $x'' - 2x' + x = 3e^{2t}$,

   (ii) $4x'' - 8x' + 5x = e^t$.

5. When $d(t) = A\sin Bt$ or $A\cos Bt$ or their linear combination in equation (2.43), assume a particular solution $x_p(t)$ in the form $x(t) = C\sin Bt + D\cos Bt$. Determine the constants C and D which yield the required particular solution. Find the general solution of

   (i) $x'' - 3x' + 2x = \sin 2t$,

   (ii) $x'' - x' - 2x = 3\cos t$.

6. Solve

   (i)$2x'' + x = 2t^2 + 3t + 1, \quad x(0) = x'(0) = 0$,

   (ii)$x'' + 2x' + 3x = t^4 + 3, \quad x(0) = 0, \quad x'(0) = 1$,

   (iii) $x'' + 3x' = 2t^3 + 5$,

   (iv) $4x'' - x' = 3t^2 + 2t$.

7. Consider an equation with constant coefficients of the form

$$x'' + \alpha x' + \beta x = 0.$$

   (i) Prove that every solution of the above equation approaches zero if and only if the roots of the characteristic equation have strictly negative real parts.

   (ii) Prove that every solution of the above equation is bounded if and only if the roots of the characteristic polynomial has non-positive real parts and roots with zero real part have multiplicity one.

Module 3

System of Linear Differential equations

Lecture 22

3.1 Introduction

The systems of linear differential equations occurs at many branches of engineering and science. Its importance needs very little emphasis. In this module, we try a modest attempt to present the various facets of linear systems. Linear Algebra is a prerequisite. To get a better insight on the calculation of the exponential of a matrix, one needs a knowledge of the Jordan canonical decomposition. We try our best to keep the description as self contained as possible. We do not venture into these proofs of results from Linear Algebra.

3.2 Systems of First Order Equations

In general non-linear differential equation of order one of the form

$$x' = f(t, x), \quad t \in I, \tag{3.1}$$

where I is an interval and where $x : I \to \mathbb{R}$ and $f : I \times \mathbb{R} \to \mathbb{R}$. The first order linear non-homogeneous equation

$$x' + a(t)x = b(t), \quad t \in I, \tag{3.2}$$

is a spacial case of (3.1). In fact, we can think of a more general set-up, where (3.1) and (3.2) are spacial cases.

Let n be a positive integer. Let

$$f_1, f_2, \cdots, f_n : I \times D \to \mathbb{R},$$

be n real valued functions defined on an open connected set $D \subset \mathbb{R}^n$. Consider a system of equations

$$x_i' = f_i(t, x_1, x_2, \cdots, x_n), \quad i = 1, 2, \cdots, n, \tag{3.3}$$

where $x_1, x_2, \cdots, x_n$ are real valued functions to be determined. The existence problem associated with the system (3.3) is to find an interval I and n functions $\phi_1, \phi_2, \cdots, \phi_n$ defined on I such that

(i) $\phi_1'(t), \phi_2'(t), \cdots, \phi_n'(t)$ exists for each $t \in I$,

(ii) $(t, \phi_1(t), \phi_2(t), \cdots, \phi_n(t)) \in I \times D$ for each t in I, and

(iii) $\phi_i'(t) = f_i(t, \phi_1(t), \phi_2(t), \cdots, \phi_n(t)),\ t \in I, i = 1, 2, \cdots, n$

$(\phi_1, \phi_2, \cdots, \phi_n)$ is called a solution of system (3.3).

Definition 3.2.1. Suppose $(t_0, \alpha_1, \alpha_2, \cdots, \alpha_n)$ is a point in $I \times D$. Then, the IVP for the system (3.3) is to find a solution $(\phi_1, \phi_2, \cdots, \phi_n)$ of (3.3) such that

$$\phi_i(t_0) = \alpha_i, i = 1, 2, \cdots, n.$$

The system of n equations has a concise form if we use vector notation. Let x denote a point in an n-dimensional real Euclidean space with co-ordinates $(x_1, x_2, \cdots, x_n)$. Define

$$f_i(t, x) = f_i(t, x_1, x_2, \cdots, x_n),\ i = 1, 2, \cdots, n.$$

The equation (3.3) can be written as

$$x_i' = f_i(t, x), \quad i = 1, 2, \cdots, n. \tag{3.4}$$

Now define a vector f by

$$f(t, x) = (f_1(t, x), f_2(t, x), \cdots, f_n(t, x)).$$

With this notation, system (3.4) assumes the form

$$x' = f(t, x). \tag{3.5}$$

We note that the equation (3.1) and (3.5) looks alike except for notations. The system (3.5) is (3.1), when $n = 1$.

Example 3.2.2. The system of two equations

$$x_1' = x_1^2,\ x_2' = x_1 + x_2,$$

has the vector form

$$x' = f(t, x),$$

where $x = (x_1, x_2)$ and $f = \big(f_1(x_1, x_2), f_2(x_1, x_2)\big) = (x_1^2, x_1 + x_2)$. Let $\varphi = (\phi_1, \phi_2)$ be the solution of the system with initial conditions

$$\varphi(t_0) = (\phi_1(t_0), \phi_2(t_0)) = (\alpha, \beta), \quad \alpha > 0.$$

The solution φ in this case is

$$\varphi(t) = \big[\phi_1(t), \phi_2(t)\big] = \Big[\frac{\alpha}{1 - \alpha(t - t_0)},\ \beta \exp(t - t_0) + \int_{t_0}^{t} \frac{e^{t-s} ds}{1 - \alpha(s - t_0)}\Big]$$

existing on the interval $t_0 \le t < t_0 + \frac{1}{\alpha}$.

In the above example we have seen a concise way of writing a system of two equations in a vector form. Normally it is useful to use column vectors rather than row vectors (as done in Example 3.2). The column vector representation of x or f is compatible, when linear systems of equations are under focus. Depending on the context we should be able to decipher whether x or f is a row or a column vector. In short, the context clarifies whether x or f is a row or a column vector. Now we concentrate on a linear system of n equations in this chapter. Let $I \subseteq \mathbb{R}$ be an interval. Let the functions $a_{ij}, b_j : I \to \mathbb{R}$, $i, j = 1, 2, \cdots, n$ be given. Consider a system of equations

$$\begin{array}{l} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{array} \qquad (t \in I). \tag{3.6}$$

Equation (3.6) is called a (general) non-homogeneous system of n equations. By defining

$$f_i(t, x_1, x_2, \cdots, x_n) = a_{i1}(t)x_1 + a_{i2}(t)x_2 + \cdots + a_{in}(t)x_n + b_i(t)$$

for $i = 1, 2, \cdots, n$, we see that the system (3.6) is a special case of the system (3.3). Define a matrix $A(t)$ by the relation

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

and the vectors $b(t)$ and $x(t)$ by

$$b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix} \quad \text{and} \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

respectively. With these notations (3.6) is

$$x' = A(t)x + b(t), \quad t \in I. \tag{3.7}$$

It is easy to observe that the right side of system (3.6) is linear in $x_1, x_2, \cdots, x_n$ when $b(t) \equiv 0$. Equation (3.7) is a vector matrix representation of a linear non-homogeneous system (3.6). If $b(t) \equiv 0$ on I, then the system (3.7) reduces to a system

$$x' = A(t)x, \quad t \in I, \tag{3.8}$$

which is called Linear homogeneous systems of equations.

Example 3.2.3. Consider a system of equations

$$\begin{array}{l} x_1' = 5x_1 - 2x_2 \\ x_2' = 2x_1 + x_2 \end{array}$$

which has the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It is easy to verify that a solution is given by

$$x_1(t) = (c_1 + c_2 t)e^{3t}, \quad x_2(t) = (c_1 - \tfrac{1}{2}c_2 + c_2 t)e^{3t}.$$

The n-th Order Equation

let us recall that a general n-th order IVP is

$$x^{(n)} = g(t, x, x', \cdots, x^{(n-1)}), \quad t \in I \tag{3.9}$$

$$x(t_0) = \alpha_0, x'(t_0) = \alpha_1, \cdots, x^{(n-1)}(t_0) = \alpha_{n-1}, \quad t_0 \in I, \tag{3.10}$$

where $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ are given real constants. The n-th order equation can be represented a system of n equations as follows. Define $x_1, x_2, \cdots, x_n$ by

$$x_1 = x, \quad x' = x_2, \cdots, x^{(n-1)} = x_n.$$

Then

$$x_1 = x, \quad x_1' = x_2, \cdots, x_{n-1}' = x_n, \; x_n'(t) = g(t, x_1, x_2, \cdots, x_n) \tag{3.11}$$

Let $\varphi = (\phi_1, \phi_2, \cdots, \phi_n)$ be a solution of (3.11). Then

$$\begin{aligned} \phi_2 = \phi_1', \quad \phi_3 = \phi_2' = \phi_1'', \cdots, \phi_n = \phi_1^{(n-1)}, \\ g(t, \phi_1(t), \phi_2(t), \cdots, \phi_n(t)) &= g(t, \phi_1(t), \phi_1'(t), \cdots, \phi_1^{(n-1)}(t)) \\ &= \phi_1^{(n)}(t). \end{aligned}$$

Clearly the first component ϕ_1 is a solution of (3.9). Conversely, let ϕ_1 be a solution of (3.9) on I then, the vector $\varphi = (\phi_1, \phi_2, \cdots, \phi_n)$ is a solution of (3.11). Thus, the system (3.11) is equivalent to (3.9). Further, if

$$\phi_1(t_0) = \alpha_0, \phi_1'(t_0) = \alpha_1, \cdots, \phi_1^{(n-1)}(t_0) = \alpha_{n-1}$$

then the vector $\varphi(t)$ also satisfies $\varphi(t_0) = \alpha$ where $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$. It is time to observe that (3.11) is a special case of

$$x' = f(t, x).$$

In particular, an equation of n-th order of the form

$$a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + \cdots + a_n(t)x = b(t), \quad t \in I$$

is called a linear non-homogeneous n-th order equation. In case $a_0(t) \neq 0$ for any $t \in I$ then,the above equation is equivalent to

$$x^{(n)} + \frac{a_1(t)}{a_0(t)}x^{(n-1)} + \cdots + \frac{a_n(t)}{a_0(t)}x = \frac{b(t)}{a_0(t)}. \tag{3.12}$$

Now (3.12) is represented in the form of a system by defining

$$x_1 = x, \quad x_1' = x_2, \cdots, x_{n-1}' = x_n$$

$$x_n'(t) = -\frac{a_n(t)}{a_0(t)}x_1 - \frac{a_{n-1}(t)}{a_0(t)}x_2 - \cdots - \frac{a_1(t)}{a_0(t)}x_n + \frac{b(t)}{a_0(t)}.$$

With the notations

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ b(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{b(t)}{a_0(t)} \end{bmatrix}$$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \frac{-a_{n-2}}{a_0} & \cdots & \frac{-a_1}{a_0}. \end{bmatrix}$$

The system (3.12) is

$$x' = A(t)x + b(t), \quad t \in I. \tag{3.13}$$

Thus, the two systems (3.12) and (3.13) are equivalent. The representations (3.7) and (3.13) gives us a considerable simplicity in handling the systems of n equations.

Lecture 15

Example 3.2.4. For illustration we consider a linear equation

$$x''' - 6x'' + 11x' - 6x = 0.$$

Denote

$$x_1 = x, \quad x_1' = x_2 = x', \quad x_2' = x'' = x_3.$$

Then, the given equation is equivalent to the system $x' = A(t)x$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

Notice that the first component x_1 of the system is a required solution of the given equation. It is easy to check, in the present case, that $x_1(t) = c_1e^t + c_2e^{2t} + c_3e^{3t}$, where c_1, c_2 and c_3 are arbitrary constants.

EXERCISES

1. Find a differential system for which the vector $y(t)$ is a solution, where

$$y(t) = \begin{bmatrix} t^2 + 2t + 5 \\ \sin^2 t \end{bmatrix}, \quad t \in I.$$

2. Consider the IVP $x_1' = x_2^2 + 3, x_2' = x_1^2,\ x_1(0) = 0, x_2(0) = 0$. Represent the problem in the vector form

$$x' = f(t, x),\ x(0) = x_0.$$

Show that the vector function f is continuous. Find a value of M such that

$$|f(t, x)| \le M \text{ on } R = \{(t, x) : |t| \le 1, |x| \le 1\}.$$

3. The system of three equations is given by

$$(x_1, x_2, x_3)' = (4x_1 - x_2, 3x_1 + x_2 - x_3, x_1 + x_3).$$

Then,

(i) show that the above system is linear in x_1, x_2 and x_3;

(ii) find the solution of the system.

4. Let f be a vector valued function defined on the rectangle

$$R = \{(t, x) : |t| \leq a, |x| \leq b, a > 0, b > 0, x \in \mathbb{R}^2\},$$

as follows:

$$f(t, x) = (x_1^2 + 3, t + x_2^2).$$

Find an upper bound for $f(t, x)$ on the rectangle.

5. Represent the linear system of equations

$$\begin{aligned} x_1' &= e^{-t}x_1 + \sin tx_2 + tx_3 + \tfrac{1}{t^2+1}, \\ x_2' &= -\cos tx_3 + e^{-2t}, \\ x_3' &= \cos tx_1 + e^{-t}\sin tx_2 + t. \end{aligned}$$

in the vector form.

6. Write the equation

$$(1 + t^2)w''' + \sin tw'' + (1 + t)w' + \cos tw = e^{-2t}\cos t,$$

in the form of a system.

7. Write the system

$$\begin{aligned} u'' + 3v' + 4u + 5v &= 6t, \\ v'' - u' + 4u + v &= \cos t, \end{aligned}$$

in the vector matrix form.

8. Consider the system of equations

$$x_1' + ax_1 + bx_2, \quad x_2' = cx_1 + dx_2,$$

where $a, b, c,$ and d are constants.

(i) Prove that x_1 satisfies the second order equation

$$x_1'' - (a + d)x_1' + (ad - bc)x_1 = 0.$$

(ii) Show that the above equation has a solution of the form

$$\phi(t) = \alpha e^{rt}(\alpha = \text{constant})$$

where r is a root of the equation $r^2 - r(a + d) + ad - bc = 0$.

9. Solve

   (i) $x' = 2x_1 + x_2, \quad x_2' = 3x_1 + 4x_2$;

   (ii) $x_1' + x_1 + 5x_2 = 0, \quad x_2' - x_1 - x_2 = 0$.

10. Show that for any two of differentiable matrices X and Y

    (i) $\frac{d}{dt}(XY) = (\frac{d}{dt}X)Y + X(\frac{d}{dt}Y)$;

    (ii) $\frac{d}{dt}(X^{-1}) = -X^{-1}(\frac{d}{dt}X)X^{-1}$.

3.3 Fundamental Matrix

Many times it is convenient to construct a matrix with solutions of

$$x' = A(t)x, \ t \in I, \tag{3.14}$$

as columns. In other words consider a set of n solutions of the system (3.14) and define a matrix Φ whose columns are these n solutions. This matrix is called a "solution matrix" since it satisfies the matrix differential equation

$$\Phi' = A(t)\Phi, \quad t \in I. \tag{3.15}$$

If the columns are linearly independent the matrix Φ thus obtained is called a *fundamental matrix* for the system (3.14). We associate with system (3.14) a matrix differential equation

$$X' = A(t)X, \quad t \in I. \tag{3.16}$$

Obviously Φ is a solution of (3.16). Once the notion of a fundamental matrix is gained the next question is whether a characterization exists for a solution matrix to be fundamental. The answer is indeed in the affirmative. Before going into the details the following is needed.

Theorem 3.3.1. *Let $A(t)$ be $n \times n$ matrix which is continuous on I. Suppose a matrix Φ satisfies* (3.16)*. Then* $\det \Phi$ *satisfies the first order equation*

$$(\det \Phi)' = (trA)(\det \Phi), \tag{3.17}$$

or in other words, for $\tau \in I$,

$$\det \Phi(t) = \det \Phi(\tau) \exp \int_\tau^t trA(s)ds. \tag{3.18}$$

Proof. By definition the n columns of Φ are n solutions $\varphi_1, \varphi_2, \cdots, \varphi_n$ of (3.14). Denote

$$\varphi_i = \{\phi_{i1}, \phi_{i2}, \cdots, \phi_{in}\}, \quad i = 1, 2, \cdots, n.$$

Let $a_{ij}(t)$ be the (i, j)-th element of $A(t)$. Then,

$$\phi_{ij}'(t) = \sum_{k=1}^{n} a_{ik}(t)\phi_{kj}(t); \quad i, j = 1, 2, \cdots, n \tag{3.19}$$

Now

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}$$

and so it is seen that

$$(\det \Phi)' = \begin{vmatrix} \phi'_{11} & \phi'_{12} & \cdots & \phi'_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi'_{21} & \phi'_{22} & \cdots & \phi'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi'_{n1} & \phi'_{n2} & \cdots & \phi'_{nn} \end{vmatrix}$$

Substituting the values of $\phi'_{11}, \phi'_{12}, \cdots \phi'_{1n}$ from (3.19), the first term on the right side of the above equation reduces to

$$\begin{vmatrix} \sum_{k=1}^{n} a_{1k}\phi_{k1} & \sum_{k=1}^{n} a_{1k}\phi_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}\phi_{kn} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

which is $a_{11}\det\Phi$. Carrying this out for the remaining terms it is seen that

$$(\det\ \Phi)' = (a_{11} + a_{22} + \cdots + a_{nn})\det\Phi = (trA)\det\Phi.$$

The equation thus obtained is a linear differential equation. The proof of the theorem is complete since it is known that the solution of this equation is given by (3.18). □

Theorem 3.3.2. *A solution matrix* Φ *of* (3.16) *on* I *is a fundamental matrix of* (3.14) *on* I *if and only if* $\det\Phi \neq 0$ *for* $t \in I$.

Proof. Let $\Phi(t)$ be a solution matrix such that $\det\Phi(t) \neq 0, t \in I$. Then, the columns of Φ are linearly independent on I. Hence, Φ is a fundamental matrix.

Conversely, let $\Phi(t)$ be a fundamental matrix and let $\varphi_j, j = 1, 2, \cdots, n$ be the columns of Φ. Let φ be any nonzero solution of (3.14). Then there exist constants $c_1, c_2, \cdots, c_n$ not all zero, such that

$$\varphi = \sum_{i=1}^{n} c_i\varphi_i = \Phi \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \Phi c, \text{ where } \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

This is a system of linear equations for the unknowns $c_1, c_2, \cdots, c_n$. For a fixed $\tau \in I$ the above system has a solution and hence $\det\Phi(\tau) \neq 0$. Now from Theorem 3.3.1 it is clear that $\det\Phi(t) \neq 0, t \in I$, which completes the proof. □

Some of the useful properties of the fundamental matrix are established in the following results.

Theorem 3.3.3. *Let Φ be a fundamental matrix for the system* (3.14) *and let C be a constant non-singular matrix. Then, ΦC is also a fundamental matrix for* (3.14)*. In addition, every fundamental matrix of* (3.14) *is of this type for some non-singular matrix C.*

Proof. The first part of the theorem is a single consequence of Theorem 3.3.2 and the fact that the product of non-singular matrices is non-singular.

Let Φ_1 and Φ_2 be two fundamental matrices for (3.14) and let $\Phi_2 = \Phi_1\Psi$. Then $\Phi_2' = \Phi_1\Psi' + \Phi_1'\Psi$. Equation (3.16) now implies that $A\Phi_2 = \Phi_1\Psi' + A\Phi_1\Psi = \Phi_1\Psi' + A\Phi_2$. Thus it is seen that $\Phi_1\Psi' = 0$ which shows that $\Psi' = 0$. Hence $\Psi = C$, where C is a constant matrix. Since Φ_1 and Φ_2 are non-singular so is C. □

Lecture 16

We consider now a special case of the linear homogeneous system (3.14). Suppose the matrix $A(t)$ is a constant matrix. A consequence of this assumption is that the fundamental matrix satisfies the law of exponentials.

Theorem 3.3.4. *Let $\Phi(t), t \in I$, denote a fundamental matrix of the system*

$$x' = Ax, \tag{3.20}$$

such that $\Phi(0) = E$, where A is a constant matrix. Here E denotes the identity matrix. Then, Φ satisfies

$$\Phi(t+s) = \Phi(t)\Phi(s), \tag{3.21}$$

for all values of t and $s \in I$.

Proof. By the uniqueness theorem there exists a unique fundamental matrix $\Phi(t)$ for the given system such that $\Phi(0) = E$. It is to be noted here that $\Phi(t)$ satisfies the matrix equation

$$X' = AX \tag{3.22}$$

Define for any real number s,

$$Y(t) = \Phi(t+s)$$

Then,

$$Y'(t) = \Phi'(t+s) = A\Phi(t+s) = AY(t).$$

Hence $Y(t)$ is a solution of the matrix equation (3.22) such that $Y(0) = \Phi(s)$. Now suppose $Z(t) = \Phi(t)\Phi(s)$, for all t and s. Observe that $Z(t)$ is solution of (3.22). Clearly $Z(0) = \Phi(0)\Phi(s) = E\Phi(s) = \Phi(s)$. So there are two solutions $Y(t)$ and $Z(t)$ of (3.22) such that $Y(0) = Z(0) = \Phi(s)$. By uniqueness property therefore it must be seen that $Y(t) \equiv Z(t)$, whence the relation (3.21). The proof of the theorem is complete. □

Example 3.3.5. Consider the linear system (3.14) where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A(t) = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

It is easy to verify that the matrix

$$\Phi(t) = \begin{bmatrix} e^{-3t} & te^{-3t} & e^{-3t}(t^2/2!) \\ 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

is a fundamental matrix.

EXERCISES

1. If Φ is a fundamental matrix for (3.14) and C is any constant non-singular matrix then show that $C\Phi$ is not, in general, a fundamental matrix.

2. Let $\Phi(t)$ be a fundamental matrix for the system (3.14), where $A(t)$ is a real matrix. Then, show that the matrix $(\Phi^{-1}(t))^T$ satisfies the equation

$$\frac{d}{dt}(\Phi^{-1})^T = -A^T(\Phi^{-1})^T,$$

and hence, show that $(\Phi^{-1})^T$ is a fundamental matrix for the system

$$x' = -A^T(t)x, \; t \in I. \tag{3.23}$$

System (3.23) is called the "adjoint" system to (3.14) and vice versa.

3. Let Φ be a fundamental matrix for Eq.(3.14), with $A(t)$ being a real matrix. Then, show that Ψ is a fundamental matrix for its adjoint (3.23) if and only if $\Psi^T\Phi = C$, where C is a constant non-singular matrix.

4. Consider a matrix P defined by

$$P(t) = \begin{bmatrix} f_1(t) & f_2(t) \\ 0 & 0 \end{bmatrix}, \; t \in I,$$

where $f_1(t)$ and $f_2(t)$ are any two linearly independent functions on I. Then, show that $\det[P(t)] \equiv 0$ on I, but the columns of $P(t)$ are linearly independent. Can it be then concluded that the columns of matrices of the type $P(t)$ cannot be solutions of linear homogeneous systems of equations of the form (3.14) in the light of Theorem 4.4?

5. Find the determinant of fundamental matrix $\Phi(t)$ which satisfies $\Phi(0) = E$ for the system (3.20) where

(a)

$$A = \begin{bmatrix} -1 & 3 & 4 \\ 0 & 2 & 0 \\ 1 & 5 & -1 \end{bmatrix};$$

(b)

$$A = \begin{bmatrix} 1 & 3 & 8 \\ -2 & 2 & 1 \\ -3 & 0 & 5 \end{bmatrix}$$

6. Can the following matrices $\Phi(t)$ be candidates for fundamental matrices for some linear system of the form

$$x' = A(t)x,\ t \in I,$$

where $A(t)$ is a matrix continuous in $t \in I$? If not, why ?

(i)

$$\Phi(t) = \begin{bmatrix} e^t & 1 & 0 \\ 1 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii)

$$\Phi(t) = \begin{bmatrix} 1 & t & t^2 \\ 0 & e^{2t} & 1 \\ 0 & 1 & e^{-2t} \end{bmatrix}.$$

Lecture

3.4 Non-homogeneous linear Systems

Assume in this section that $A(t)$ is an $n \times n$ matrix that is continuous on I. The system

$$x' = A(t)x + b(t), \quad t \in I, \tag{3.24}$$

is called a non-homogeneous linear system of order n. Here b is a continuous function defined on I and taking values in $\mathbb{R}^n$. An inspection shows that if $b(t) \equiv 0$, then (3.24) reduces to (3.14). The term $b(t)$ in (3.24) often goes by the name "forcing term" or "perturbation" for the system (3.14). The system (3.24) is a perturbed state of (3.14). The nature of the solution of (3.24) is quite closely connected with the solution of (3.14) and to some extent it is brought out in this section. Before proceeding further, it may be remarked here that the continuity of A and b ensures the existence and uniqueness of a solution for IVP on I for the system (3.24). The proof is postponed for the present and is dealt with in Module 4.

To express the solution (3.24) in term of (3.14) it becomes necessary to resort to the method of variation of parameters. Let $\Phi(t)$ be a fundamental matrix for the system (3.14) on I. Let $\Psi(t)$ be a solution of (3.24) such that for some $t_0 \in I, \psi(t_0) = 0$. Now let it be assumed that $\psi(t)$ is given by

$$\psi(t) = \Phi(t)u(t), \quad t \in I, \tag{3.25}$$

where $u(t)$ is an unknown vector function mapping I into $\mathbb{R}^n$ such that $u(t)$ is differentiable and $u(t_0) = 0$. The solution ψ is determined by finding $u(t)$ in terms of known quantities $\Phi(t)$ and $b(t)$. Substituting (3.25) in (3.24) notice that for $t \in I$,

$$\psi'(t) = \Phi'(t)u(t) + \Phi(t)u'(t) = A(t)\Phi(t)u(t) + \Phi(t)u'(t).$$

It is also seen that

$$\psi'(t) = A(t)\psi(t) + b(t) = A(t)\Phi(t)u(t) + b(t).$$

Equating the two expressions for $\psi'(t)$ it is concluded that $\Phi(t)u'(t) = b(t)$. Note that $\Phi(t)$, being a fundamental matrix, is non-singular on I and so

$$u'(t) = \Phi^{-1}(t).b(t)$$

or

$$u(t) = 0 + \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds, \quad t, t_0 \in I \tag{3.26}$$

Substituting the value of $u(t)$ in (3.25), we get,

$$\psi(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)b(s)ds, \quad t \in I \tag{3.27}$$

It can easily be verified that (3.27) is indeed a solution of (3.24). This discussion so far is now summed up in Theorem 3.4.1.

Theorem 3.4.1. *Let $\Phi(t)$ be a fundamental matrix for the system* (3.14) *for $t \in I$. Then ψ, defined by* (3.27)*, is a solution of the IVP*

$$x' = A(t)x + b(t), \; x(t_0) = 0. \tag{3.28}$$

Now let us assume that $x_h(t)$ is the solution of the IVP

$$x' = A(t)x,\ x(t_0) = c,\ \ t, t_0 \in I. \tag{3.29}$$

Then, a consequence of Theorem 3.4.1 is that

$$\psi(t) = x_h(t) + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds, \quad t \in I \tag{3.30}$$

is a solution of

$$x' = A(t)x + b(t);\ x(t_0) = c.$$

Thus with a prior knowledge of the solution of (3.29), the solution of (3.28) is computable from (3.30).

EXERCISES

1. Prove that the equation (3.27) can also be written as
   (i) $\Psi(t) = \Phi(t) \int_{t_0}^{t} \Psi^T(s)b(s)ds, \ \ t \in I$ provided $\Psi^T(t)\Phi(t) = E$;
   (ii) $\Psi(t) = (\Psi^{-1})^T \int_{t_0}^{t} \Psi^T(s)b(s)ds, \ \ t \in I$, where Ψ is a fundamental matrix for the adjoint system $x' = -A^T(t)x$. Assume that $A(t)$ is a real matrix.

2. Consider the system $x' = Ax + b(t)$, where

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } b(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}.$$

   Show that

$$\Phi(t) = \begin{bmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

   is a fundamental matrix of $x' = Ax$. Compute the solution $y(t)$ of the non-homogeneous system for which $y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Consider the system $x' = Ax$ given that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.Show that a fundamental matrix is $\Phi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$. Let $b(t) = \begin{bmatrix} \sin at \\ \cos bt \end{bmatrix}$. Find the solution $\Psi(t)$ of the non-homogeneous equation $x' = Ax + b(t)$ for which $\Psi(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Lecture 17

3.5 Linear Systems with Constant Coefficients

In previous sections, the existence and uniqueness of solutions of linear systems of the type

$$x' = A(t)x,\ x(t_0) = x_0,\ \ t, t_0 \in I, \tag{3.31}$$

has been proved. However, when trying to find the solution of such systems in an explicit form several difficulties are encountered. In fact, there are very few situation when the solution can be found explicitly. The aim of this article is to develop a method to find the solution of (3.31) with the assumption that $A(t)$ is a constant matrix. The method involves first finding the characteristic values of the matrix A. If the characteristic values of the matrix A are known then, in general, a solution can be obtained in an explicit form. Note that when the matrix $A(t)$ is variable, it is usually difficult to find solutions.

Before proceeding further, recall the definition of the exponential of a given-matrix A. It is defined as follows:

$$\exp A = E + \sum_{p=1}^{\infty} \frac{A^p}{p!}$$

Also, if A and B are two matrices which commute then,

$$\exp(A + B) = \exp A . \exp B$$

For the present assume the proofs of the convergence of the sum through which $\exp A$ is defined and the result stated above. So by definition

$$\exp(tA) = E + \sum_{p=1}^{\infty} \frac{t^p A^p}{p!},\ t \in I$$

Here it is noted that the infinite series for $\exp(tA)$ converges uniformly on every compact interval of I.

Now consider a linear homogeneous system with a constant matrix, namely,

$$x' = Ax, \quad t \in I, \tag{3.32}$$

where I is an interval in $\mathbb{R}$. From Module 1 recall that the solution of (3.32), when A and x are scalars, is $x(t) = ce^{tA}$ for an arbitrary constant c. A similar situation prevails when we deal with (3.32). This leads to Theorem 3.5.1.

Theorem 3.5.1. *The general solution of the system* (3.32) *is* $x(t) = e^{tA}c$*, where* c *is an arbitrary constant column matrix. Further, the solution of* (3.32) *with the initial condition* $x(t_0) = x_0, t_0 \in I$*, is*

$$x(t) = e^{(t-t_0)A}x_0, \quad t \in I \tag{3.33}$$

Proof. Let $x(t)$ be any solution of (3.32). Define a vector $u(t)$ by, $u(t) = e^{-tA}x(t), \ \ t \in I$. Then, it follows that

$$u'(t) = e^{-tA}(-Ax(t) + x'(t)), \ \ t \in I.$$

Since x is a solution of (3.32) it is easy to observe that $u'(t) \equiv 0$, which means that $u(t) = c, t \in I$, where c is some constant vector. Substituting the value c for $u(t)$, it is seen that $x(t) = e^{tA}c$. Clearly $c = e^{-t_0 A}x_0$, and so we have $x(t) = e^{tA}e^{-t_0 A}x_0, t \in I$. Since A commutes with itself, it is seen that $e^{tA}e^{-t_0 A} = e^{(t-t_0)A}$, and thus, (3.33) follows. This completes the proof. $\square$

In particular, let us choose $t_0 = 0$ and n linearly independent vectors $e_j, j = 1, 2, \cdots, n$, the vector e_j being the vector with 1 at the jth component and zero elsewhere. In this case, we get n linearly independent solutions corresponding to the set of n vectors $(e_1, e_2, \cdots, e_n)$. Thus a fundamental matrix for (3.32) is

$$\Phi(t) = e^{tA}E = e^tA, \quad t \in I, \tag{3.34}$$

since the matrix with columns represented by $e_1, e_2, \cdots, e_n$ is the identity matrix E. Thus e^{tA} solves the matrix differential equation

$$X' = AX, \quad x(0) = E; \quad t \in I. \tag{3.35}$$

Example 3.5.2. Find a fundamental matrix for the system $x' = Ax$, where

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

where α_1,α_2 and α_3 are scalars.

The fundamental matrix is e^{tA}. It is very easy to verify that

$$A^k = \begin{bmatrix} \alpha_1^k & 0 & 0 \\ 0 & \alpha_2^k & 0 \\ 0 & 0 & \alpha_3^k \end{bmatrix}$$

Hence,

$$e^{tA} = \begin{bmatrix} \exp \alpha_1 t & 0 & 0 \\ 0 & \exp \alpha_2 t & 0 \\ 0 & 0 & \exp \alpha_3 t \end{bmatrix}.$$

Example 3.5.3. Consider a similar example to determine a fundamental matrix for $x' = Ax$, where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$. Notice that

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}.$$

By the remark which followed Theorem 3.5.1, it is known that the fundamental matrix in this case is given by

$$\exp(tA) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} t. \exp \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} t,$$

since $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$ commute. But

$$\exp \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} t = \exp \begin{bmatrix} 3t & 0 \\ 0 & 3t \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

It is left as an exercise to the readers to verify that

$$\exp\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} t = \frac{1}{2}\begin{bmatrix} e^{2t}+e^{-2t} & e^{-2t}-e^{2t} \\ e^{-2t}-e^{2t} & e^{2t}+e^{-2t} \end{bmatrix}.$$

Thus $e^{tA} = \frac{1}{2}\begin{bmatrix} e^{5t}+e^{t} & e^{t}-e^{5t} \\ e^{t}-e^{5t} & e^{5t}+e^{t} \end{bmatrix}$.

From Theorem 3.5.1 we know that the general solution of the system (3.32) is $e^{tA}c$ but we have still not computed e^{tA}. Once e^{tA} determined, the solution of (3.32) is completely determined.

In order to be able to do this the procedure given below is followed. Choose a solution of (3.32) in the form

$$x(t) = e^{\lambda t}c, \tag{3.36}$$

where c is a constant vector and λ is a scalar. x is determined if λ and c are known. Substituting (3.36) in (3.32), we get

$$(\lambda E - A)c = 0. \tag{3.37}$$

which is a system of algebraic homogeneous linear equations for the unknown c. The system (3.37) has a non-trivial solution c if and only if λ satisfies $\det(\lambda E - A) = 0$. Let

$$P(\lambda) = \det(\lambda E - A).$$

Actually $P(\lambda)$ is a polynomial of degree n normally called the "characteristic polynomial" of the matrix A and the equation

$$P(\lambda) = 0 \tag{3.38}$$

is called the "characteristic equation" for A. Since (3.38) is an algebraic equation, it admits n roots which may be distinct, repeated or complex. The roots of (3.38) are called the "eigenvalues" or the "characteristic values" of A. Let λ_1 be an eigenvalue of A and corresponding to this eigen value, let c_1 be the non-trivial solution of (3.37). The vector c_1 is called an "eigenvector" of A corresponding to the eigenvalue λ_1. Note that any nonzero constant multiple of c_1 is also an eigenvector corresponding to λ_1. Thus, if c_1 is an eigenvector corresponding to an eigenvalue λ_1 of the matrix A then,

$$x_1(t) = e^{\lambda_1 t}c_1$$

is a solution of the system (3.32). Let the eigenvalues of A be $\lambda_1, \lambda_2, \cdots, \lambda_n$(not necessarily distinct) and let $c_1, c_2, \cdots, c_n$ be linearly independent eigenvectors corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively. Then, it is clear that

$$x_k(t) = e^{\lambda_k t}c_k (k = 1, 2, \cdots, n),$$

are n linearly independent solutions of the system (3.32). Here we stress that the eigenvectors corresponding to the eigenvalues are linearly independent. Thus, $\{x_k\}, k = 1, 2, \cdots, n$ is a set of n linearly independent solutions of (3.32). So by the principle of superposition the general solution of the linear system is

$$x(t) = \sum_{k=1}^{n} e^{\lambda_k t}c_k. \tag{3.39}$$

Now let Φ be a matrix whose columns are the vectors

$$e^{\lambda_1 t}c_1, e^{\lambda_2 t}c_2, \cdots, e^{\lambda_n t}c_n$$

So by construction Φ has n linearly independent columns which are solutions of (3.32) and hence, Φ is a fundamental matrix. Since e^{tA} is also a fundamental matrix, from Theorem 3.4, we therefore have

$$e^{tA} = \Phi(t)D,$$

where D is some non-singular constant matrix. A word of caution is warranted namely that the above discussion is based on the assumption that the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are linearly independent.

Example 3.5.4. Let

$$x' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} x.$$

The characteristic equation is given by

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

whose roots are

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3.$$

Also the corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix},$$

respectively. Thus, the general solution of the system is

$$x(t) = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} e^{2t} + \alpha_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t}$$

where α_1, α_2 and α_3 are arbitrary constants. Also a fundamental matrix is

$$\begin{bmatrix} \alpha_1 e^t & 2\alpha_2 e^{2t} & \alpha_3 e^{3t} \\ \alpha_1 e^t & 4\alpha_2 e^{2t} & 3\alpha_3 e^{3t} \\ \alpha_1 e^t & 8\alpha_2 e^{2t} & 9\alpha_3 e^{3t} \end{bmatrix}.$$

Lecture 18

When the eigenvalues of A are not distinct, the problem of finding a fundamental matrix is not that easy. The next step is to find the nature of the fundamental matrix in the case of repeated eigenvalues of A. Let $\lambda_1, \lambda_2, \cdots, \lambda_n (m < n)$ be the distinct eigenvalues of A with multiplicities $n_1, n_2, \cdots, n_m$, respectively, where $n_1 + n_2 + \cdots + n_m = n$. Consider the system of equations, for an eigenvalue λ_i with multiplicity n_i,

$$(\lambda_i E - A)^{n_i} x = 0, \quad i = 1, 2, \cdots, m. \tag{3.40}$$

Let X_i be the subspace of $\mathbb{R}^n$ generated by the solutions of the system (3.40) for each $\lambda_i,\ i = 1, 2, \cdots, m$. From linear algebra it is known that for any $x \in \mathbb{R}^n$, there exist unique vectors $y_1, y_2, \cdots, y_m$, where $y_i \in X_i, (i = 1, 2, \cdots, m)$, such that

$$x = y_1 + y_2 + \cdots + y_m. \tag{3.41}$$

It is common in linear algebra to speak of $\mathbb{R}^n$ as a "direct sum" of the subspaces $X_1, X_2, \cdots, X_m$.

Consider the problem of determining e^{tA} discussed earlier. Let x be a solution of (3.32) with $x(0) = \alpha$. By the result which was quoted, unique vectors $\alpha_1, \alpha_2, \cdots, \alpha_m$ are obtained, such that

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m.$$

It is also known from Theorem 3.5.1 that the solution $x(t)$ of (3.32) with $x(0) = \alpha$ is

$$x(t) = e^{tA}\alpha = \sum_{i=1}^{m} e^{tA}\alpha_i$$

But,

$$e^{tA}\alpha_i = \exp(\lambda_i t)\exp[t(A - \lambda_i E)]\alpha_i$$

By the definition of the exponential function, we get

$$e^{tA}\alpha_i = \exp(\lambda_i t)[E + t(A - \lambda_i E) + \cdots + \frac{t^{n_i - 1}}{(n_i - 1)!}(A - \lambda_i E)^{n_i - 1} + \cdots]\alpha_i.$$

It is to be noted here that the terms of form

$$(A - \lambda_i E)^k \alpha_i = 0 \ \text{ if } k \geq n_i,$$

because recall that the subspace X_i is generated by the vectors, which are solutions of $(A - \lambda_i E)^{n_i} x = 0$, and that $\alpha_i \in X_i, i = 1, 2, \cdots, m$. Thus,

$$x(t) = e^{tA}\sum_{i=1}^{m}\alpha_i = \sum_{i=1}^{m}\exp(\lambda_i t)\Big[\sum_{j=0}^{n_i - 1}\frac{t^j}{j!}(A - \lambda_j E)^j\Big]\alpha_j, \quad t \in I. \tag{3.42}$$

Indeed one might wonder whether (3.42) is the desired solution. To start with we were aiming at $\exp(tA)$ but all we have in (3.42) is $\exp(tA).\alpha$, where α is an arbitrary vector. But a simple consequence of (3.42) is the deduction of $\exp(tA)$ which is done as follows. Note that

$$\begin{aligned}\exp(tA) &= \exp(tA)E \\ &= [\exp(tA)e_1, \exp(tA)e_2, \cdots, \exp(tA)e_n].\end{aligned}$$

$\exp(tA)e_i$ can be obtained from (3.42), $i = 1, 2, \cdots, n$ and hence $\exp(tA)$ is determined. It is important to note that (3.42) is useful provided all the eigenvalues are known along with their multiplicities.

Example 3.5.5. Let $x' = Ax$ where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic equation is given by

$$\lambda^3 = 0.$$

whose roots are

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Since the rank of the co-efficient matrix A is 2, there is only one eigenvector namely

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The other two generalized eigenvectors are determined by the solution of

$$A^2x = 0 \quad \text{and} \quad A^3x = 0.$$

The other two generalized eigenvectors are

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Since

$$A^3 = 0,$$

$$e^{At} = I + At + \frac{A^2t^2}{2}$$

or

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & t & 0 \end{bmatrix}.$$

We leave it as exercice to find the e^{At} given

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Lecture 19

In this elementary study, we wish to draw the phase portraits for a system of two linear ordinary differential equations. In order to make life easy, we first go through a bit of elementary linear algebra.Parts A and B are more or less a revision ,,which hopefully helps the readers to draw the phase portraits. We may skip Parts A and B in case we are familiar with curves and elementary canonical forms for real matrices.

Part A: Preliminaries.

$\mathbb{R}$ denotes the real line. By $\mathbb{R}^n$, we mean the standard or the usual Euclidean space of dimension n, $n \geq 1$. A $n \times n$ matrix A is denoted by $(a_{ij})_{n\times n}$, $a_{ij} \in \mathbb{R}$. The set of all such real matrices A is denoted by $M_n(\mathbb{R})$. A also induces a linear operator on $\mathbb{R}^n$(now understood as column vectors) defined as $x \xrightarrow{A} A(x)$ or $A\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by $A(x) = Ax$(matrix multiplication). The set of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ is denoted by $L(\mathbb{R}^n)$. For a $n \times n$ real matrix A, we some times use $A \in M_n(\mathbb{R})$ or $A \in L(\mathbb{R}^n)$ if there is no confusion . Let $T \in L(\mathbb{R}^n)$. The $Ker(T)$ or $N(T)$ (read as kernel of T or Null space of T) is defined as

$$Ker(T) = N(T) \colon = \{x \in \mathbb{R}^n \colon Tx = o\}$$

The dimension of $Ker(T)$ is called the nullity of T and is denoted by $\nu(T)$. The dimension of range of T is called the rank of T and is denoted by $\rho(T)$. If $T \in L(\mathbb{R}^n)$, then the Rank Nullity Theorem asserts

$$\nu + \rho = n.$$

Consequently for $T \in L(\mathbb{R}^n)$ (i.e. $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear.) T is one-one iff T is onto. Let us now prove the following result.

1. **Theorem** : Given $T \in L(\mathbb{R}^n)$ and given $t_0 \geq 0$, the series

$$\sum_{k=0}^{\infty} \frac{T^k}{k!} t^k$$

is absolutely and uniformly convergent for all $|t| \leq t_0$.
Proof : We let $\| T \| = a$. We know

$$\| \frac{T^k\, t^k}{k!} \| \leq \frac{a^k\, t_0^k}{k!}$$

and

$$\sum_{k=0}^{\infty} \frac{a^k\, t_0^k}{k!} = e^{a\, t_0}$$

By comparison test the series

$$\sum_{k=0}^{\infty} \frac{T^k\, t^k}{k!}$$

is absolutely and uniformly convergent for all $|t| \leq t_0$.

2. **Definition** : Let $T \in L(\mathbb{R}^n)$. The exponential e^T of T is defined by

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

Note

(a) It is clear that $e^T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator and $\| e^T \| \leq e^{\|T\|}$.

(b) For a matrix $A \in M_n\mathbb{R}$ and for $t \in \mathbb{R}$, we define

$$e^{At} : = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k.$$

3. **Some consequences**

(a) Let P, $T \in L(\mathbb{R}^n)$ and $S = PTP^{-1}$. Then

$$e^S = Pe^T P^{-1}$$

(b) For $A \in M_n(\mathbb{R})$, if $P^{-1}AP = \mathrm{diag}(\lambda_1, ..., \lambda_n)$, then

$$e^{At} = P\ diag(e^{\lambda_1 t}, ..., e^{\lambda_n t})P^{-1}.$$

(c) If $S, T \in L(\mathbb{R}^n)$ and commute (i.e. $ST = TS$), then

$$e^{S+T} = e^S\, e^T.$$

(d) (c) $\Rightarrow (e^T)^{-1} = e^{-T}$.

4. **Lemma**: Let $A = \begin{bmatrix} c & -b \\ b & a \end{bmatrix}$. Then

$$e^{At} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

Proof : The eigen values of A are $a \pm ib$. Let $\lambda = a \pm ib$. By induction for $k \in \mathbb{N}$

$$\begin{bmatrix} c & -b \\ b & a \end{bmatrix}^k = \begin{bmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{bmatrix}$$

$$or\ \ e^A = \sum_{k=0}^{\infty} \begin{bmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{bmatrix} = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

5. **Exercise** : Supply the details for the proof of Lemma 4.

6. In Lemma 5, e^A is a rotation through b when $a = 0$.

7. **Lemma**: Let $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$; $a, b \in \mathbb{R}$. Then

$$e^A = e^a \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

Proof : Exercise.

Conclusion : Let $A = M_n(\mathbb{Z})$. Then $B = P^{-1}AP$ where

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \ \ or \ \ B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \ \ or \ \ B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and a consequence is

$$e^{Bt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \ \ or \ \ e^{Bt} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \ \ or \ \ e^{Bt} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

and $e^{At} = Pe^{Bt}P^{-1}$.

8. **Lemma** : For $A \in M_n(\mathbb{R})$

$$\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A, \ \ t \in \mathbb{R} \tag{3.43}$$

Proof :

$$\begin{aligned} \frac{d}{dt} e^{At} &= \lim_{h \to 0} \frac{e^{(t+h)A} - e^{tA}}{h}, \ \ (|h| \leq 1) \\ &= e^{At} \lim_{h \to 0} \lim_{k \to 0} (A + \frac{A^2 h}{2!} + \cdots + \frac{A^k h^{k-1}}{k!} \ldots) \\ &= e^{At} A = Ae^{At} \end{aligned} \tag{3.44}$$

the last two step follows since the series for e^{Ah} converges uniformly for $|h| \leq 1$.

Part B : Linear Systems of ODE

We recall the following for clarity :

Let $A \in M_n(\mathbb{R})$. Consider the system of n linear ordinary differential equations

$$\frac{dx}{dt} = \dot{x} = Ax, \ t \in \mathbb{R} \tag{3.45}$$

with an initial condition

$$x(0) = x_0 \tag{3.46}$$

where $x_0 \in \mathbb{R}^n$.

9. **Theorem** (Fundamental Theorem for Linear ODE).
Let $A \in M_n(\mathbb{R})$ and $x_0 \in \mathbb{R}^n$ (column vector). The unique solution of the IVP (3.51) and (3.52) is given by

$$x(t) = e^{At} x_0 \tag{3.47}$$

Figure 3.1:

Proof : Let $y(t) = e^{At}x_0$. Then by lemma 9,

$$\frac{d}{dt}y(t) = \dot{y}(t) = Ae^{At}x_0 = e^{At}Ax_0 = e^{At}y(t)$$

and $y(0) = x_0$. Thus, $e^{At}x_0$ is a solution of the IVP (3.51) and (3.52) and by the Picard's Theorem
$x(t) \colon = e^{At}x_0$
is the unique solution of (3.51) and (3.52).
Remark : Let $y(t) = e^{-At}x(t)$ which implies $y'(t) = -Ae^{-At}x(t) + e^{-At}Ax(t) = 0,\ t \in \mathbb{R}$ or $y(t) = c$ for $t \in \mathbb{R}$.
$y(0) = x_0 \Rightarrow c = x(0) = x_0$ or $x(t) = e^{At}x_0$.

10. **Example** : Let us solve the IVP

$$\dot{x}_1 = -2x_1 - x_2,\ \ x_1(0) = 1$$

$$\dot{x}_2 = x_1 - 2x_2,\ \ x_2(0) = 0.$$

Note that the above system can be rewritten as

$$\dot{x} = Ax, x(0) = (1,0)^T,\ \ \textit{where}\ \ A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}.$$

It is easy to show that $2 \pm i$ are the eigenvalues of A and so by

$$\begin{aligned} x(t) &= e^{At}x_0 \\ &= e^{-2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-2t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}. \end{aligned} \tag{3.48}$$

Consequences :

(a) $|x(t)| = e^{-2t} \to 0$ as $t \to \infty$.
(b) $\theta(t) \colon = \tan^{-1}(\frac{x_2(t)}{x_1(t)}) = t$.
(c) Perimetrically $(x_1(t), x_2(t))^T$ describe a curve in $\mathbb{R}^2$ which spirals into $(0,0)$ as shown in figure 1.

Exercise : Supply the details for Example 10.

Lecture 20
Phase Portraits in $\mathbb{R}^2$

In this part, we undertake an elementary study of the Phase Portraits in $\mathbb{R}^2$ for a system of two linear ordinary differential equations, viz,

$$\dot{x} = Ax \tag{3.49}$$

Here A is a 2×2 real matrix (i.e. an element of $M_2(\mathbb{R})$) and $x \in \mathbb{R}^2$ is a column vector. The tuple $(x_1(t), x_2(t))$ for $t \in \mathbb{R}^2$ represents a curve C in $\mathbb{R}^2$ in a parametric form; the curve C is called the phase portrait of . It is easier to draw the curve when A is in its canonical form. However, in its original form (i.e. when A is not in the canonical form) these portraits have similar (but distorted) diagrams. The following example clarifies the same ideas.

Example : Let $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$. The canonical form B is $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, i.e., $A = P^{-1}BP$. The equation (3.49) with $y = Px$, is

$$y' = By \tag{3.50}$$

Equation (3.50) is sometimes is referred to (3.49), when A is in its canonical form. The phase Portrait for (3.50) is (fig2) Figure 3.2:

while the phase portrait of (3.49) is (fig3)

Figure 3.3:

Supply the details for drawing Figure 2 and 3.

In general, it is easy to write/draw the phase portrait of (3.49) when A in its canonical form. Coming back to (3.49), let P be an invertible 2×2 matrix such that $B = P^{-1}AP$, where B is a canonical form of A. We now consider the system

$$y' = By \tag{3.51}$$

By this time it is clear that phase portrait for (3.49) is the phase portrait of (3.50) under the transformation $x = Py$. We also write that B has one of the following form.

$$(a)\ \ B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \qquad (b)\ \ B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad (c)\ \ B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Let y_0 be an initial condition for (3.50), i.e.,

$$y(0) = y_0 \tag{3.52}$$

Then the solution of the IVP (3.51) and (3.52) is

$$y(t) = e^{Bt}y_0 \tag{3.53}$$

and for the 3 different choices of B, we have

$$(a)\ \ y(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} y_0 \quad (b)\ \ y(t) = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} y_0 \quad (c)\ \ B = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} y_0$$

With the above representation of y, we are now ready to draw the phase Portrait. Let us discuss the cases when $\lambda > 0,\ \mu > 0$; $\lambda > 0,\ \mu < o$; $\lambda = \mu$(with $\lambda > 0$ or $\lambda < 0$) and finally the case when λ is a complex number.

Case 1: Let $\lambda \leq \mu < 0$ with $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ or with $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. In this case the Phase Portrait of (3.50) looks like the following (figure 4):

Figure 3.4:

In drawing these diagrams, we note the following :

(a) $y_1(t), y_2(t) \to 0$ as $t \to \infty$ $(\lambda_1 < 0,\ \mu < 0)$

(b) $\lim\limits_{t\to\infty} \frac{y_2(t)}{y_1(t)} = 0\ \ if\ \ \lambda < \mu\ (\lambda < 0,\ \mu < 0)$

(c) $\lim_{t\to\infty} \frac{y_2(t)}{y_1(t)} = \infty \;\; if \;\; \lambda > \mu \; (\lambda < 0, \; \mu < 0)$

(d) $\lim_{t\to\infty} \frac{y_2(t)}{y_1(t)} = c$ if $\lambda = \mu, \;\; \lambda < 0.$

and hence an arrow is indicated to note that $y_1(t) \to 0$ and $y_2(t) \to 0$ as $t \to \infty$, in all the diagram. In this case, every solution tends to zero as $t \to \infty$ and in such a case the origin is called a stable node.

In case $\lambda \geq \mu > 0$ or $\mu \geq \lambda > 0$, the phase portrait essentially remains the same as shown in Figure 5 except the direction of the arrows are reversed.

The solutions are repelled away from origin. In this case the origin is referred to as an unstable node.

Case 1 essentially deals with real non-zero eigenvalues of B which are either both positive or negative. Below we consider the case when both the eigenvalues are real nonzero but of opposite signs.

Case 2 : Let $B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ with $\lambda < 0 < \mu$. The figure 5 (below) depicts the phase portrait.

Figure 3.5:

When $\mu < 0 < \lambda$,we have a similar diagram but with arrows in opposite directions. The origin is called, in this case, a Saddle Point. The four non-zero trajectories OA,OB,OC and OD are called the separatrices, two of them (OA and OB) approaches to the origin as $t \to \infty$ and the remaining two (namely OC and OD) approaches the origin as $t \to -\infty$. It is an exercise to draw the phase portrait when $\mu < 0 < \lambda$.

Lecture 21

Now we move to the case when A has complex eigenvalues $a \pm ib, b \neq 0$.

Case 3 : $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a < 0$.

Since the root is not real, we have $b \neq 0$ and so with $b > 0$ or $b < 0$. The phase portraits for this case are as shown in Fig 6 for $b > o$ and (b) for $b < 0$).

Figure 3.6:

In this case the origin is called a stable focus. We also note that it spirals around the origin and it tends to origin as $t \to \infty$.

When $a > 0$, the trajectories looks similar the one shown in Figure 7 and they are spiralling and moving away from the origin. When $a > 0$, the origin is called an unstable focus.

Case 4 : This case deals with the case when A has purely imaginary eigenvalues i.e. $\pm bi, \ (b \neq 0)$.

The canonical form of A is $B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$, $b \neq 0$. Equation (3.50) is

$$y_1' = -by_2 \ \ and \ \ y_2' = by_1$$

which leads to

$$y_1(t) = A\cos bt + B\sin bt \ \ and \ \ y_2(t) = -A\sin bt + B\cos bt$$

$$or \quad y_1^2 + y_2^2 = A^2 + B^2$$

which are concentric circles with center at origin. The phase portraits are as shown in Figure 7.

Figure 3.7:

Also, we note that the phase portraits for (3.49) is a family of ellipses as shown in Figure 8.

Figure 3.8:

In this case the origin is called the center for the system (3.49). We end this short discussion with an example.

Example : Consider the linear system

$$\dot{x}_1 = -4x_2 \ ; \ \ \dot{x}_2 = x_1$$

$$or \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

It is easy to verify that A has two non-zero (complex) eigenvalues $\pm 2i$. With usual notations

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = P^{-1}AP = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

The general solution is

$$\begin{aligned} x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= P \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} P^{-1}C \\ &= \begin{bmatrix} \cos 2t & -2\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

where C is an arbitrary constant vector in $\mathbb{R}^2$. It is left as an exercise to show

$$x_1^2 + x_2^4 = c_1^2 + c_2^2$$

or the phase portrait is a family of ellipses.

Module 4

Oscillations and Boundary Value Problems

Lecture 22

3.6 Introduction

Qualitative properties of solutions of differential equations assume importance in the absence of closed form solutions. In case the solutions are not expressible in terms of the usual "known functions", an analysis of the equation is necessary to find the various facets of the solutions. One such qualitative property, which has wide applications, is the oscillation of solutions. We again stess that it is but natural to expect to know the solution in an explicit form which unfortunately is not always possible. A rewarding alternative is to resort to qualitative study. The point is asserted once again to justify the inclusion of qualitative theory to students who think that it is otherwise out of place.

Before proceeding further, some definitions and their consequences are looked into as a part of the ground work. Consider a second order equation

$$x'' = f(t, x, x'), \quad t \geq 0, \tag{4.1}$$

and let x be a solution of equation (4.1) existing on $[0, \infty)$. Unless or otherwise mentioned we understannd (in this chapter) that a solution means a non-trivial solution.

Definition 3.6.1 (Definition 4.1). *A point $t = t^* \geq 0$ is called a zero of a solution x of the equation* (4.1) *if $x(t^*) = 0$.*

Definition 3.6.2 (Definition 4.2). *(a) Equation* (4.1) *is called "non-oscillatory" if for every solution x there exists $t_0 > 0$ such that x does not have a zero in $[t_0, \infty)$*

(b) Equation (4.1) *is called "oscillatory" if* (a) *is false.*

Example 3.6.3 (Example 4.3). *Consider the linear equation*

$$x'' - x = 0, t \geq 0.$$

It is an easy matter to show that the above equation is non-oscillatory once we recognize that the general solution is $Ae^t + Be^{-t}$ where A and B are constants.

Example 3.6.4 (Example 4.4). *The equation*

$$x'' + x = 0$$

is oscillatory. The general solution in this case is

$$x(t) = A \cos t + B \sin t, t \geq 0$$

and without loss of generality we assume that both A and B are non-zero constants; otherwise x is trivially oscillatory. It is easy to show that x has a zero at

$$n\pi + \tan^{-1}(A/B), n = 0, 1, 2, \cdots$$

and so the equation is oscillatory.

In this chapter we restrict our attention to only second order linear homogeneous equations. There are results concerning higher order equations. We conclude the introduction with a few basic results concerning linear equations.

$$x'' + a(t)x' + b(t)x = 0, \quad t \geq 0, \tag{4.2}$$

where a and b are real valued continuous functions defined on $[0, \infty)$

Theorem 3.6.5 (Theorem 4.5). *Assume that a' exists and is continuous for $t \geq 0$. Equation* (4.2) *is oscillatory if, and only if, the equation*

$$x'' + c(t)x = 0 \tag{4.3}$$

is oscillatory, where

$$c(t) = b(t) - \frac{1}{2}a^2(t) - \frac{a'(t)}{2}.$$

The equation (4.3) *is called the "normal" form of equation* (4.2).

Proof. Let x be any solution of (4.2). Consider a transformation

$$x(t) = v(t)y(t)$$

where v and y are twice differentiable functions. The computation of x', x'' and their substitution in (4.2) gives us

$$vy'' + (2v' + a(t)v)y' + (v'' + a(t)v' + b(t)v)y = 0.$$

Thus equating the coefficients of y' to zero, it is seen that

$$v(t) = \exp(-\frac{1}{2}\int_0^t a(s)ds)$$

.

Therefore $y(t)$ satisfies a differential equation

$$y'' + c(t)y = 0, \quad t \geq 0$$

where $c(t) = b(t) - \frac{1}{2}a^2(t) - \frac{a'(t)}{2}$. So it is concluded that if $x(t)$ is a solution of (4.2), then

$$y(t) = x(t)\exp(\frac{1}{2}\int_0^t a(s)ds)$$

is a solution of (4.3). Similarly if $y(t)$ is a solution of (4.3) then

$$x(t) = y(t)\exp(-\frac{1}{2}\int_0^t a(s)ds)$$

is a solution of (4.2). Thus the theorem holds. □

Remark We note that (4.2) is oscillatory if and only if (4.3) is oscillatory. Although the proof of the Theorem 4.5 is elementary the conclusion simplifies subsequent work to a great extent.

The following two theorems are of interest in themselves.

Theorem 3.6.6 (Theorem 4.6)**.** *Let x_1 and x_2 be two linearly independent solutions of* (4.2)*. Then x_1 and x_2 do not admit common zeros.*

Proof. Suppose $t = a$ is a common zero of x_1 and x_2. Then the Wronskian of x_1 and x_2 vanishes at $t = a$. Thus, it follows that x_1 and x_2 are linearly dependent which is a contradiction to the hypothesis or else x_1 and x_2 cannot have common zeros. □

Theorem 3.6.7 (Theorem 4.7)**.** *The zeros of a solution of* (4.2) *are isolated.*

Proof. Let $t = a$ be a zero of a solution x of (4.2). Then $x(a) = 0$ and $x'(a) \neq 0$, otherwise $x \equiv 0$, which is not the case, since x is a non-trivial solution.

There are two cases.

Case 1: $x'(a) > 0$

Since the derivative of x is continuous and positive at $t = a$ it follows that x is strictly increasing in some neighbourhood of $t = a$ which means that $t = a$ is the only zero of x in that neighbourhood. This shows that the zero $t = a$ of x is isolated.

Case 2: $x'(a) < 0$

The proof is similar to that of case 1 with minor changes. □

EXERCISES

1. Prove that the equation (4.2) is non-oscillatory if and only if the equation (4.3) is non-oscillatory.

2. If $t_1, t_2, \cdots, t_n, \cdots$ are zeros of a solution x of (4.2) in $(0, \infty)$, then show that $\lim t_n = \infty$ as $n \to \infty$.

3. Prove that any solution x of (4.2) has at most a countable number of zeros in $(0, \infty)$.

4. Show that the equation

$$x'' + a(t)x' + b(t)x = 0, \quad t \geq 0 \qquad (*)$$

transforms into an equation of the form

$$(p(t)x')' + q(t)x = 0, \quad t \geq 0 \qquad (**)$$

by multiplying (*) throughout by $\exp(\int_0^t a(s)ds)$, where $a(t)$ and $b(t)$ are continuous functions on $[0, \infty)$,

$$p(t) = \exp(\textstyle\int_0^t a(s)ds), q(t) = b(t)p(t).$$

State and prove a theorem similar to Theorem 4.5 for equation (*) and (**). Also show that if $a(t) \equiv 0$, then (**) reduces to $x'' + q(t)x = 0, t \geq 0$.

3.7 Sturm's Comparison Theorem

The phrase "comparison theorem" for differential equation is used in the sense stated below:

' If a solution of a differential equation has a certain known property P then the solution of a second differential equation have the same or some related property P under certain hypothesis.'

Sturm's comparison theorem is a result in this direction concerning zeros of solutions of a pair of linear homogeneous differential equations. Sturm's theorem has varied interesting implications in the theory of oscillations.

Theorem 3.7.1 (Theorem 4.8). *(Sturm's Comparison Theorem)*
Let p, r_1, r_2 and p be continuous functions on (a,b) and $p>0$. Assume that x and y are real solutions of

$$(px')' + r_1 x = 0, \tag{4.4}$$

$$(py')' + r_1 y = 0 \tag{4.5}$$

respectively on (a,b). If $r_2(t) \geq r_1(t)$ for $t \in (a,b)$ then between any two consecutive zeros t_1, t_2 of x in (a,b) there exists at least *one zero of y (unless $r_1 \equiv r_2$) in $[t_1,t_2]$. Moreover, when $r_1 \equiv r_2$ in $[t_1,t_2]$ the conclusion still holds if x and y are linearly independent .*

Proof. If possible, let $y(t)$ be positive in (t_1,t_2). Without loss of generality let us assume that $x(t)>0$ on (t_1,t_2) . Multiplying (4.4) and (4.5) by y and x respectively and subtraction leads to

$$(px')'y - (py')'x - (r_2 - r_1)xy = 0,$$

which, on integration gives us

$$\int_{t_1}^{t_2} [(px')'y - (py')'x] \; dt = \int_{t_1}^{t_2} (r_2 - r_1)xy \; dt.$$

If $r_2 \neq r_1$ on (t_1,t_2), then $r_2(t) > r_1(t)$ in a small interval of (t_1,t_2) and therefore

$$\int_{t_1}^{t_2} [(px')'y - (py')'x] > 0. \tag{4.6}$$

Using the identity

$$\tfrac{d}{dt}[p(x'y - xy')] = (px')'y - (py')'x,$$

now the inequality (4.6) implies

$$p(t_2)x'(t_2)y(t_2) - p(t_1)x'(t_1)y(t_1) > 0, \tag{4.7}$$

since $x(t_1) = x(t_2) = 0$. However $x'(t_1) > 0$ and $x'(t_2) < 0$ as x is a non-trivial solution which is positive in (t_1,t_2). As py is positive at t_1 as well as at t_2, (4.7) leads to a contradiction.

Again, if $r_1 \equiv r_2$ on $[t_1,t_2]$, then in place of (4.7), we have

$$p(t_2)y(t_2)x'(t_2) - p(t_1)y(t_1)x'(t_1) \geq 0.$$

which again leads to a contradiction as above unless y is a multiple of x. This completes the proof. □

Remark What Sturm's comparison theorem asserts is that the solution y has *at least* one zero between two successive zeros t_1 and t_2 of x. Many times y may vanish more than once between t_1 and t_2. As a special case of Theorem 4.8,we have

Theorem 3.7.2 (Theorem 4.9). *Let r_1 and r_2 be two continuous functions such that $r_2 \geq r_1$ on (a,b). Let x and y be solutions of equations*

$$x'' + r_1(t)x = 0 \tag{4.8}$$

and

$$y'' + r_2(t)y = 0 \tag{4.9}$$

on the interval (a,b). Then $y(t)$ has at least a zero between any two successive zeros t_1 and t_2 of x in (a,b) unless $r_1 \equiv r_2$ on $[t_1,t_2]$. Moreover, in this case the conclusion remains valid if the solution $y(t)$ is linearly independent of $x(t)$.

Proof. the proof is immediate if we let $p \equiv 1$ in Theorem 4.8. Notice that the hypotheses of Theorem 4.8 are satisfied. □

The celebrated Sturm's separation theorem is an easy consequence of Sturm's comparison theorem as shown below.

Theorem 3.7.3 (Theorem 4.10)**.** *(Sturm's Separation Theorem) Let $x(t)$ and $y(t)$ be two linearly independent real solutions of*

$$x'' + a(t)x' + b(t)x = 0, t \geq 0 \tag{4.10}$$

where a, b are real valued continuous functions on $(0, \infty)$. Then, the zeros of x and y separate each other, i.e. between any two consecutive zeros of x there is one and only one zero of y. (Note that the roles of x and y are interchangeable.)

Proof. First we note that all the hypotheses of Theorem 4.8 are satisfied by letting

$$r_1(t) \equiv r_2(t) = b(t)\exp(\int_0^t a(s)ds)$$
$$p(t) = \exp(\int_0^t a(s)ds)$$

So between any two consecutive zeros of x, there is at least one zero of y. By repeating the argument with x in place of y, it is clear that between any two consecutive zeros of y there is a zero of x which completes the proof. □

By setting $a \equiv 0$ in Theorem 4.10 gives us the following result.

Corollary 3.7.4 (Corollary 4.11)**.** *Let r be a continuous function on $(0, \infty)$ and let x and y be two linearly independent solutions of*

$$x'' + r(t)x = 0.$$

Then, the zeros of x and y separate each other.

After having dealt with some of the implications of Theorem 4.8, a few comments are warrented on the hypotheses of Theorem 4.8. Example 4.12 shows that Theorem 4.8 fails if the condition $r_2 \geq r_1$ is dropped.

Example 3.7.5 (Example 4.12)**.** *Consider the equations*

(i) $x'' + x = 0, r_1(t) \equiv +1, t \geq 0,$

(ii) $x'' - x = 0, r_2(t) \equiv -1, t \geq 0.$

All the conditions of Theorem 4.8 are satisfied except that r_2 is not greater than r_1. We note that between any consecutive zeros of a solution $x($ of (i), any solution y of (ii) does not admit a zero. Thus, Theorem 4.8 may not hold true if the condition $r_2 \geq r_1$ is dropped.

Assuming the hypotheses of Theorem 4.8, let us pose a question: is it true that between any two zeros of a solution y of equation (4.5) there is a zero of a solution x of equation (4.4) ? The answer to this question is in the negative as is clear from example 4.13.

Example 3.7.6 (Example 4.13)**.** *Consider*

$$\begin{gathered} x'' + x = 0, r_1(t) \equiv 1 \\ y'' + 4y = 0, r_2(t) \equiv 4. \end{gathered}$$

Note that $r_2 \geq r_1$ and also that the remaining conditions of Theorem 4.8 are satisfied. $x(t) = \sin t$ is a solution of the first equation and $y(t) = \sin(2t)$ is a solution of the second equation which has zero at $t_1 = 0$ and $t_2 = \pi/2$. It is obvious that $x(t) = \sin t$ does not vanish at any point in $(0, \pi/2)$. This clearly shows that, under the hypotheses of Theorem 4.8, between two successive zeros of y there need not exist a zero of x.

EXERCISES

1. Let r be a positive continuous function and let m be a real number. Show that the equation
$$x'' + (m^2 + r(t))x = 0, t \geq 0$$
is oscillatory.

2. Assume that the equation
$$x'' + r(t)x = 0, t \geq 0$$
is oscillatory. Prove that the equation
$$x'' + (r(t) + s(t))x = 0, t \geq 0$$
is oscillatory, given that r, s are continuous functions and $s(t) \geq 0$.

3. Let r be a continuous function (for $t \geq 0$) such that $r(t) > m^2 > 0$, where m is an integer. A solution y of

$$y'' + r(t)y = 0, t \geq 0$$

then prove that y must vanish in any interval of length π/m.

4. Show that the normal form of Bessel's equation

$$t^2x'' + tx' + (t^2 - p^2)x = 0 \qquad (*)$$

is given by

$$y'' + (1 + \tfrac{1-4p^2}{4t^2})y = 0 \qquad (**)$$

(a) Show that the solution $J_p(t)$ of (*) and $Y_p(t)$ of (**) have common zeros for $t > 0$.

(b)

(i) If $0 \leq p < \frac{1}{2}$, show that every interval of length π contains at least one zero of $J_p(t)$;

(ii) If $p = \frac{1}{2}$ then prove that every zero of $J_p(t)$ is at a distance of π from its successive zero.

(c) Suppose t_1 and t_2 are two consecutive zeros of $J_p(t), 0 \leq p < \frac{1}{2}$. Show that $t_2 - t_1 < \pi$ and that $t_2 - t_1$ approaches π in the limit as $t_1 \to \infty$. What is your comment when $p = \frac{1}{2}$ in this case ?

3.8 Elementary Linear Oscillations

Presently we restrict our discussion to a class of second order equation of the type

$$x'' + a(t)x = 0, t \geq 0, \tag{4.11}$$

where a is a real valued continuous function defined for $t \geq 0$. A very interesting implication of Sturm's separation theorem is

Theorem 3.8.1 (Theorem 4.14:)**.** *(a) The equation* (4.11) *is non-oscillatory if, and only if, it has no solution with finite number of zeros in* $[0, \infty)$*. (b) Equation* (4.11) *is either oscillatory or non-oscillatory but cannot be both.*

Proof. (a) *Necessity* It has an immediate consequence of the definition.
Sufficiency Let $z(t)$ be the given solution which does not vanish on (t^*, ∞) where $t^* \geq 0$. Then any non-trivial solution $x(t)$ of (4.11) can vanish utmost once in (t^*, ∞), i.e, there exists $t_0 (> t^*)$ such that $x(t)$ does not have a zero in $[t_0, \infty)$.

The proof of (b) is obvious. □

We conclude this section with two elementary results.

Theorem 3.8.2 (Theorem 4.15)**.** *Let x be a solution of* (4.11) *existing on* $(0, \infty)$*. If $a < 0$ on* $(0, \infty)$*, then x has utmost one zero.*

Proof. Let t_0 be a zero of x. It is clear that $x'(t_0) \neq 0$ for $x(t) \not\equiv 0$. Without loss of generality let us assume that $x'(t_0) > 0$ so that x is positive in some interval to the right of t_0. Now $a < 0$ implies that x'' is positive on the same interval which in turn implies that x' is an increasing function, and so, x does not vanish to the right of t_0. A similar argument shows that x has no zero to the left of t_0. Thus, x has utmost one zero. $\square$

Remark Theorem 4.15 can also be seen as a corollary of Sturm's comparison theorem. Consider the equation $y'' = 0$. It is known that any non-zero constant function $y(t) \equiv k$ is a solution. Thus, if this equation is compared with the equation (4.11) (observe that all the hypotheses of Theorem 4.8 are satisfied) then $x(t)$ vanishes utmost once, for otherwise $x(t)$ vanishes twice and $y(t)$ necessarily vanishes at least once by Theorem 4.8. So $x(t)$ cannot have more than one zero. From Theorem 4.15 the question arises: If $a(t)$ is continuous and $a(t) > 0$ on $(0, \infty)$, is the equation (4.11) oscillatory ? A partial answer is given in the following theorem.

Theorem 3.8.3 (Theorem 4.16)**.** *Let $a(t)$ be continuous and positive on $(0, \infty)$ with*

$$\int_1^\infty a(s)ds = \infty. \tag{4.12}$$

Also assume that $x(t)$ is any solution of (4.11) existing for $t \geq 0$. Then $x(t)$ has infinite zeros in $(0, \infty)$.

Proof. Assume, on the contrary, that $x(t)$ has only a finite number of zeros in $(0, \infty)$. Then there exist a point $t_0 > 1$ such that $x(t)$ does not vanish on $[t_0, \infty)$. Without loss of generality it can be assumed that $x(t) > 0$ for all $t \geq t_0$. Thus

$$v(t) = +\frac{x'(t)}{x(t)}, t \geq t_0$$

is well defined. It now follows that

$$v'(t) = -a(t) - v^2(t).$$

Integration on the above leads to

$$v(t) - v(t_0) = -\textstyle\int_{t_0}^t a(s)ds - \int_{t_0}^t v^2(s)ds.$$

The condition (4.12) now implies that there exist two constants A and T such that $v(t) < A(< 0)$ if $t \geq T$ since $v^2(t)$ is always non-negative and $v(t) \leq v(t_0) - \int_{t_0}^t a(s)ds$.

This means that $x'(t)$ is negative for large t. Let $T(\geq t_0)$ be so large that $x'(T) < 0$. Then on $[T, \infty)$ notice that $x(t) > 0, x'(t) < 0$ and $x''(t) < 0$. But $\int_T^t x''(s)ds = x'(t) - x'(T) \leq 0$ integrating once again it is seen that

$$x(t) - x(T) \leq x'(T)(t - T), t \geq T \geq t_0. \tag{4.13}$$

Since $x'(T)$ is negative, the right hand side of (4.13) tends to $-\infty$ as $t \to \infty$ while the left hand side of (4.13) either tends to a finite limit(because $x(T)$ is finite) or tends to $+\infty$ (in case $x(t) \to \infty$ as $t \to \infty$). Thus in either case a contradiction is reached. So the assumption that $x(t)$ has a finite number of zeros in $(0, \infty)$ is false. Hence $x(t)$ has infinite number of zeros in $(0, \infty)$, which completes the proof. $\square$

It is not possible to do away with the condition (4.12) as shown by the following example.

Example 3.8.4 (Example 4.17)**.** *Consider Euler's equation* $x'' + \frac{2}{9t^2}x = 0$. $x(t) = t^{1/3}$ *is a solution of this equation which does not vanish anywhere in* $(0, \infty)$ *and hence the equation is non-oscillatory. Also in this case*

$$a(t) = \frac{2}{9t^2} > 0; \int_1^\infty \frac{2}{9t^2}dt = \frac{2}{9} < \infty$$

.

Thus all the conditions of Theorem 4.16 *are satisfied except the condition* (4.12)*.*

EXERCISES

1. Prove (b) part of Theorem 4.14.
2. Suppose $a(t)$ is a continuous function on $(0, \infty)$ such that $a(t) < 0$ for $t \geq \alpha, \alpha$ is a finite real number. Show that $x'' + a(t)x = 0$ is non-oscillatory.

3. Check for the oscillations or non-oscillations of:

   (i) $x'' - (t - \sin t)x = 0, \quad t \geq 0$

   (ii) $x'' + e^t x = 0, \quad t \geq 0$

   (iii) $x'' - e^t x = 0, \quad t \geq 0$

   (iv) $x'' - \frac{t}{\log t}x = 0, \quad t \geq 1$

   (v) $x'' + (t + e^{-2t})x = 0, \quad t \geq 0$

4. Prove that Euler's equation $x'' + \frac{k}{t^2}x = 0$

   (a) is oscillatory if $k > \frac{1}{4}$

   (b) is non-oscillatory if $k \leq \frac{1}{4}$

5. The normal form of Bessel's equation $t^2x'' + tx' + (t^2 - p^2)x = 0, t \geq 0$, is

$$x'' + (1 + \tfrac{1-4p^2}{4t^2})x = 0, t \geq 0. \qquad (*)$$

   (i) Show that Bessel's equation is oscillatory for all values of p.

   (ii) If $p > \frac{1}{2}$ show that $t_2 - t_1 > \pi$ and approaches π as $t_1 \to \infty$, where t_1, t_2(with $t_1 < t_2$) are two successive zeros of Bessel's function $J_p(t)$.
   (Hint: Show that $J_p(t)$ and the solution $Y_p(t)$ of (*) have common zeros. Then compare (*) with $x'' + x = 0$, successive zeros of which are at a distance of π.)
   (Exercise 4 of sec. 2 and Exercise 5 above justify the assumption of the existence of zeros of Bessel's functions

6. Decide whether the following equations are oscillatory or non-oscillatory:

   (i) $(tx')' + x/t = 0$,

   (ii) $x'' + x'/t + x = 0$,

   (iii) $tx'' + (1 - t)x' + nx = 0, n$ is a constant(Laguerre's equation),

   (iv) $x'' - 2tx' + 2nx = 0, n$ is a constant(Hermite's equation),

   (v) $tx'' + (2n + 1)x' + tx = 0, n$ is a constant,

   (vi) $t^2x + ktx' + nx = 0, k, n$ are constants.

3.9 Boundary Value Problems

In Chapter 1, the definition of a boundary value problem (BVP) was introduced. BVPs appear in various branches of science and engineering. A problem in calculus of variation leads to a BVP. Solutions to the problems of vibrating strings and membranes are the outcome of solutions of certain BVPs. Thus the importance of the study of BVP, both in mathematics and in the applied sciences, needs no emphasis.

Speaking in general, BVPs pose many difficulties in comparison with IVPs. The problem of existence, both for linear and nonlinear equations with boundary conditions, requires discussions which are quite complicated. Needless to say nonlinear BVPs are far tougher to solve than linear BVPs.

In this chapter attention is focused on some aspects of the regular BVP of the second order. Picard's theorem on the existence of a unique solution to a nonlinear BVP is also dealt with in the last section.

Consider a second order linear equation

$$L(x) = a(t)x'' + b(t)x' + c(t)x = 0, \quad A \leq t \leq B. \tag{4.14}$$

It is tacitly assumed throughout this chapter that a, b, c are continuous real valued functions defined on $[A, b]$. To proceed further the concepts of linear forms in necessary. Let x_1, x_2, x_3, x_4 be four variables. Then, for any scalars a_1, a_2, a_3, a_4

$$V(x_1, x_2, x_3, x_4) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$$

is called a "linear form" in the variables x_1, x_2, x_3, x_4. $V(x_1, x_2, x_3, x_4)$ is denoted in short by V. Two linear forms V_1 and V_2 are said to be linearly dependent if there exists a scalar K such that $V_1 = KV_2$ for all x_1, x_2, x_3, x_4. V_1 and V_2 are called linearly independent if V_1 and V_2 are not linearly dependent.

Definition 3.9.1 (Definition 4.18)**.** *(Linear Homogeneous BVP) Consider an equation of type* (4.14)*. Let V_1 and V_2 be two linearly independent linear forms in the variables $x(A), x(B), x'(A)$ and $x'(B)$. Then a linear homogeneous BVP is the problem of finding a function x defined on $[A, B]$ which satisfies*

$$\begin{aligned} &L(x) = 0, \quad t \in [A.B] \text{ and} \\ &V_i[x(A), x(B), x'(A), x'(B)] = 0, \quad i = 1, 2 \end{aligned} \tag{4.15}$$

simultaneously. The condition 4.15 *is called a "linear homogeneous boundary condition" stated at $t = A$ and $t = B$.*

Definition 3.9.2 (Definition 4.19)**.** *(Linear Non-homogeneous BVP) Let $d(t)$ be a given continuous real valued function on $[A, B]$. A linear non-homogeneous BVP is the problem of finding a function x defined on $[A, B]$ satisfying*

$$\begin{aligned} &L(x) = d(t), \quad t \in [A.B] \text{ and} \\ &V_i[x(A), x(B), x'(A), x'(B)] = 0, \quad i = 1, 2 \end{aligned} \tag{4.16}$$

simultaneously where V_i are two given linear forms and the operator L is defined by equation (4.14)*.*

Example 3.9.3 (Example 4.20)**.** *(i) Consider*

$$L(x) = x'' + x' + x = 0 \text{ and}$$

$$\begin{aligned} V_1[x(A), x'(A), x(B), x'(B)] = x(A) \\ V_2[x(A), x'(A), x(B), x'(B)] = x(B). \end{aligned}$$

Then any solution $x(t)$ of $L(x) = 0, A \leq t \leq B$ which satisfies $x(A) = x(B) = 0$ is a solution of the given BVP.

(ii) $L(x) = x'' + e^tx' + 2x = 0, 0 \leq t \leq 1$, with boundary conditions $x(0) = x(1)$ and $x'(0) = x'(1)$ is a linear homogeneous BVP. In this case

$$\begin{aligned} V_1[x(0), x'(0), x(1), x'(1)] = x(0) - x(1) \\ V_2[x(0), x'(0), x(1), x'(1)] = x'(0) - x'(1). \end{aligned}$$

Also $L(x) = \sin 2\pi t, 0 \leq t \leq 1$, along with boundary conditions $x(0) = x(1)$ and $x'(0) = x'(1)$ constitute a linear non-homogeneous BVP.

Definition 3.9.4 (Definition 4.21)**.** *(Periodic Boundary Conditions) The boundary conditions $x(A) = x(B)$ and $x'(A) = x'(B)$ are usually known as periodic boundary conditions stated at $t = A$ and $t = B$.*

Definition 3.9.5 (Definition 4.22)**.** *(Regular Linear BVP) A linear BVP, homogeneous or non-homogeneous, is called a regular BVP if A and B are finite and in addition to that $a(t) \neq 0$ for all t in $[A, B]$.*

Definition 3.9.6 (Definition 4.23)**.** *(Singular Linear BVP) A linear BVP which is not regular is called a singular linear BVP.*

Lemma 3.9.7 (Lemma 4.24)**.** *A linear BVP* (4.14) *and* (4.15) *(or* (4.16) *and* (4.15)*) is singular if and only if one of the following conditions holds:*

(a) Either $A = -\infty$ or $B = \infty$.

(b) Both $A = -\infty$ and $B = \infty$.

(c) $a(t) = 0$ for at least one point t in $[A, B]$.

The proof is obvious.

In this chapter, the discussions are confined to only regular BVPs. The definitions listed so far lead to the definition of a nonlinear BVP.

Definition 3.9.8 (Definition 4.25)**.** *A BVP which is not a linear BVP is called a nonlinear BVP.*

A careful analysis of the above definition shows that the nonlinearity in a BVP may be introduced because

(i) the differential equation may be nonlinear;

(ii) the given differential equation may be linear but the boundary conditions may not be linear homogeneous.

The assertion made in (i) and (ii) above is further clarified in example 4.7.

Example 3.9.9 (Example 4.26)**.** *(i) Consider a differential equation $x'' + |x| = 0, 0 \leq t \leq \pi$ with boundary conditions $x(0) = x(\pi) = 0$. Notice that this BVP is not linear because of the presence of $|x|$.*

(ii) $x'' - 4x = e^t, \leq t \leq 1$ with boundary conditions $x(0).x(1) = x'(0), x'(1) = 0$ is a nonlinear BVP because one of the boundary conditions is not linear homogeneous.

EXERCISES

1. State with reasons whether the following BVPs are linear homogeneous, linear non-homogeneous or non-linear.

   (i) $x'' + \sin x = 0, \quad x(0) = x(2\pi) = 0.$

   (ii) $x'' + x = 0, \quad x(0) = x(\pi), \quad x'(0) = x'(\pi).$

   (iii) $x'' + x = \sin 2t, \quad x(0) = x(\pi) = 0.$

   (iv) $x'' + x = \cos 2t, \quad x^2(0) = 0, \quad x^2(\pi) = x'(0).$

2. Are the following BVPs regular ?

   (i) $2tx'' + x' + x = 0, \quad x(-1) = 1, \quad x(1) = 1.$

   (ii) $2x'' - 3x' + 4x = 0, \quad x(-\infty) = 0, \quad x(0) = 1.$

(iii) $x'' - 9x = 0, \quad x(0) = 1, \quad x(\infty) = 0.$

3. Find a solution of

    (i) BVP (ii) of Exercise 2;

    (ii) BVP (iii) of Exercise 2.

3.10 Sturm-Liouville Problem

The Sturm-Liouville problem represents a class of linear BVPs. The importance of these problems lies in the fact that they generate sets of orthogonal functions (indeed complete sets of orthogonal functions). The sets of orthogonal functions are useful in the expansion of a certain class of function. Few examples of such sets of functions have already been studied in Chapter 3, namely the Legendre and Bessel functions. In all of what follows, we consider a differential equation of the form

$$(px')' + qx + \lambda rx = 0, \quad A \le t \le B \tag{4.17}$$

where p', q and r are real valued continuous functions on $[A, B]$ and λ is a real parameter. We focus our attention on second order equations with a special kind of boundary condition. Let us consider two sets of boundary conditions, namely

$$m_1 x(A) + m_2 x'(A) = 0, \tag{4.18a}$$

$$m_3 x(B) + m_4 x'(B) = 0, \tag{4.18b}$$

$$x(A) = x(B), \quad x'(A) = x'(B), \quad p(A) = p(B), \tag{4.19}$$

where at least one of m_1 and m_2 and at least one of m_3 and m_4 are non-zero. A glance at the boundary conditions (4.18) shows that the two conditions are separately stated on A ans B. Relation (4.19) is the periodic boundary condition at A and B.

A BVP consisting of equation (4.17) with (4.18) or equation (4.17) with (4.19) is called a Sturm-Liouville boundary value problem. It is trivial to show that the identically zero functions on $[A, B]$ is always a solution of Sturm-Liouville problem. It is of interest to examine the existence of a non-trivial solution and its properties.

Suppose that for a value of λ, x_λ is a non-trivial solution of (4.17) with (4.18) or (4.17) with (4.19). Then λ is called an "eigenvalue" and x_λ is called an "eigenfunction" (corresponding to λ) of the Sturm-Liouville problem of (4.17) with (4.18) or with (4.19) respectively. The following theorem is of fundamental importance whose proof is beyond the scope of this book.

Theorem 3.10.1 (Theorem 4.27). *Assume that*

(i) A, B are finite real numbers;

(ii) the functions $p'(t), q(t)$ and $r(t)$ are real valued continuous functions on $[A, B]$; and

(iii) m_1, m_2, m_3 and m_4 are real numbers.

Then the Sturm-Liouville problem (4.17) *with* (4.18) *or* (4.17) *with* (4.19) *has countably many eigenvalues with no finite limit point. Corresponding to each eigenvalue there exists an eigenfunction.*

Theorem 4.27 just guarantees the existence of solutions. Some properties of such eigenfunctions which can be exploited in expansion theorems are discussed. One such is the orthogonal property.

Definition 3.10.2 (Definition 4.28). *Two distinct functions x and y, defined and continuous on $[A, B]$ are said to be orthogonal with respect to a continuous weight function $r(t)$ if*

$$\int_A^B r(s)x(s)y(s)ds = 0. \tag{4.20}$$

Theorem 3.10.3 (Theorem 4.29). *Let all the assumptions of Theorem 4.27 hold. For the parameters $\lambda, \mu(\lambda \neq \mu)$ let x and y be the corresponding solutions of (4.17) such that $\Big[pW(x,y)\Big]_A^B = 0$, where $W(x,y)$ is the Wronskian of x and y and $\Big[Z\Big]_A^B$ means $Z(B)-Z(A)$. Then*

$$\int_A^B r(s)x(s)y(s)ds = 0$$

.

Proof. From the hypotheses of theorem it is seen that

$$(px')' + qx + \lambda rx = 0,$$
$$(py')' + qy + \mu ry = 0.$$

The above two equations imply that

$$(\lambda - \mu)rxy = (py')'x - (px')'y,$$

that is

$$(\lambda - \mu)rxy = \frac{d}{dt}[(py')x - (px')y]. \tag{4.21}$$

Integrating Eq. (4.21), the following is obtained.

$$(\lambda - \mu)\int_A^B r(s)x(s)y(s)ds = \Big[(py')x - (px')y\Big]_A^B = \Big[pW(x,y)\Big]_A^B$$

.

Since $\lambda \neq \mu$ it follows from the assumptions that $\int_A^B r(s)x(s)y(s)ds = 0$. The proof is complete. □

From Theorem 4.29 it is clear that if a search is made for conditions which imply

$$\Big[pW(x,y)\Big]_A^B = 0,$$

then the desired orthogonal property is obtained. Also notice that till now the boundary conditions (4.18) or (4.19) have not been made use of. It can be shown that (4.18) or (4.19) implies that $\Big[pW(x,y)\Big]_A^B = 0$.

Theorem 3.10.4 (Theorem 4.30). *Let the hypotheses of Theorem 4.27 be satisfied. In addition let x_m and x_n be two eigenfunctions of the BVP (4.17) and (4.18) corresponding to two distinct eigenvalues λ_m and λ_n. Then*

$$\Big[pW(x_m, x_n)\Big]_A^B = 0. \tag{4.22}$$

If $p(A) = 0$ then (4.22) holds without the use of (4.18a). If $p(B) = 0$, then (4.22) holds with (4.18b) being deleted.

Proof. Let $p(A) \neq 0, p(B) \neq 0$. From (4.18a) it is seen that

$$m_1 x_n(A) + m_2 x_n'(A) = 0, \qquad m_1 x_m(A) + m_2 x_m'(A) = 0.$$

Without loss of generality it can be assumed that $m_1 \neq 0$. Elimination of m_2 from the above two equation leads to

$$m_1[x_n(A)x_m'(A) - x_m(A)x_m'(A)] = 0.$$

Since $m_1 \neq 0$, it is seen that

$$x_n(A)x_m'(A) - x_m(A)x_n'(A) = 0. \tag{4.23}$$

Similarly if $m_4 \neq 0$ (or $m_3 \neq 0$) in (4.18b), it is seen that

$$x_n(B)x_m'(B) - x_n'(B)x_m(B) = 0. \tag{4.24}$$

From the relations (4.23) and (4.24) it is obvious that (4.22) is satisfied.

If $p(A) = 0$, then the relation (4.22) holds since

$$\Big[pW(x_m, x_n)\Big]_A^B = p(B)[x_n(B)x_m'(B) - x_n'(B)x_m(B)] = 0,$$

in view of the equation (4.24). Similar is the case when $p(B) = 0$. This completes the proof. □

The following theorem deals with periodic boundary conditions given in (4.19).

Theorem 3.10.5 (Theorem 4.31)**.** *Let the assumptions of theorem* 4.27 *be true. Suppose x_m and x_n are eigenfunctions of BVP* (4.17) *and* (4.19) *corresponding to the distinct eigenvalues λ_m and λ_n respectively. Then x_m and x_n are orthogonal with respect to the weight function $r(t)$.*

Proof. In this case

$$\Big[pW(x_n, x_m)\Big]_A^B = p(B)[x_n(B)x_m'(B) - x_n'(B)x_m(B) - x_n(A)x_m'(A) + x_n'(A)x_m(A)].$$

The expression inside the brackets is zero once we use the periodic boundary condition (4.19). □

The following theorem ensures that the eigenvalues of (4.17), (4.18) or (4.17), (4.19) are real if $r(t) > 0$ (or $r(t) < 0$) on $[A, B]$.

Theorem 3.10.6 (Theorem 4.32)**.** *Let the hypotheses of Theorem* 4.27 *hold. Suppose that $r(t)$ is positive on $[A, B]$ or $r(t)$ is negative on $[A, B]$. Then all the eigenvalues of BVP* (4.17), (4.18) *or* (4.17), (4.19) *are real.*

Proof. Let $\lambda = a + ib$ be an eigenvalue and let $x(t) = m(t) + in(t)$ be a corresponding eigenfunction. It is clear that $a, b, m(t)$ and $n(t)$ are real. So, it is seen that

$$(pm' + pin')' + q(m + in) + (a + ib)r(m + in) = 0.$$

Equating the real and imaginary parts, the following is obtained

$$(pm')' + (q + ar)m - brn = 0$$

and

$$(pn')' + (q + ar)n + brm = 0.$$

Elimination of $(q + ar)$ in the above two equations implies

$$-b(m^2 + n^2)r = m(pn')' - n(pm')' = \tfrac{d}{dt}[(pn')m - (pm')n].$$

Thus, by integrating, we get

$$-b \int_A^B (m^2(s) + n^2(s))r(s)ds = \Big[(pn')m - (pm')n\Big]_A^B. \tag{4.25}$$

Since m and n satisfy either of the boundary conditions (4.18) or (4.19), we have, as shown earlier,

$$\Big[p(n'm - m'n)\Big]_A^B = \Big[pW(m,n)\Big]_A^B = 0. \tag{4.26}$$

Also, $\int_A^B [m^2(s) + n^2(s)]r(s)ds \neq 0$ by the assumptions. Hence from (4.25) and (4.26) it is clear that $b = 0$, which means that λ is real. This completes the proof. □

An important application of the previous result is Theorem 4.33.

Theorem 3.10.7 (Theorem 4.33)**.** *(Eigenfunction expansion) Let $g(t)$ be a piecewise continuous function defined on $[A, B]$ satisfying the boundary conditions* (4.18) *or* (4.19)*. Let $x_1, x_2, \cdots, x_n, \cdots$ be the set of eigenfunctions of the Sturm-Liouville problem* (4.17) *and* (4.18) *or* (4.17) *and* (4.19)*. Then*

$$g(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + \cdots \tag{4.27}$$

where c_n's are given by

$$c_n \int_A^B r(s)x_n^2(s)ds = \int_A^B r(s)g(s)x_n(s)ds, \quad n = 1, 2, \cdots \tag{4.28}$$

note that $r(s)x_n^2(s) > 0$ on $[A, B]$ so that c_n's in (4.28) *are well defined.*

Example 3.10.8 (Example 4.34)**.** *(i) Consider the BVP $x'' + \lambda x = 0, x(0) = 0, x'(1) = 0$. Note that this BVP is a Sturm-Liouville problem with $p \equiv 1, q \equiv 0, r \equiv 1; A = 0$ and $B = 1$. Hence by Theorem 4.29 the eigenfunctions are pairwise orthogonal. It is easy to show that the eigenfunctions are*

$$x_n(t) = \sin \frac{(2n+1)}{2}\pi t, \quad n = 0, 1, 2, \cdots; 0 \leq t \leq 1. \tag{4.29}$$

Thus, if $g(t)$ is any function such that $g(0) = 0$ and $g'(1) = 0$, then there exist constants $c_1, c_2, \cdots$ such that

$$g(t) = c_0x_0(t) + c_1x_1(t) + \cdots + c_nx_n(t) + \cdots \tag{4.30}$$

where c_n's are determined by the relation (4.28)*.*

(ii) It is known, from Chapter 3, that the Legendre polynomials $P_n(t)$ are the solutions of the Legendre equation

$$\tfrac{d}{dt}[(1 - t^2)x'] + \lambda x = 0, \lambda = n(n+1), -1 \leq t \leq 1.$$

The polynomials $P_n(t)$ form an orthogonal set of functions on $[-1,1]$. In this case $p(t) = (1-t^2), q \equiv 0, r \equiv 1$. Also notice that $p(1) = p(-1) = 0$ so that the boundary conditions are not needed for establishing the orthogonality of $P_n(t)$. Hence, if $g(t)$ is any piece-wise continuous function, then the eigenfunction expansion of $g(t)$ is

$$g(t) = c_0 p_0(t) + c_1 p_1(t) + \cdots + c_n p_n(t) + \cdots,$$

where

$$c_n = \tfrac{2n+1}{2} \int_{-1}^{1} g(s) P_n(s) ds, n = 0, 1, 2, \cdots$$

since

$$\int_{-1}^{1} P_n^2(s)ds = \frac{2}{2n+1}, n = 0, 1, 2, \cdots$$

EXERCISES

1. Show that corresponding to an eigenvalue the Sturm-Liouville problem (4.17), (4.18) or (4.17), (4.19) has a unique eigenfunction.

2. Show that the eigenvalues for the BVP $x'' + \lambda x = 0, x(0) = 0$ and $x(\pi) + x'(\pi) = 0$ satisfies the equation $\sqrt{\lambda} = -\tan \pi\sqrt{\lambda}$. Prove that the corresponding eigenfunctions are $\sin(t\sqrt{\lambda_n})$ where λ_n is an eigenvalue.

3. Consider the equation $x'' + \lambda x = 0, 0 < t \le \pi$. Find the eigenvalues and eigenfunctions in the following cases:

   (i) $x'(0) = x'(\pi) = 0$;

   (ii) $x(0) = 0, x'(\pi) = 0$;

   (iii) $x(0) = x(\pi) = 0$;

   (iv) $x'(0) = x(\pi) = 0$.

3.11 Green's Functions

The aim of this article is to construct what is known as Green's Function and then use it to solve a non-homogeneous BVP. We start with

$$L(x) + f(t) = 0, \quad a \le t \le b \tag{4.31}$$

where L is a differential operator defined by $L(x) = (px')' + qx$. Here p, p' and q are given real valued continuous functions defined on $[a, b]$ such that $p(t)$ is non-zero on $[a, b]$. Equation (4.31) is considered with separated boundary conditions

$$m_1 x(a) + m_2 x'(a) = 0 \tag{4.32a}$$

$$m_3 x(b) + m_4 x'(b) = 0 \tag{4.32b}$$

with the usual assumptions that at least one of m_1 and m_2 and one of m_3 and m_4 are non-zero.

Definition 3.11.1 (Definition 4.35)**.** *A function $G(t, s)$ defined on $[a, b] \times [a, b]$ is called Green's function for $L(x) = 0$ if, for a given $s, G(t, s) = G_1(t, s)$ if $t < s$ and $G(t, s) = G_2(t, s)$ for $t > s$ where G_1 and G_2 are such that:*

(i) G_1 satisfies the boundary condition (4.32a) at $t = a$ and $L(G_1) = 0$ for $t < s$;

(ii) G_2 satisfies the boundary condition (4.32b) at $t = b$ and $L(G_2) = 0$ for $t > s$;

(iii) The function $G(t, s)$ is continuous at $t = s$;

(iv) The derivative of G with respect to t has a jump discontinuity at $t = s$ and

$$\left[\frac{\partial G_2}{\partial t} - \frac{\partial G_1}{\partial t}\right]_{t=s} = -\frac{1}{p(s)}.$$

With this definition, the Green's function for (4.31), (4.32) is constructed. Let $y(t)$ be a non-trivial solution of $L(x) = 0$ satisfying the boundary condition (4.32a). Also let $z(t)$ be a non-trivial solution of $L(x) = 0$ which satisfies the boundary condition (4.32b).

Assumption Let y and z be linearly independent solutions of $L(x) = 0$ on (a, b). For some constants c_1 and c_2 define $G_1 = c_1 y(t)$ and $G_2 = c_2 z(t)$. Let

$$G(t,s) = \begin{cases} c_1 y(t) & \text{if } t \le s, \\ c_2 z(t) & \text{if } t \ge s. \end{cases} \tag{4.33}$$

Choose c_1 and c_2 such that

$$\begin{aligned} c_2 z(s) - c_1 y(s) &= 0 \\ c_2 z'(s) - c_1 y'(s) &= -1/p(s). \end{aligned} \tag{4.34}$$

With this choice of c_1 and $c_2, G(t,s)$ defined by the relation (4.33) has all the properties of the Green's function. Since y and z satisfy $L(x) = 0$ it follows that

$$y(pz')' - z(py')' \equiv \frac{d}{dt}[p(yz' - y'z)] = 0. \tag{4.35}$$

Hence

$$p(t)[y(t)z'(t) - y'(t)z(t)] = A \text{ for all } t \text{ in } [a,b]$$

where A is a non-zero constant (because y and z are linearly independent solutions of $L(x) = 0$). In particular it is seen that

$$y(s)z'(s) - y'(s)z(s)] = A/p(s), A \neq 0 \tag{4.36}$$

From equation (4.34) and (4.36) it is seen that

$$c_1 = -z(s)/A, c_2 = -y(s)/A.$$

Hence the Green's function is

$$G(t,s) = \begin{cases} -y(t)z(s)/A & \text{if } t \le s, \\ -y(s)z(t)/A & \text{if } t \ge s. \end{cases} \tag{4.37}$$

The main result of this article is Theorem 4.36.

Theorem 3.11.2 (Theorem 4.36)**.** *Let $G(t,s)$ be given by the relation* (4.37) *then $x(t)$ is a solution of* (4.31)*,*
(4.32) *if and only if*

$$x(t) = \int_a^b G(t,s)f(s)ds. \tag{4.38}$$

Proof. Let the relation (4.38) hold. Then

$$x(t) = -\Big[\int_a^t z(t)y(s)f(s)ds + \int_t^b y(t)z(s)f(s)ds\Big]\Big/A. \tag{4.39}$$

Differentiating (4.39) with respect to t yields

$$x'(t) = -\Big[\int_a^t z'(t)y(s)f(s)ds + \int_t^b y'(t)z(s)f(s)ds\Big]\Big/A. \tag{4.40}$$

Next on computing $(px')'$ from (4.40) and adding to qx in view of y and z being solutions of $L(x) = 0$ it follows that

$$L(x(t)) = -f(t) \tag{4.41}$$

Further, from the relations (4.39) and (4.40), it is seen that

$$\begin{cases} Ax(a) = -y(a)\int_a^b z(s)f(s)ds, \\ Ax'(a) = -y'(a)\int_a^b z(s)f(s)ds. \end{cases} \tag{4.42}$$

Since $y(t)$ satisfies the boundary condition given in ($4.32a$), it follows from (4.42) that $x(t)$ also satisfies the boundary condition ($4.32a$). Similarly $x(t)$ satisfies the boundary condition ($4.32b$). This proves that $x(t)$ satisfies (4.31) and (4.32).

Conversely, let $x(t)$ satisfy (4.31) and (4.32). Then from (4.31) it is clear that

$$-\int_a^b G(t,s)L(x(s))ds = \int_a^b G(t,s)f(s)ds \tag{4.43}$$

The left side of (4.43) is

$$-\int_a^t G_1(t,s)L(x(s))ds - \int_t^b G_2(t,s)L(x(s))ds. \tag{4.44}$$

Now a well-known result is used that if u and v are two functions which admit continuous derivatives in $[t_1, t_2]$, then

$$\int_{t_1}^{t_2} u(s)L(v(s))ds = \int_{t_1}^{t_2} v(s)L(u(s))ds + \Big[p(s)(u(s)v'(s) - u'(s)v(s))\Big]_{t_1}^{t_2} \tag{4.45}$$

Applying the identity (4.45) in (4.44) and using the properties of $G_1(t,s)$ and $G_2(t,s)$ the left side of (4.43) becomes

$$-p(t)\Big\{\Big[G_1(t,t)x'(t) - \frac{\partial G_1(t,s)}{\partial t}\Big|_{s=t}x(t)\Big] - \Big[G_2(t,t)x'(t) - \frac{\partial G_2(t,s)}{\partial t}\Big|_{s=t}x(t)\Big]\Big\} \tag{4.46}$$

The first and third term in (4.46) cancel each other because of continuity of $G(t,s)$ at $t = s$. The condition (iv) in the definition of Green's function now shows that the value of the expression (4.46) is $x(t)$. But (4.46) is the left side of (4.43) which means $x(t) = \int_a^b G(t,s)f(s)ds$. This completes the proof. □

Example 3.11.3 (Example 4.37)**.** *Consider the BVP*

$$x'' = f(t); x(0) = x(1) = 0. \tag{4.47}$$

It is easy to verify that the Green's function $G(t,s)$ is given by

$$G(t,s) = \begin{cases} t(1-s) & \text{if } t \le s, \\ s(1-t) & \text{if } t \ge s. \end{cases} \tag{4.48}$$

Thus the solution of (4.47) *is given by $x(t) = -\int_0^1 G(t,s)f(s)ds$.*

EXERCISES

1. In theorem 4.36 establish the relations (4.41), (4.45) and (4.46). Also show that if x satisfies (4.38), then x also satisfies the boundary conditions (4.32).

2. Prove that the Green's function defined by (4.37) is symmetric, that is $G(t,s) = G(s,t)$.

3. Show that the Green's function for $L(x) = x'' = 0, x(1) = 0; x'(0) + x'(1) = 0$ is

$$G(t,s) = \begin{cases} 1-s & \text{if } t \le s, \\ 1-t & \text{if } t \ge s. \end{cases}$$

   Hence solve the BVP

$$x'' = f(t), x(0) + x(1) = 0, \quad x'(0) + x'(1) = 0$$

where

(i) $f(t) = \sin \pi t;$

(ii) $f(t) = e^t; \quad 0 \le t \le 1$

(iii) $f(t) = t.$

4. Consider the BVP $x'' + f(t, x, x') = 0, x(a) = 0, x(b) = 0$. Show that $x(t)$ is a solution of the above BVP if and only if

$$x(t) = \int_a^b G(t,s) f(s, x(s), x'(s)) ds,$$

where $G(t, s)$ is the Green's function given by

$$(b-a)G(t,s) = \begin{cases} (b-t)(s-a) & \text{if } a \le s \le t \le b, \\ (b-s)(t-a) & \text{if } a \le t \le s \le b. \end{cases}$$

Also establish that

(i) $0 \le G(t,s) \le \frac{b-a}{4}$

(ii) $\int_a^b G(t,s)ds = \frac{(b-t)(t-a)}{2}$

(iii) $\int_a^b G(t,s)ds \le \frac{(b-a)^2}{8}$

(iv) $G(t, s)$ is symmetric.

5. Consider the BVP $x'' + f(t, x, x') = 0, x(a) = 0, x'(b) = 0$. Show that x is a solution of this BVP if, and only if, x satisfies

$$x(s) = \int_a^b H(t,s) f(s, x(s), x'(s)) ds, \quad a \le t \le b$$

where $H(t, s)$ is the Green's function defined by

$$H(t,s) = \begin{cases} s-a & \text{if } a \le s \le t \le b, \\ t-a & \text{if } a \le t \le s \le b. \end{cases}$$

Module 5
Aysmptotic behavior and Stability Theory
Lecture

3.12 Introduction

Once the existence of a solution for a differential equation is established, the next question is :" How does a solution grow with time "? It is all the more necessary to investigate such a behavior of solutions in the absence of an explicit solution. One of the way out is to find suitable criteria, in terms of the known quantities, to establish the asypmtotic behavior. A few such criteria are studied below. More or less a detailed analysis for linear systems is known.

In this chapter the asymtotic behavior of n-th order equations, autonomous systems of order two, linear homogeneous and non-homogeneous systems with constant and variable coefficients are dealt . The study includes the behavior of solutions for arbitrary large values of t as well as phase plane analysis. They are some kind of stability propeties for the concerned equatons..

3.13 Linear Systems with Constant Coefficients

Consider a linear system

$$x' = Ax, \quad 0 \le t < \infty, \tag{3.54}$$

where A is an $n \times n$ constant matrix. It is clear that the *priori* knowledge of eigenvalues of the matrix A kcompletely determines all solutions of (5.1) . So much so, the eigenvalues determine the behavior of solutions as $t \to \infty$. A suitable upper bound for the solutions of (5.1) is obtained below.

Theorem 3.13.1. *Let $\lambda_1, \lambda_1, \cdots, \lambda_m$ $(m \le n)$ be the distinct eigenvalues of the matrix A and λ_j be repeated n_j times $(n_1 + n_2 + \cdots + n_m = n)$. Let*

$$\lambda_j = \alpha_j + i\beta_j \quad (i = \sqrt{-1}, j = 1, 2, \cdots, m), \tag{3.55}$$

and $\eta \in \mathbb{R}$ be a number such that

$$\alpha_j > \eta, \quad (j = 1, 2, \cdots, m). \tag{3.56}$$

Then, there exists a real constant $M > 0$ such that

$$|e^{At}| \le Me^{\eta t}, \quad 0 \le t < \infty. \tag{3.57}$$

Proof. Let e_j be the n-vector with 1 in the j-th place and zero elsewhere. Then,

$$\varphi_j(t) = e^{At}e_j, \tag{3.58}$$

denotes the j-th column of the matrix e^{At}. From the previous modules on systems of equations , we know that

$$e^{At}e_j = \sum_{r=1}^{m}(c_{r1} + c_{r2}t + \cdots + c_{rn_r}t^{n_r - 1})e^{\lambda_r t}, \tag{3.59}$$

where $c_{r1}, c_{r2}, \cdots, c_{rn_r}$ are constant vectors. From (5.5) and (5.6) we have

$$|\varphi_j(t)| \le \sum_{r=1}^{m}(|c_{r1}| + |c_{r2}|t + \cdots + |c_{rn_r}|t^{n_r-1})|\exp(\alpha_r + i\beta_r)t| = \sum_{r=1}^{m} P_r(t)e^{\alpha_r t} \tag{3.60}$$

where P_r is a polynomial in t. By (5.3),

$$t^k e^{\alpha_r t} < e^{\eta t}, \tag{3.61}$$

for sufficiently large values of t. In view of (5.7) and (5.8) there exists $M_j > 0$ such that $|\varphi_j(t)| \le M_j e^{\eta t}$, $0 \le t < \infty$; $(j = 1, 2, \cdots, n)$. Now

$$|e^{At}| \le \sum_{j=1}^{n} |\varphi_j(t)| \le (M_1 + M_2 + \cdots + M_n)e^{\eta t} = Me^{\eta t} \quad (0 \le t < \infty),$$

where $M = M_1 + M_2 + \cdots + M_n$ which proves the inequality (5.4). □

Actually we have estimated an upper bound for the fundamental matrix e^{At} for the equation (5.1) in terms of an exponential function given through the inequality (5.4). Theorem 5.2.2 proved subsequently is a direct consequence of Theorem 5.2.1 . It tells us about a necessary and sufficient conditions for the solutions of (5.1) decaying to zero as $t \to \infty$. In other words, it chaterizes a certain asymptotic behavior of solutions of (5.1)

Theorem 3.13.2. *Every solution of the equation* (5.1) *tends to zero as* $t \to +\infty$ *if and only if the real parts of all the eigenvalues of* A *are negative.*

Obviously, if the real part of an eigenvalue is positive and if φ is a solution corresponding to this eigenvalue then $|\varphi(t)| \to +\infty$ as $t \to \infty$.

The system

$$x' = Ax + b(t), \tag{3.62}$$

where A is an $n \times n$ constant matrix, is a perturbed system with the perturbation term b, where $b : [0, \infty) \to \mathbb{R}$ is assumed to be continuous . Since the fundamental matrix for the system (5.1) is e^{tA} any solution of (5.9) is (by the method of variation of parameters) is

$$x(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A}b(s)ds, \quad t \geq t_0 \geq 0,$$

satisfies the equation (5.9). Here x_0 is an n-vector such that $x(t_0) = x_0$ and e^{At} is the fundamental matrix of (5.1). Taking the norm on both sides it is seen

$$|x(t)| \leq |e^{(t-t_0)A}x_0| + \int_{t_0}^{t} |e^{(t-s)A}||b(s)|ds, \quad 0 \leq t_0 \leq t < \infty.$$

Suppose $|x_0| \leq K$ and η is a number such that $\eta > R \exp \lambda_i$, $i = 1, 2, \cdots, m$, where λ_i are the eigenvalues of the matrix A. Now, in view of (5.4) it is seen that

$$|x(t)| \leq KMe^{\eta(t-t_0)} + M\int_{t_0}^{t} e^{\eta(t-t_0)}|b(s)|ds. \tag{3.63}$$

The inequality (5.10) has been obtained by using the conclusion arrived at in Theorem 5.2.1 . We note that the right side is independent of x. It depends on the constants K, M and η and the function b. The inequality (5.10) is a pointwise estimate. The nature of solution x for large value of t would depend on the sign of the constant φ and the nature of the function b. In the following theorem, b is assumed to satisfy a certain growth condition.

Theorem 3.13.3. *Suppose the function* b *is such that*

$$|b(t)| \leq pe^{at}, \quad t \geq T \geq 0, \tag{3.64}$$

where p *and* a *are constants with* $p \geq 0$. *Then every solution* $x(t)$ *of the system* (5.9) *satisfies*

$$|x(t)| \leq Le^{qt} \tag{3.65}$$

where L *and* q *are constants.*

Proof. Since b is continuous on $0 \leq t < \infty$, it is clear that every solution $x(t)$ of (5.9) exists on $0 \leq t < \infty$. Further

$$x(t) = e^{At}c + \int_{0}^{t} e^{(t-s)A}b(s)ds, \quad 0 \leq t < \infty, \tag{3.66}$$

where c is a suitable constant vector. From Theorem 5.2.1 it is clear that there exists M and η such that

$$|e^{At}| \leq Me^{\eta t}, \ 0 \leq t < \infty. \tag{3.67}$$

For some T $(0 \leq T < \infty)$ Rewrite (5.13) as,

$$x(t) = e^{At}c + \int_0^t e^{(t-s)A}b(s)ds + \int_T^t e^{(t-s)A}b(s)ds. \tag{3.68}$$

Define

$$M_1 = \sup\{|b(s)| : 0 \leq s \leq T\}. \tag{3.69}$$

Now from the relation (5.11), (5.14) and (5.16) the following is follows:

$$\begin{aligned}|x(t)| &\leq M|c|e^{\eta t} + M\int_0^T e^{\eta(t-s)}M_1 ds + M\int_T^t e^{\eta(t-s)}pe^{as}ds \\ &= Me^{\eta t}\Big[|c| + \int_0^T e^{-\eta s}M_1 ds + \int_T^t e^{(a-\eta)s}p\ ds\Big].\end{aligned}$$

Assume that $a \neq \eta$. Then

$$|x(t)| \leq Me^{\eta t}\Big[|c| + \int_0^T e^{-\eta s}M_1 ds + \frac{p}{|a-\eta|}e^{(a-\eta)T}\Big] + \frac{pM}{a-\eta}e^{at}$$

Now, by choosing $q = \max(\eta, a)$we have

$$|x(t)| \leq M\Big[|c| + \int_0^T e^{-\eta s}M_1 ds + \frac{p}{|a-\eta|}e^{(a-\eta)T} + \frac{pM}{a-\eta}\Big]e^{at} = Le^{qt}$$

where

$$L = M\Big[|c| + \int_0^T e^{-\eta s}M_1 ds + \frac{p}{|a-\eta|}e^{(a-\eta)T} + \frac{pM}{a-\eta}\Big]$$

and the above inequality yeilds the desired estimate of a solution x. Thus t he behavior of the solution for the large values of t depends on the the q and on L. □

Example 3.13.4. *Consider*

$$\begin{aligned}x_1' &= -3x_1 - 4x_2, \\ x_2' &= 4x_1 - 9x_2.\end{aligned}$$

The characteristic equation is

$$\lambda^2 + 12\lambda + 43 = 0.$$

whose roots are $\lambda_1 = -6 + 7i$, $\lambda_2 = -6 - 7i$. The real parts of the roots are negative. Hence, all solutions tend to zero at $t \to +\infty$.

Example 3.13.5. *Consider*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 2 & 3 & 1 \\ -3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 9\lambda - 8 = 0$$

whose roots are

$$\lambda_1 = 1,\ \lambda_2 = \frac{1+\sqrt{31}i}{2},\ \lambda_3 = \frac{1-\sqrt{31}i}{2}.$$

The real parts of the roots are positive. All non-trivial solutions of the system are unbounded.

EXERCISES

1. Complete the proof of Theorem 5.2.2.

2. Determine the nature of the solutions as $t \to +\infty$ for the system $x' = Ax$ where

   (i) $A = \begin{bmatrix} -9 & 19 & 4 \\ -3 & 7 & 1 \\ -7 & 17 & 2 \end{bmatrix}$;

   (ii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$;

   (iii) $A \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

3. Determine the behavior of solutions and their first two derivatives as $t \to +\infty$ for the following equations:

   (i) $x''' + 4x'' + x' - 6x = 0$;

   (ii) $x''' + 5x'' + 7x' = 0$;

   (iii) $x''' + 4x'' + x' + 6x = 0$.

4. Find all solutions of the following nonhomogeneous system and discuss their behavior as $t \to +\infty$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

   where

   (i) $b_1(t) = \sin t, \quad b_2(t) = \cos t$;

   (ii) $b_1(t) = 0, \quad b_2(t) = 1$; and

   (iii) $b_1(t) = t, \quad b_2(t) = 0$.

Lecture

3.14 Linear Systems with Variable Coefficients

Consider a linear system

$$x' = A(t)x, \quad t \geq 0 \tag{3.70}$$

where for each $t \in \ (0 \leq t < \infty)$, $A(t)$ is a real valued, continuous $n \times n$ matrix . We intende to find the behavior of solution of (5.17) as $t \to +\infty$. Two such results proved below which depends on the eigenvalues of the matrix $A(t) + A^T(t)$, where $A^T(t)$ is the transpose of matrix $A(t)$. Obviously, the eigenvalues are functions of t.

Theorem 3.14.1. *for each $t \in \ (0 \le t < \infty)$ let $A(t)$ be a real valued, continuous $n \times n$ matrix . Let $M(t)$ be the largest eigenvalues of $A(t) + A^T(t)$. If*

$$\lim_{t \to +\infty} \int_{t_0}^{t} M(s)ds = -\infty \quad (t_0 > 0 \ \ \textit{is fixed}); \tag{3.71}$$

then every solution of (5.17) *tends to zero as $t \to +\infty$.*

Proof. Let φ be a solution of (5.17). Then, $|\varphi(t)|^2 = \varphi^T(t)\varphi(t)$. Differentiation leads to

$$\begin{aligned}\frac{d}{dt}|\varphi(t)|^2 &= \varphi^T(t)\varphi'(t) + \varphi^{T'}(t)\varphi(t)\\ &= \varphi^T(t)A(t)\varphi(t) + \varphi^T(t)A^T(t)\varphi(t)\\ &= \varphi^T(t)[A(t) + A^T(t)]\varphi(t)\end{aligned}$$

The matrix $A(t) + A^T(t)$ is symmetric and since $M(t)$ is the largest eigenvalue we have

$$|\varphi^T(t)[A(t) + A^T(t)]\varphi(t)| \le M(t)|\varphi(t)|^2$$

$$0 \le |\varphi(t)|^2 \le |\varphi(t_0)|^2 \Big(\exp\Big(\int_{t_0}^{t} M(s)ds \Big)\Big). \tag{3.72}$$

By the condition (5.18) the right side tends to zero. Hence

$$\lim \varphi(t) = 0 \ \text{ as } t \to \infty$$

which completes the proof. □

Theorem 3.14.2. *Let $m(t)$ be the smallest eigenvalue of $A(t) + A^T(t)$. If*

$$\limsup_{t \to +\infty} \int_{t_0}^{t} m(s)ds = +\infty \quad (t_0 > 0 \ \textit{is fixed}); \tag{3.73}$$

then every nonzero solution of (5.17) *is unbounded as $t \to +\infty$.*

Proof. As in the proof of Theorem 5.2.1 we have

$$\frac{d}{dt}|\varphi(t)|^2 \ge m(t)|\varphi(t)|^2.$$

Thus

$$\frac{d}{dt}\Big[\exp\Big(-\int_{t_0}^{t} m(s)ds\Big)|\varphi(t)|^2\Big] = \exp\Big(-\int_{t_0}^{t} m(s)ds\Big)\Big[\frac{d}{dt}|\varphi(t)|^2 - m(t)|\varphi(t)|^2\Big] \ge 0$$

whence

$$|\varphi(t)|^2 \ge |\varphi(t_0)|^2 \exp\Big(-\int_{t_0}^{t} m(s)ds$$

By (5.20) we note that the expression on the right hand side tends to $+\infty$ as $t \to \infty$ or else

$$\lim |\varphi(t)| = \infty, \ \text{as } t \to \infty.$$

□

Example 3.14.3. *Consider the system*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1/t^2 & t^2 \\ -t^2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Comparing the system with (5.17)*, we get*

$$A(t) + A^T(t) = \begin{bmatrix} 2/t^2 & 0 \\ 0 & -2 \end{bmatrix}$$

So

$$M(t) = \frac{2}{t^2}, \quad m(t) = -2, \quad \lim_{t\to\infty} \int_{t_0}^{t} \frac{2}{s^2} ds = \frac{2}{t_0} > -\infty.$$

The exponential term remains bounded as $t \to \infty$ due to (5.19)*. Thus, $\varphi(t)$ is bounded as $t \to +\infty$.*

Example 3.14.4. *For the system $x' = A(t)x$, where*

$$A(t) = \begin{bmatrix} -1/t & t^2+1 \\ -(t^2+1) & -2 \end{bmatrix}, \quad A(t) + A^T(t) = \begin{bmatrix} -2/t & 0 \\ 0 & -4 \end{bmatrix}$$

for which $M(t) = -2/t$ for $t > \frac{1}{2}$ and $m(t) = -4$. Now

$$\lim_{t\to+\infty} \int_{t_0}^{t} \frac{-2}{s} ds = \lim_{t\to\infty} (-2\log t + 2\log t_0) = -\infty.$$

The condition (5.18) *holds and so the solutions tends to zero as $t \to +\infty$.*

Theorem 5.3.5 provides a criterion for boundedness of the inverse of a fundamental matrix.

Theorem 3.14.5. *Let Φ be a fundamental matrix of* (5.17) *which is uniformly bounded over* $[0, \infty)$*. Suppose*

$$\liminf \int_{t_0}^{t} tr A(s) ds > -\infty \quad as \quad t \to \infty. \tag{3.74}$$

Then $|\Phi^{-1}(t)|$ is uniformly bounded on $[0, \infty)$*.*

Proof. Let Φ be a fundamental matrix of (5.17). By Able's formula

$$\det \Phi(t) = \det \Phi(0) \exp \int_{t_0}^{t} tr\ A(s) ds. \tag{3.75}$$

Now the relations (5.21) and (5.22) imply $\det \Phi(t) \neq 0$, $t \in [0, \infty)$. Further, since $|\Phi(t)|$ is uniformly bounded so is $\det\ \Phi(t)$. Now it is known that

$$\Phi^{-1}(t) = \frac{adj\ [\Phi(t)]}{\det \Phi(t)}$$

or else we have a bound $k > 0$ such that

$$\big| adj\ [\Phi(t)] \big| \leq k, \quad t \in [0, \infty).$$

Thus, $|\Phi^{-1}(t)|$ is well-defined and uniformly bounded. □

also let us note that (since $\det \Phi(t) \neq 0$) for all values of t none of the solutions ϕ which form the fundamental matrix, can tend to zero as $t \to +\infty$. Thus, no solution except the null solution of the equation (5.17) tends to zero as $t \to +\infty$.

It is interesting to note that the above Theorem (5.3.5) can be used to study the behavior of solutions of an equation of the form

$$x' = B(t)x, \quad t \in [0, \infty), \tag{3.76}$$

where B is a continuous $n \times n$ matrix defined on $[0, \infty)$. Let ψ denote a solution of (5.23). Suppose

$$\int_0^\infty |A(t) - B(t)|dt < \infty. \tag{3.77}$$

The following is a reult on boundedness of solutions.

Theorem 3.14.6. *Let the hypotheses of Theorem* **??** *and the condition* (5.24) *hold. Then, any solution $\psi(t)$ of* (5.23) *is bounded on* $[0, \infty)$.

Proof. Let $\varphi(t)$ be a solution of (5.17). It is easy to verify that $\psi(t)$ is a solution of the equation

$$x' = A(t)x + [B(t) - A(t)]x.$$

Hence, by using the variation of parameters formula, we obtain

$$\psi(t) = \varphi(t) + \Phi(t) \int_0^t \Phi^{-1}(s)(B(s) - A(s))\psi(s)ds.$$

With the norm on either side

$$|\psi(t)| \leq |\varphi(t)| + |\Phi(t)| \int_0^t |\Phi^{-1}(s)||B(s) - A(s)||\psi(s)|ds$$

Using the more general form of the Gronwall's inequality,we have

$$|\psi(t)| \leq |\varphi(t)| + \int_0^t |\Phi(t)||\phi(s)||\Phi^{-1}(s)||B(s) - A(s)|\Big(\exp \int_s^t |\Phi(u)||\Phi^{-1}(u)||B(u) - A(u)|du\Big)ds.$$

$$= |\varphi(t)| + |\Phi^{-1}(t)| \int_0^t |\varphi(s)||\Phi^{-1}(s)||B(s) - A(s)|\Big(\exp \int_s^t |B(u) - A(u)|du\Big)ds.$$

By (5.24), observe that the right side is bounded. Thus, $\psi(t)$ is bounded on $[0, \infty)$. □

Theorem 3.14.7. *Let the hypotheses of Theorems 5.3.5 and 5.3.6 hold. Then, corresponding to any solution φ of* (5.17) *there exists a unique solution ψ of* (5.23)*, such that $|\psi(t) - \varphi(t)| \to 0$ as $t \to \infty$.*

Proof. Let φ be a given solution of (5.17). Any solution ψ of (5.23) may be written in the form

$$\psi(t) = \varphi(t) - \int_t^\infty \Phi(t)\Phi^{-1}(s)(B(s) - A(s))\varphi(s)ds, \quad t \in [0, \infty).$$

The above relation determines uniquely the solution ψ of (5.23). Clearly under the given conditions

$$\lim |\psi(t) - \varphi(t)| = 0 \text{ as } t \to \infty.$$

□

The above theorem establishes a kind of equivalence between the two systems (5.17) and (5.23). This relationship between the two systems is many times known as asymptotic equivalence. It is not intended here to go into details of this concept.
Perturbed Sysems: The equation

$$x' = A(t)x + B(t), \quad 0 \le t < \infty, \tag{3.78}$$

is called a perturbed system of (5.17), where B is a continuous n-column vector function defined on $0 \le t < \infty$. The behavior of the solutions of such a system (5.25) is closely related to the behavior of solution of the system (5.17).

Theorem 3.14.8. *Suppose every solution of* (5.17) *tends to zero as* $t \to +\infty$*. If one solution of* (5.25) *is bounded then, all of its solutions are bounded.*

Proof. Let ψ_1 and ψ_2 be any two solutions of (5.25). Then $\varphi = \psi_1 - \psi_2$ is a solution of (5.17). By Noting $\psi_1 = \psi_2 + \varphi$ then, clearly $\psi_1(t)$ is bounded, if $\psi_2(t)$ is bounded, since $\varphi(t) \to 0$ as $t \to +\infty$. This completes the proof. □

From the Theorem 5.3.8 it is clear that if $\psi_2(t) \to \infty$ as $t \to \infty$ then $\psi_1(t) \to \infty$ as $t \to \infty$. If $\psi_2(t) \to 0$ as $t \to \infty$ then $\psi_1(t) \to 0$ as $t \to \infty$. The next comparison theorem asserts the boundedness of solutions of (5.25).

Theorem 3.14.9. *Let the matrix* $A(t)$ *in* (5.17) *be such that*

$$\liminf_{t\to\infty} \int_0^t tr\ A(s)ds > -\infty \tag{3.79}$$

and let $\int_0^\infty |b(s)|ds < \infty$*. If every solution of* (5.17) *is bounded on* $[0,\infty)$ *then, every solution of the equation* (5.25) *is bounded.*

Proof. Let $\varphi(t)$ be any solution of (5.25). Then

$$\varphi(t) = \Phi(t)C + \Phi(t)\int_0^t \Phi^{-1}(s)b(s)ds.$$

Here Φ represents a fundamental matrix of the equation (5.17) and C is a constant vector. Since every solution of (5.17) is bounded on $[0,\infty)$, there is a constant K such that $\Phi(t) \le K$ for $t \in [0,\infty)$. Hence, $\Phi(t)$ is uniformly bounded on $[0,\infty)$. The condition in (5.26) implies, as in Theorem **??**, that $\Phi^{-1}(t)$ is bounded. Taking the norm on either side we have

$$|\varphi(t)| \le |\Phi(t)||C| + |\Phi(t)|\int_0^t |\Phi^{-1}(s)||b(s)|ds$$

Now each term on the right side is bounded which shows that $\varphi(t)$ is also bounded. □

EXERCISES

1. Show that any solution of $x' = A(t)x$ tend to zero as $t \to 0$ where,

(i) $A(t) = \begin{bmatrix} -t & 0 & 0 \\ 0 & -t^2 & 0 \\ 0 & 0 & -t^2 \end{bmatrix}$;

(ii) $A(t) = \begin{bmatrix} -e^t & -1 & -\cos t \\ 1 & -e^{2t} & t^2 \\ \cos t & -t^2 & -e^{3t} \end{bmatrix}$;

(iii) $A(t) = \begin{bmatrix} -t & \sin t \\ 0 & e^{-t} \end{bmatrix}$.

2. Consider a system $x' = A(t)x$. Let $M(t)$ be the largest eigenvalue of $A(t) + A^T(t)$ such that $\int_{t_0}^{\infty} M(s)ds < \infty$. Show that all the solutions of $x' = A(t)x$ are bounded.

3. Prove that all the solutions of $x' = A(t)x$ are bounded, where $A(t)$ is given by

$$\text{(i)} \begin{bmatrix} e^t & -1 & -2 \\ 1 & e^{-2t} & 3 \\ 2 & -3 & e^{-3t} \end{bmatrix}, \ \text{(ii)} \begin{bmatrix} (1+t)^{-2} & \sin t & 0 \\ -\sin t & 0 & \cos t \\ 0 & -\cos t & 0 \end{bmatrix} \ \text{and} \ \text{(iii)} \begin{bmatrix} e^{-t} & 0 \\ 0 & -1 \end{bmatrix}.$$

4. What can you say about the boundedness of solutions of the system $x' = A(t)x + f(t)$ on $(0, \infty)$ when a particular solution x_p, the matrix $A(t)$ and the function f are as given below:

(i) $x_p(t) = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}$, $A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $f(t) = \begin{bmatrix} e^{-t} & \cos t \\ -e^{-t} & \sin t \end{bmatrix}$,

(ii) $x_p(t) = \begin{bmatrix} \frac{1}{2}(\sin t - \cos t) \\ 0 \\ 0 \end{bmatrix}$, $A(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -t^2 & 0 \\ 0 & 0 & -t^2 \end{bmatrix}$, $f(t) = \begin{bmatrix} \sin t \\ 0 \\ 0 \end{bmatrix}$.

5. Show that the solutions of $x' = A(t)x + f(t)$ are bounded on $[0, \infty)$ for the following cases:

(i) $A(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$, $f(t) = \begin{bmatrix} \sin t \\ \sin t^2 \end{bmatrix}$;

(ii) $A(t) = \begin{bmatrix} (1+t)^{-2} & \sin t & 0 \\ -\sin t & 0 & t \\ 0 & -t & 0 \end{bmatrix}$, $f(t) = \begin{bmatrix} 0 \\ (1+t)^{-2} \\ (1+t)^{-3} \end{bmatrix}$.

3.15 Second Order Linear Differential Equations

Hitherto we have considered the asymptotic behavior and boundedness of solutions of a linear system. Now we glace at asymptotoic behavior of solutions of second order linear differential equations. A large number of methods are available for such a study the behavior of solutions of second order linear differential equations. In this section, we consider some of them mainly concentrating on the equation

$$x'' + a(t)x = 0, \quad 0 \le t < \infty \tag{3.80}$$

where $a : [0, \infty] \to \mathbb{R}$ is a continuous function . The following results through light on the boundedness of solutions of (5.27).

Theorem 3.15.1. *Let a be a non-decreasing continuous function such that*

$$a(t) \to \infty \text{ as } t \to \infty.$$

Then, all solutions of (5.27) *are bounded.*

Proof. Multiply (5.27) by x' to get

$$x'x'' + a(t)xx' = 0.$$

Integration leads to

$$\int_0^t x'(s)x''(s)ds + \int_0^t a(s)x(s)x'(s)ds = c_1$$

which is the same as

$$\frac{1}{2}x'^2(t) + \frac{1}{2}a(t)x^2(t) - \int_0^t \frac{x^2(s)}{2}da(s) = c_1.$$

The first term on the left side is nonnegative. Consequently

$$a(t)\frac{x^2(t)}{2} \le c_1 + \frac{1}{2}\int_0^t x^2(s)da(s).$$

Now an application of Gronwall's inequality gives us

$$a(t)\frac{x^2(t)}{2} \le c_1 \exp \int_0^t \frac{da(s)}{a(s)} \le c_1 a(t)$$

which shows that $x^2(t) \le 2|c_1|$ thereby completing the proof. $\square$

Theorem 3.15.2. *Let x be a solution of the equation* (5.27) *and let*

$$\int_0^\infty t|a(t)|dt < \infty.$$

Then $\lim x'$ *exists and further the general solution of* (5.27) *is asymptotic to $a_0 + a_1 t$, where a_0 and a_1 are constants simultaneously not equal to zero.*

Proof. We integrate (5.27) twice to get

$$x(t) = c_1 + c_2 t - \int_1^t (t-s)a(s)x(s)ds \tag{3.81}$$

from which we have, for $t \ge 1$,

$$|x(t)| \le (|c_1| + |c_2|)t + t\int_1^t |a(s)||x(s)|ds.$$

That is,

$$\frac{|x(t)|}{t} \le (|c_1| + |c_2|) + \int_1^t s|a(s)|\frac{|x(s)|}{s}\,ds.$$

Gronwall's inequality now implies

$$\frac{|x(t)|}{t} \le (|c_1| + |c_2|) \exp \int_1^t s|a(s)|ds \le c_3, \tag{3.82}$$

in view of the hypothyses of the theorem. Differentiation of (5.28) now yields

$$x'(t) = c_2 - \int_1^t a(s)x(s)ds.$$

Now the estimate (5.29) gives us

$$|x'(t)| \leq |c_2| + \int_1^t |a(s)||x(s)|ds \leq |c_2| + c_3 \int_1^t s|a(s)|ds < \infty. \tag{3.83}$$

Thus, $\limsup |x'(t)|$ as $t \to \infty$ exists.
Let $\limsup |x'(t)| \neq 0$ as $t \to \infty$. Then, from (5.29) we have

$$x(t) \sim a_1 t \text{ as } t \to \infty \quad (a_1 \neq 0).$$

The second solution of (5.27) is

$$u(t) = x(t) \int_t^\infty \frac{ds}{x^2(s)} \sim a_1 t \int_t^\infty \frac{ds}{a_1^2 s^2} \sim \frac{1}{a_1} = a_0 \ (say).$$

Hence, the general solution of (5.27) is asymptotic to $a_0 + a_1 t$. □

Remark : In the above proof it is assumed that $\lim_{t\to\infty} |x'(t)| \neq 0$. Such a choice is always possible. For this purpose, choose $c_2 = 1$ and the lower limit t_0 in place of 1. Let $1 - c_3 \int_{t_0}^\infty s|a(s)|ds > 0$. Clearly, $\lim_{t\to\infty} |x'(t)| \neq 0$.

EXERCISES

1. Prove that, if $a(t) > 0$ and $a'(t)$ exists for all $t \geq 0$ then, any solution of $x'' + a(t)x = 0$ satisfies the inequality

$$x^2 t \leq \frac{c_1}{a(t)} \exp\Big(\int_0^t \frac{a'(t)}{a(t)} dt\Big), \quad t \geq 0.$$

2. If $\int_0^\infty |a(t)|dt < \infty$, prove that all the solutions of $u'' + a(t)u = 0$ cannot be bounded.

3. Show that the equation $x'' - \phi(t)x = 0$ can have no non-trivial solutions bounded for $-\infty < t < \infty$, if $\phi(t) > \alpha > 0$ for $-\infty < t < \infty$.

4. Prove that if all solutions of $x'' + a(t)x = 0$ are bounded then all solutions of $x'' + [a(t) + b(t)]x = 0$ are also bounded if $\int_0^\infty |b(s)|ds < \infty$.

5. Prove that all solutions of $x'' + [1 + a(t) + b(t)]x = 0$ are bounded provided that
   (i) $\int_0^\infty |a(s)|ds < \infty$,
   (ii) $\int_0^\infty |b(s)|ds < \infty$, $\quad b(t) \to 0$ as $t \to \infty$.

Stability of Nonlinear Systems

Introduction

We have studied the behavior of solutions of linear systems when the time increased indefinitely. It was mentioned that the behavior of solutions as $t \to \infty$ is a kind of stability property. So far the notion of stability has not been precisely defined. We devote the rest of this module to introduce the concept of stability of solutions. Before proceeding, let us examine the following problem.

Suppose a physical phenomenon is governed by a differential equation. Fix a stationary state of the system (which is also known as the unperturbed state). Let an external force act on the system which results in perturbing the stationary state. The question now is whether this perturbed state will be "close" enough to the unperturbed state. In other words, what is the order of the magnitude of the change from the stationary state ? Usually this change is estimated by a norm which also measures the size of the perturbation.

A system is called stable if the change is small provided at the time of starting the size of the perturbation is small enough. If the perturbed system moves away from the stationary state in spite of the size of the perturbation being small at the initial time, then it is customary to label such a system as unstable. The following example illustrates further the concept of stability.

Let us consider the oscillation of a pendulum of a clock. When we start a clock we deflect the pendulum away from its vertical position. If the pendulum is given a small deflection then after some time it returns to its vertical position. If the deflection is sufficiently large then oscillations start and after some time the amplitude of the oscillations retains a fairly constant value. The clock then works for a long time with this amplitude. Now the oscillations of a pendulum can be described by a system of equation. This system has two equilibrium states (stationary solutions), one being the position of rest and the other the normal periodic motion. For any perturbation of the pendulum a new motion is obtained which is also a solution of the system. This new solution approaches fast to either of these two stationary solutions and after some time they almost coincide with them. In this case it is said that both the stationary solutions are stable.

This chapter is devoted to the study of the stability of stationary solutions. Below, the definitions of stability due to Lyapunov are listed. Among the methods known today, to study the stability properties, the direct or the second method due to Lyapunov is important and useful. This method involves a construction of a scalar function satisfying certain conceivable conditions. Further it does not depend on the knowledge of solutions in a closed form.

Stability Definitions

In many of the problems the main interest revolves round the stability behavior of solutions of nonlinear differential equations which describes the problem. Such a study turns out to be difficult due to the lack of closed form solutions of such equations. The study is more or less concerned with the family of motions defined by a differential equation

$$x' = f(t, x), \quad t \geq t_0 \geq 0, \tag{3.84}$$

where x and f are n-vectors. The following notion is used:

$$I = [t_0, \infty), \quad \text{for } \rho > 0, \quad S_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}. \tag{3.85}$$

Let us assume that the function f in the equation (5.32) is defined and in continuous on $I \times S_\rho$. Let (5.32) posses a unique solution $x(t; t_0, x_0)$ in S_ρ passing through a point (t_0, x_0) on I and let it continuously depend on (t_0, x_0). For simplicity, the solution $x(t; t_0, x_0)$ is denoted by $x(t)$. $x(t)$ is treated as a special solution the stability of which is under consideration. In the physical sense this implies that $x(t)$ is an equilibrium position of an object the motion of which is determined by the equation (5.32). It is to be noted that we are assuming the existence of a unique solution of (5.32). The following definitions distinguish between various types of behavior of solutions.

Definition 3.15.3 (Definition 6.17)**.** *(i) A solution $x(t)$ is said to be* stable *if for each $\epsilon > 0 (\epsilon < \rho)$ there exists a positive number $\delta = \delta(\epsilon)$ such that any solution $y(t)$ of* (5.32) *existing on I satisfies*

$$|y(t) - x(t)| < \epsilon, t \geq t_0 \quad \textit{whenever} \quad |y(t_0) - x(t_0)| < \delta.$$

(ii) A solution $x(t)$ is said to be asymptotically stable *if it is stable and if there exists a number $\delta_0 > 0$ such that any other solution $y(t)$ of* (5.32) *existing on I is such that*

$$|y(t) - x(t)| \to 0 \text{ as } t \to \infty \quad \textit{whenever} \quad |y(t_0) - x(t_0)| < \delta_0.$$

(iii) A solution $x(t)$ is said to be unstable *if it is not stable.*

In this definition the existence of a solution $x(t)$ of (5.32) is assumed. In general, there is no loss of generality if we consider this special solution to be the zero solution. This assumption would be at once clear if we consider the transformation

$$z(t) = y(t) - x(t), \tag{3.86}$$

where $y(t)$ is any solution of (5.32). Since $y(t)$ satisfies (5.32), it is seen that

$$y'(t) = z'(t) + x'(t) = f(t, z(t) + x(t)).$$

Hence $z'(t) = f(t, z(t) + x(t)) - x'(t)$. Set $\tilde{f}(t, z(t)) = f(t, z(t) + x(t)) - x'(t)$. Hence

$$z'(t) = \tilde{f}(t, z(t)). \tag{3.87}$$

Clearly, in view of (5.32), it is seen that

$$\tilde{f}(t, 0) = f(t, x(t)) - x'(t) \equiv 0$$

Thus the resulting system (5.34) possesses a trivial solution or a zero solution. It is important to note that the transformation (5.33) does not change the character of the stability of a solution of (5.32). In subsequent discussions it is assumed that (5.32) possesses a trivial or a null solution which is the state of equilibrium.

The stability Definition 6.17 becomes clearer when it is viewed geometrically. Figure 6.1 depicts this behavior and is drawn in phase space for $n = 2$. Time axis can be considered as a line perpendicular to the plane at the origin. The solution represented in the figure are the projections of solutions $y(t)$ on the phase space. Let us assume that the origin is the unperturbed state.

Consider a circle with origin at the center and radius ϵ where $\epsilon < \rho$. The definition 6.17 for stability states that a circle with radius δ exists such that if $y(t_0)$ is in S_δ then $y(t)$ remains in S_ϵ for all $t \geq t_0$. Further, it never reaches the boundary point of S_ϵ. Clearly in this case $\delta \leq \epsilon$ (Refer to Fig. 6.1).

Now let the origin be stable. Let the starting point $y(t_0)$ lie in S_{δ_0}, $\delta_0 > 0$. Let $y(t)$ approach the origin as time increases indefinitely. In this case the origin is asymptotically stable.

Further consider an S_ϵ region and any arbitrary number $\delta (\delta < \epsilon)$ however small. Let $y(t)$ be a solution through any point of S_δ. If the system is unstable $y(t)$ reaches the boundary of S_ϵ for some t in I.

The stability definitions given above are due to Lyapunov. We have mentioned above only few stability behaviors of solutions of (5.32). It is to be remarked that several other stability properties have been introduced and investigated in detail and voluminous literature is now available on this topic. However, only the above mentioned stability properties are discussed in this chapter.

The following examples illustrates Definition 6.17.

Example 3.15.4 (Example 6.18). *Let $x' = 0$. Then $y(t) = c$ where c is an arbitrary constant is a solution. Let the solution $x(t) \equiv 0$ be the unperturbed state. For a given $\epsilon > 0$, for stability it is necessary to have $|y(t) - x(t)| = |y(t) - 0| = |c| < \epsilon$ for $t \geq t_0$ whenever $|y(t_0) - x(t_0)| = |c - 0| = |c| < \delta$. If we choose $\delta \leq \epsilon$, then the criterion for stability is satisfied. Note that $x(t) \equiv 0$ is not asymptotically stable.*

Example 3.15.5 (Example 6.19). *Let $x' = -x$. Then $y(t) = ce^{-(t-t_0)}$ is a solution. We first study the stability of the origin and so compute $|y(t) - x(t)| = |ce^{-(t-t_0)}| < \epsilon$ for $t \geq t_0$. Let $\delta < \epsilon$. By choosing $|y(t_0) - x(t_0)| = |c| < \delta$, it is clear that $x(t) \equiv 0$ is stable. Further, let $\delta = \delta_0$ so that $|c| < \delta_0$. Clearly $|ce^{-(t-t_0)}| \to 0$ as $t \to \infty$. Hence it is concluded that $x(t) \equiv 0$ is asymptotically stable.*

Example 3.15.6 (Example 6.20). *Consider the equation $x' = x$. Then any solution through (t_0, η) is $y(t) = \eta \exp(t - t_0)$. Choose any $\eta > 0$. Clearly as t increases indefinitely this solution escapes out of any neighborhood of the origin. The origin, in this case, is unstable.*

EXERCISES

1. Show that the system $x' = y,\ y' = -x$ is stable but not asymptotically stable.

2. Prove that $x' = -x,\ y' = -y$ is asymptotically stable; however, the system $x' = x,\ y' = y$ is unstable.

3. Determine the stability of the origin in the following cases:

   (i) $x''' + 6x'' + 11x' + 6x = 0$,

   (ii) $x''' - 6x'' + 11x' - 6x = 0$,

   (iii) $x''' + ax'' + bx' + cx = 0$, for all possible values of a, b and c.

4. Consider the system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

   Show that no non-trivial solution of this system tends to zero as $t \to \infty$. Is every solution bounded ? Is it periodic ?

5. Prove that for $1 < \alpha < \sqrt{2}$, $x' = (\sin \log t + \cos \log t - \alpha)x$ is asymptotically stable.

6. Consider the equation $x' = a(t)x$. Show that the origin is asymptotically stable if and only if $\int_0^\infty a(s)ds = -\infty$. Under what condition is it stable ?

3.16 Stability of Quasi-linear Systems

In many physical problems the equation (5.32) may be written in a more useful form

$$x' = A(t)x + f(t, x). \tag{3.88}$$

The equation (5.35) simplifies the work since it is closely related with the system

$$x' = A(t)x. \tag{3.89}$$

Many properties of (5.36) have already been discussed. Under some restrictions on A and f, stability properties of (5.35) are very similar to those of (5.36). It is assumed that

(i) the matrix $A(t)$ is an $n \times n$ matrix which is continuous on I;

(ii) f is a n-vector and it is continuous on $I \times S_\alpha$ and $f(t,0) \equiv 0, \ t \in I$.

These two conditions guarantee the existence of solutions of (5.35) on some interval. The solutions may not be unique. However, for stability it is assumed that solutions of (5.35) uniquely exist on I. Let $\Phi(t)$ denote a fundamental matrix of (5.36) such that $\Phi(t_0) = E$, where E is the identity matrix. As a first step, we obtain necessary and sufficient conditions for the stability of (5.36). Note that $x(t) = 0, \ t \in I$ satisfies (5.36).

Theorem 3.16.1 (Theorem 6.21)**.** *The null solution of equation* (5.36) *is stable if and only if a positive constant k exists such that*

$$|\Phi(t)| \le k, \quad t \ge t_0. \tag{3.90}$$

Proof. The solution $y(t)$ of (5.36) which takes the value c at $t_0 \in I$ is given by

$$y(t) = \Phi(t)c \ \ (\Phi(t_0) = E).$$

Let the inequality (5.37) hold. Then for $t \in I, \ |y(t)| = |\Phi(t)c| \le k|c| < \epsilon$, if $|c| < \epsilon/k$. The origin is thus stable.

Conversely, let $|y(t)| = |\Phi(t)c| < \epsilon, \ t \ge t_0$ for all c such that $|c| < \delta$. Then $|\Phi(t)| < \epsilon/\delta$. Choose $k = \epsilon/\delta$. Hence the inequality (5.37) follows. The proof is complete. □

Theorem 3.16.2 (Theorem 6.22)**.** *The null solution of the system* (5.36) *is asymptotically stable if and only if*

$$|\Phi(t)| \to 0 \quad as \quad t \to \infty. \tag{3.91}$$

Proof. The condition in (5.37) is a special case of (5.38). Hence the origin is obviously stable. Further, since $|\Phi(t)| \to 0$ as $t \to \infty$ in view of (5.38) $|y(t)| \to 0$ as $t \to \infty$. The asymptotic stability follows. □

The stability of (5.36) has already been considered when $A(t) = A$ is a constant matrix. It is known that if the characteristic roots of the matrix A have negative real parts then a solution of (5.36) tends to zero as $t \to \infty$. In fact, this is asymptotic stability. It has been already proved that the fundamental matrix $\Phi(t)$ is given by

$$\Phi(t) = e^{(t-t_0)A}, \quad t_0, t \in I. \tag{3.92}$$

When the characteristic roots of the matrix A have negative real parts then two positive constants M and ρ can be found such that

$$|e^{(t-t_0)A}| \le Me^{-\rho(t-t_0)}, \quad t_0, t \in I. \tag{3.93}$$

Let the function $f(t,x)$ satisfy the condition

$$|f(t,x)| = o(|x|) \tag{3.94}$$

uniformly in t for $t \in I$. This implies that for x in a sufficiently small neighborhood of the origin, $\dfrac{|f(t,x)|}{|x|}$ can be made arbitrary small. The proof of the following result depends on the use of Gronwall's inequality.

Theorem 3.16.3 (Theorem 6.23). *In equation* (5.35)*, let $A(t)$ be a constant matrix A and let all the characteristic roots of A have negative real parts. Assume further that f satisfies the condition* (5.8.3)*. Then the origin for the system* (5.35) *is asymptotically stable.*

Proof. The solution of the equation (5.35) when $A(t) = A$ is a constant matrix exists on some subset of the interval I provided $y(t_0) = y_0$ is sufficiently small. We assume this result. By the variation of parameters formula, the solution $y(t)$ of the equation (5.35) passing through (t_0, y_0) satisfies the integral equation

$$y(t) = e^{(t-t_0)A}y_0 + \int_{t_0}^{t} e^{(t-s)A} f(s, y(s))ds. \tag{3.95}$$

The inequality (5.40) together with (5.42) yields

$$|y(t)| \leq M|y_0|e^{-\rho(t-t_0)} + M\int_{t_0}^{t} e^{-\rho(t-s)}|f(s, y(s))|ds. \tag{3.96}$$

which takes the form

$$|y(t)|e^{\rho t} \leq M|y_0|e^{\rho t_0} + M\int_{t_0}^{t} e^{\rho s}|f(s, y(s))|ds.$$

Let $|y_0| < \alpha$. Then the relation (5.42) is true in any interval $[t_0, t_1)$ for which $|y(t)| < \alpha$.

In view of the condition (5.8.3), for a given $\epsilon > 0$ we can find a positive number δ such that

$$|f(t, x)| \leq \epsilon|x|, \quad t \in I, \tag{3.97}$$

for $|x| < \delta$. Let us assume that $|y_0| < \delta$. Then, there exists a number T such that $|y(t)| < \delta$ for $t \in [t_0, T]$. Using (5.44) in (5.43), we obtain

$$e^{\rho t}|y(t)| \leq M|y_0|e^{\rho t_0} + M\epsilon\int_{t_0}^{t} e^{\rho s}|y(s)|ds, \tag{3.98}$$

for $t_0 \leq t < T$. Applying Gronwall's inequality to (5.45), it is seen that

$$e^{\rho t}|y(t)| \leq M|y_0|e^{\rho t_0}.e^{M\,\epsilon(t-t_0)}. \tag{3.99}$$

Hence, for $t_0 \leq t < T$, we obtain

$$|y(t)| \leq M|y_0|e^{(M\,\epsilon-\rho)(t-t_0)}. \tag{3.100}$$

Choose $M\epsilon < \rho$ and $y(t_0) = y_0$. If $|y_0| < \delta/M$, then (5.47) yields

$$|y(t)| < \delta, \quad t_0 \leq t < T.$$

The solution $y(t)$ of the equation (5.35) exists locally at each point (t, y), $t \geq t_0$, $|y| < \alpha$. Since the function $f(t, x)$ is defined on $I \times S_\alpha$, we can extend the solution $y(t)$ interval by interval by preserving the bound δ. Hence given any solution $y(t) = y(t; t_0, y_0)$ with $|y_0| < \delta/M$, it is defined on $t_0 \leq t < \infty$ and satisfies $|y(t)| < \delta$. In the above discussion, δ can be made arbitrarily small. Hence $y(t) \equiv 0$ is asymptotically stable when $M\epsilon < \rho$. □

When the matrix $A(t)$ is non-constant the relations between solutions of (5.35) and (5.36) still exist but now the fundamental matrix needs to satisfy some stronger conditions. Let the function f be continuous and satisfy the inequality

$$|f(t,x)| \leq r(t)|x|, \tag{3.101}$$

$(t,x) \in I \times S_\alpha$, where $r(t)$ is a non-negative continuous function such that

$$\int_{t_0}^{\infty} r(s)ds < +\infty.$$

The condition (5.48) guarantees the existence of a null solution of (5.35). Now the following result is proved on asymptotic stability of (5.35).

Theorem 3.16.4 (Theorem 6.24)**.** *Let the fundamental matrix $\Phi(t)$ satisfy the condition*

$$|\Phi(t)\Phi^{-1}(s)| \leq K, \tag{3.102}$$

where K is a positive constant and $t_0 \leq s \leq t < \infty$. Let f satisfy the hypotheses given by (5.48)*. Then, a positive constant M can be found such that if $t_1 \geq t_0$, any solution $y(t)$ of* (5.35) *is defined and satisfies $|y(t)| \leq M|y(t_1)|,\ t \geq t_1$ whenever $|y(t_1)| < \alpha/M$. Moreover, if the fundamental matrix $|\Phi(t)| \to 0$ as $t \to \infty$ then $|y(t)| \to 0$ as $t \to \infty$.*

Proof. Let $t_1 \geq t_0$ and $y(t)$ be any solution of (5.35) such that $|y(t_1)| < \alpha$. Then $y(t)$ satisfies the integral equation

$$y(t) = \Phi(t)\Phi^{-1}(t_1)y(t_1) + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s)f(s,y(s))ds. \tag{3.103}$$

for $t_1 \leq t < T$, where $|y(t)| < \alpha$ for $t_1 \leq t < T$.

In view of the conditions (5.48) and (5.49) we obtain

$$|y(t)| \leq K|y(t_1)| + K\int_{t_1}^{t} r(s)|y(s)|ds$$

The Gronwall's inequality now yields

$$|y(t)| \leq K|y(t_1)| \exp\Big(K\int_{t_1}^{t} r(s)ds\Big). \tag{3.104}$$

Note that due to the condition (5.48) the integral on the right side is bounded. Let

$$M = K\exp\Big(K\int_{t_1}^{\infty} r(s)ds\Big).$$

Then

$$|y(t)| \leq M|y(t_1)|. \tag{3.105}$$

Clearly this inequality holds if $|y(t_1)| < \alpha/M$. By following the argument as in Theorem 6.23 we can extend the solution for all $t \geq t_1$. Hence the inequality (5.52) holds for $t \geq t_1$.

The general solution $y(t)$ of (5.35) also satisfies the integral equation

$$y(t) = \Phi(t)\Phi^{-1}(t_0)y(t_0) + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)f(s,y(s))ds$$

$$= \Phi(t)y(t_0) + \int_{t_0}^{t_1} \Phi(t)\Phi^{-1}(s)f(s,y(s))ds + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s)f(s,y(s))ds.$$

Note that $\Phi(t_0) = E$. In view of the conditions (5.48), (5.49) and (5.52), we obtain

$$\begin{aligned} |y(t)| &\le |\Phi(t)||y(t_0)| + |\Phi(t)| \int_{t_0}^{t_1} |\Phi^{-1}(s)||f(s,y(s))|ds + K\int_{t_1}^{\infty} r(s)|y(s)|ds \\ &\le |\Phi(t)||y(t_0)| + |\Phi(t)| \int_{t_0}^{t_1} |\Phi^{-1}(s)||f(s,y(s))|ds + KM|y(t_1)| \int_{t_1}^{\infty} r(s)ds. \end{aligned} \quad (3.106)$$

The last term of the right side of the inequality (5.53) can be made less than (arbitrary) $\epsilon/2$ by choosing t_1 sufficiently large. By hypotheses $\Phi(t) \to 0$ as $t \to \infty$. The first two terms on the right side contain the term $|\Phi(t)|$. Hence their sum together can be made arbitrarily small by choosing t large enough, say less than $\epsilon/2$. Thus $|y(t)| < \epsilon$ for large t. This proves that $|y(t)| \to 0$ as $t \to \infty$. □

The inequality (5.52) shows that the origin is stable for $t \ge t_1$. But note that $t_1 \ge t_0$ is any arbitrary number. Here, condition (5.52) holds for any $t_1 \ge t_0$. Thus this stability is stronger than the stability of the origin defined previously. This is uniform stability. We do not propose to go into the detailed study of such types of stability behaviors.

EXERCISES

1. Prove that all solutions of the system (5.36) are stable if and only if they are bounded.
2. Consider a linear nonhomogeneous system $x'A(t)x + b(t)$, where $b(t)$ is an n-vector which is continuous for $t \ge t_0$. Prove that a solution $x(t)$ is stable, asymptotically stable, unstable, if the same holds for the null solution of the corresponding homogeneous system (5.36).
3. Prove that if the characteristic polynomial of the matrix A is stable, the matrix $C(t)$ is continuous on $0 \le t < \infty$ and $\int_0^\infty |C(t)|dt < \infty$, then all solutions of $x' = (A + C(t))x$ are asymptotically stable.
4. Prove that the system (5.36) is unstable if
$$Re\Big(\int_{t_0}^{t} tr\ A(s)ds\Big) \to \infty, \quad \text{as} \quad t \to \infty.$$
5. Define the norm of a matrix $A(t)$ by $\mu(A(t)) = \lim\limits_{h\to 0} \dfrac{|E + hA(t)| - 1}{h}$, where E is the $n \times n$ identity matrix.
   (i) Prove that μ is a continuous function of t.
   (ii) For any solution $y(t)$ of (5.36) prove that
   $$|y(t_0)| \exp\Big(-\int_{t_0}^{t} \mu(-A(s))ds\Big) \le |y(t)| \le |y(t_0)| \exp \int_{t_0}^{t} \mu(A(s))ds.$$
   [Hint : Let $r(t) = |y(t)|$. Then
   $$r'_+(t) = \lim_{h\to 0^+} \frac{|y(t) + hy'(t)| - |y(t)|}{h}.$$
   Show that $r'_+(t) \le \mu(A(t))r(t)$.]

(iii) When $A(t) = A$ a constant matrix, show that $|\exp(tA)| \leq \exp[t\mu(A)]$.

(iv) Prove that the trivial solution is stable if $\limsup_{t\to\infty} \int_{t_0}^{t} \mu(A(s))ds < \infty$.

(v) Show that the trivial solution is asymptotically stable if

$$\int_{t_0}^{t} \mu(A(s))ds \to -\infty \quad \text{as} \quad t \to \infty.$$

(vi) Establish that the solution is unstable if $\liminf_{t\to\infty} \int_{t_0}^{t} \mu(-A(s))ds = -\infty$.

3.17 Stability of Autonomous Systems

The previous section includes some stability properties of the equation (5.32). The function $f(t, x)$ in equation (5.32) depends on both variables t and x. In some physical problems the time variable does not appear explicitly. For example, the equation $x' = k\,x(k$ is a constant) representing the growth of population does not explicitly involve t. In situations of this type the equation (5.32) takes the form

$$x' = g(x) \tag{3.107}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$.

A system of the form (5.54) is called an autonomous system. Let us assume that the function g together with its first partial derivatives with respect to $x_1, x_2, \cdots, x_n$ are continuous in S_ρ. Further let $g(0) = 0$ so that (5.54) admits the trivial solution. Presently, the aim is to study the behavior of solutions of (5.54) on the interval I.

The main question which arise is how to determine the stability behavior of (5.54) when a solution cannot be obtained in a closed form. Very few methods are known to solve nonlinear differential equations to get a solution in a closed form. Lyapunov's direct method provides the study of stability of a solution without the actual knowledge of the solution. Hence it is very useful too to determine stability properties of linear and nonlinear equations. During the last twenty-five years many mathematicians have made interesting contributions to this method. The study cannot be said to be complete since many problems still remain unsolved.

This method involves the construction of a scalar function satisfying certain properties. In fact, this method is the generalization of the energy concept in classical mechanics. It is known that a mechanical system is stable if its energy(kinetic energy+ potential energy) continuously decreases. These two energies are always positive quantities and are zero when the system is completely at rest. Lyapunov thought of a generalized energy function which is known as the 'Lyapunov function'. This function is generally denoted by V. The function $V : S_\rho \to \mathbb{R}$ is said to be positive definite if the following conditions hold:

(i) $V(x)$ and $\frac{\partial V}{\partial x_j}$ $(j = 1, 2, \cdots, n)$ be continuous on S_ρ.

(ii) $V(0) = 0$.

(iii) $V(x)$ is positive for all $x \in S_\rho$ and $x \neq 0$.

The definition of negative definite function can be written similarly. The function $V(x)$ attains the minimum value at the origin. Further the origin is the only point in S_ρ at which the minimum value is attained. Since $V(x)$ has continuous first order partial derivatives, the chain rule may be used to obtain $\frac{dV(x)}{dt}$ as

$$\frac{dV(x)}{dt} = V(x) = \frac{\partial V(x)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V(x)}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial V(x)}{\partial x_n}\frac{dx_n}{dt}$$

$$= \sum_{j=1}^{n} \frac{\partial V(x)}{\partial x_j} x'_j = grad\ V(x).g(x).$$

The last step is true in view of (5.54). Observe that the derivative of V with respect to t along a solution of (5.54) is now known to us, although we do not have the explicit form of a solution. The conditions on the V function are not very stringent and it is possible to construct several functions which satisfy these conditions. $V(x) = x^2($ x scalar) or $V(x_1, x_2) = x_1^4 + x_2^4$ are some of the simple examples.

It has been assumed that the scalar function $V(x) = V(x_1, x_2, \cdots, x_n)$ is positive definite. One can visualize the nature of this function in a three dimensional space. For this purpose we consider a simple function $V(x_1, x_2) = x_1^2 + x_2^2$; clearly all the conditions (i),(ii) and (iii) hold. Let $z = x_1^2 + x_2^2$. Since $z \geq 0$ for all (x_1, x_2) the surface will always lie in the upper part of the plane OX_1X_2. Further $z = 0$ when $x_1 = x_2 = 0$. Thus the surface passes through the origin. This surface is like a parabolic mirror pointed upwards. It has an appearance as shown in Fig.6.2.

Now consider a section of this cup-like surface by a plane parallel to the plane OX_1X_2. This section is a curve $x_1^2 + x_2^2 = k, z = k$. Its projection on the OX_1X_2 plane is $x_1^2 + x_2^2 = k, z = 0$. Clearly these are circles with radius k, and the center as the origin (Fig.6.3). In a general situation, instead of circles, closed curves around the origin are obtained. The geometrical picture for any Lyapunov function in three dimensional, in a small neighborhood of the origin, is of this type. If we consider Lyapunov function in higher dimensions than three the above discussion helps us to imagine the geometrical nature of such functions.

We state below three theorems regarding the stability behavior of the system (5.54). The geometrical explanation given below these theorems describes the line of the proof. But they are not proofs in a strict mathematical sense. In the next section we provide detailed mathematical proofs of these theorems. Note further that the conditions guaranteeing the different stability behavior are only sufficient.

Theorem 3.17.1 (Theorem 6.25)**.** *If there exists in S_ρ a positive definite function $V(x)$ such that $\dot{V}(x) \leq 0$ then, the origin of the equation* (5.54) *is stable.*

Geometrical Interpretation : Let $\epsilon > 0$ be an arbitrary number such that $0 < \epsilon < \bar{\rho} < \rho$, where $\bar{\rho}$ is some number very near ρ. Consider the hypersphere S_ϵ. Find a constant $K > 0$ such that the surface $V(x) = K$ lies inside S_ϵ. Such a K always exists for each ϵ. Also a number $\delta > 0$ can be determined such that the hypersphere S_δ lies inside the oval-shaped surface $V(x) = K$. Choose $x_0 \in S_\delta$. Obviously $V(x_0) < K$. Let $x(t; t_0, x_0)$ be a solution of (5.54) through (t, x_0). Since $\dot{V} \leq 0$, i.e. V is non-decreasing along the solution, $x(t; t_0, x_0)$ will not reach the surface $V(x) = K$. This implies that the solution $x(t; t_0, x_0)$ remains in S_ϵ. This is the case for each solution of (5.54). Hence the origin is stable.(See Fig.6.4)

Now $V > 0$ is continuous on the boundary of S_ϵ. Note that the boundary of S_ϵ is a compact set. V actually attains the minimum value K on this set. Thus $V(x) \geq K$ on the boundary of S_ϵ. Since $V(x)$ is continuous and $V(0) = 0$ a positive number δ can be found sufficiently small such that $V(x) < K$ for $x \in S_\delta$. This proves the existence of δ.

Theorem 3.17.2 (Theorem 6.26). *If in S_ρ there exists a positive definite function V such that $-\dot{V}$ is also positive definite, then the origin of the equation* (5.54) *is asymptotically stable.*

The hypotheses of this theorem include the hypotheses of Theorem 6.25. It is concluded that the origin is stable. Since $-\dot{V}$ is positive definite, $V(x)$ decreases along the solution. Assume that $\lim\limits_{t\to\infty} V(x(t,t_0,x_0)) = l$ where $l > 0$. It is proved that this is impossible. This implies that $-\dot{V}$ tends to zero outside a hypersphere S_{r_1} for some $r_1 > 0$. But this cannot be true since $-\dot{V}$ is positive definite. Hence

$$\lim_{t\to\infty} V(x(t,t_0,x_0)) = 0.$$

This implies that $\lim\limits_{t\to\infty} |x(t;t_0,x_0)| = 0$. Thus the origin is asymptotically stable. (see Fig. 6.5).

Theorem 3.17.3 (Theorem 6.27*(Cetav)*). *Let $V(x)$ be given function and N a region in S_ρ such that*

(i) $V(x)$ has continuous first partial derivatives on N;

(ii) at the boundary points of N (inside S_ρ), $V(x) = 0$;

(iii) the origin is on the boundary of N;

(iv) $V(x)$ and $\dot{V}(x)$ are positive on N.

Then the origin of 6.53 *is unstable.*

Figure 6.6 explains the conditions of the theorem for a function $V(x) = V(x_1,x_2)$. Notice that the boundary of N in S_ρ is defined by $V(x) = 0$. Figure 6.6 shows the curves $V(x) = K_1,\ V(x) = K_2,\ K_2 > K_1$ (K_1 and K_2 are constants). Consider a solution $x(t;t_0,x_0)$ through (t_0,x_0). Now $V(x)$ is positive on N. The solution moves in the increasing direction of the function $V(x)$. This is the case even though x_0 is very close to the origin. The direction of increasing $V(x)$ is away from the origin which proves that the origin is unstable.

Example 3.17.4 (Example 6.28). *Consider the systems $x_1' = -x_2,\ x_2' = x_1$. The system is autonomous and possesses a trivial solution. Let $V(x_1,x_2) = x_1^2 + x_2^2$. Clearly, $V(x_1,x_2)$ is positive definite. The derivative $\dot{V}$ along the solution is $\dot{V}(x_1,x_2) = 2[x_1(-x_2) + x_2(x_1)] = 0$. The hypotheses of Theorem* 6.25 *holds. Hence the origin is stable. Geometrically it is observed that since $x_1x_1' + x_2x_2' = 0,\ x_1^2 + x_2^2 = c$($c$ is arbitrary constant) is a solution which represents circles with the origin as the center. Further* $(0,0)$ *is the only critical point.*

The solutions are represented in Fig 6.7. *Note that none of the solutions tend to zero. Hence it is not a case of asymptotic stability.*

Example 3.17.5 (Example 6.29). *Consider the system*

$$\begin{aligned} x_1' &= (x_1 - bx_2)(\alpha x_1^2 + \beta x_2^2 - 1) \\ x_2' &= (ax_1 + x_2)(\alpha x_1^2 + \beta x_2^2 - 1). \end{aligned}$$

Let $V(x_1,x_2) = ax_1^2 + bx_2^2$. When $a > 0,\ b > 0,\ V(x_1,x_2)$ is positive definite.

$$\dot{V}(x_1,x_2) = 2(ax_1^2 + bx_2^2)(\alpha x_1^2 + \beta x_2^2 - 1).$$

Let $\alpha > 0,\ \beta > 0$. If $\alpha x_1^2 + \beta x_2^2 < 1$ then $\dot{V}(x_1, x_2)$ is negative definite. The trivial solution is asymptotically stable since $V(x_1, x_2)$ satisfies the conditions of Theorem 6.26.

Example 3.17.6 (Example 6.30). *Consider the system*

$$\begin{aligned} x_1' &= x_2 - x_1 f(x_1, x_2) \\ x_2' &= -x_1 - x_2 f(x_1, x_2), \end{aligned}$$

where f is represented by a convergent power series in x_1, x_2 and $f(0,0) = 0$. Let $V = \frac{1}{2}(x_1^2 + x_2^2)$. Then $\dot{V}(x_1, x_2) = -(x_1^2 + x_2^2) f(x_1, x_2)$. Clearly if $f(x_1, x_2) \geq 0$ arbitrarily near the origin, the origin is stable. If f is positive definite in some neighborhood of the origin, the origin is asymptotically stable. If $f(x_1, x_2) < 0$ arbitrarily near the origin, the origin is unstable.

EXERCISES

1. Determine the nature of the following functions with regard to positive definiteness or negative definiteness:

   (i) $4x_1^2 + 3x_1x_2 + 2x_2^2$,

   (ii) $-3x_1^2 - 4x_1x_2 - x_2^2$,

   (iii) $10x_1^2 + 6x_1x_2 + 9x_2^2$,

   (iv) $-x_1^2 - 4x_1x_2 - 10x_2^2$.

2. Prove that $ax_1^2 + bx_1x_2 + cx_2^2$ is positive definite if $a < 0$ and $b^2 - 4ac < 0$ and negative definite if $a < 0$ and $b^2 - 4ac > 0$.

3. Consider the quadratic form $x^T R x$ where x is a n-column-vector and $R = [r_{ij}]$ is an $n \times n$ symmetric matrix. Prove that this quadratic form is positive definite if and only if $r_{11} > 0,\ r_{11}r_{22} - r_{21}r_{12} > 0$ and $\det[r_{ij}] > 0,\ i = 1, 2, \cdots; m = 3, 4, \cdots, n$.

4. Find the condition under which the following matrices are positive definite:

   (i) $\frac{1}{ab-c}\begin{bmatrix} ac & c & 0 \\ c & a^2+b & a \\ 0 & a & 1 \end{bmatrix}$

   (ii) $\frac{1}{9-a}\begin{bmatrix} \frac{6a+27}{a} & a+2a & 9-a \\ 9+2a & a(a+3) & 3a \\ 9-a & 3a & 3a \end{bmatrix}$.

5. Let $V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} f(s)ds$ where $f(x)$ is such that $f(0) = 0$, and $xf(x) > 0$ for $x \neq 0$. Show that $V(x_1, x_2)$ is positive definite.

6. Show that the trivial solution of the equation $x'' + f(x) = 0$, where f is a continuous function on $|x| < \rho,\ f(0) = 0$ and $xf(x) > 0$ is stable.

7. Show that the following systems are asymptotically stable:

   (i) $x_1' = -x_2 - x_1^3,\ \ x_2' = x_1 - x_2^3$.

   (ii) $x_1' = -x_1^3 - x_1x_2^3,\ \ x_2' = x_1^4 - x_2^3$.

   (iii) $x_1' = -x_1^3 - 3x_2,\ \ x_2' = 3x_1 - 5x_2^3$.

8. Consider the system

$$\begin{aligned} x_1' &= -x_1 + 2x_1(x_1+x_2)^2 \\ x_2' &= -x_2^3 + 2x_2^3(x_1+x_2)^2 \end{aligned}$$

Show that the origin is asymptotically stable if $|x_1| + |x_2| < 1/\sqrt{2}$.

3.18 Stability of Non-autonomous Systems

The study of the stability properties of non-autonomous systems involves some difficulties. Systems of this kind are given by (5.32). For this purpose a Lyapunov function $V(t,x)$ is needed which depends on t and x. Let f in (5.32) be such that $f(t,0) \equiv 0,\ t \in I$. Let f together with its first partial derivative be continuous on $I \times S_\rho$. This condition guarantees the existence and the uniqueness of solutions. For stability it is assumed that solutions of (5.32) exist on the entire time interval I and that the trivial solution is the equilibrium or the steady state.

Definition 3.18.1 (Definition 6.31)**.** *A real valued function ϕ is said to belong to the class $\mathscr{K}$ if*

(i) ϕ is defined and continuous on $0 \le r < \infty$,

(ii) ϕ is strictly increasing on $0 \le r < \infty$,

(iii) $\phi(0) = 0$ and $\phi(r) \to \infty$ as $r \to \infty$.

The function $\phi(r) = \alpha r^2,\ \alpha > 0$, is of class $\mathscr{K}$.

Definition 3.18.2 (Definition 6.32)**.** *A real valued function $V(t,x)$ defined on $I \times S_\rho$ is said to be positive definite if $V(t,0) = 0$ and there exists a function $\phi \in \mathscr{K}$ such that $V(t,x) \ge \phi(|x|)$, $(t,x) \in I \times S_\rho$. It is negative definite if $V(t,x) \le -\phi(|x|)$.*

The function $V(t,x) = (t^2+1)x^4$ is positive definite since $V(t,0) = 0$ and $\phi \in \mathscr{K}$ can be found ($\phi(r) = r^4$), such that $V(t,x) \ge \phi(|x|)$.

Definition 3.18.3 (Definition 6.33)**.** *A real valued function $V(t,x)$ defined on $I \times S_\rho$ is said to be decrescent if there exists a function $\psi \in \mathscr{K}$ such that in a neighborhood of the origin and for all $t \ge t_0$, $V(t,x) \le \psi(|x|)$.*

The function $V(t,x_1,x_2) = \frac{1}{t^2+1}(x_1^2+x_2^2),\ (t,x) \in I \times \mathbb{R}^2$, is decrescent. In this case, we can choose $\Psi(r) = r^2$. The function $V(t,x_1,x_2) = (1+e^{-t})(x_1^2+x_2^2)$ is both positive definite and decrescent since $x_1^2+x_2^2 \le (1+e^{-t})(x_1^2+x_2^2) \le 2(x_1^2+x_2^2)$. Choose $\phi(r) = r^2,\ \psi(r) = 2r^2$.

The following hypotheses **(H*)** is assumed:

(H*) Let $V(t,x)$ be such that $V(t,0) = 0$ for $t \in I$, $V(t,x)$ is bounded and the first order partial derivatives of V with respect to $x_i (i = 1,2,\cdots,n)$ are continuous on $I \times S_\rho$.

The chain rule is now applied to get the derivative $\dot V(t,x)$. Since **(H*)** holds, it is seen that

$$\dot V(t,x) = \frac{dV(t,x)}{dt} = \frac{\partial V(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt}.$$

Our interest is in the derivative of $V(t,x)$ along a solution $x(t)$ of the system (5.32). Indeed, we have

$$\dot{V}(t,x(t)) = \frac{\partial V(t,x(t))}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t,x(t))}{\partial x_i} f_i(t,x(t)).$$

It is to be noted that the derivative of $V(t,x)$ with respect to the system (5.32) i.e. along the solution of (5.32) does not depend directly on the knowledge of the solution.

We are now set to prove the fundamental theorems on the stability of the equilibrium of the system (5.32).

Theorem 3.18.4 (Theorem 6.34)**.** *Let a function $V(t,x)$ exists satisfying the hypotheses **(H*)** and such that it is positive definite and $\dot{V}(t,x) \le 0$; then the system* (5.32) *is stable.*

Proof. The function V is positive definite. Hence, there exists a function $\phi \in \mathscr{K}$ such that

$$0 \le \phi(|x|) \le V(t,x), \ |x| < \rho, \ t \in I. \tag{3.108}$$

Let $x(t) = x(t;t_0,x_0)$ be a solution of (5.32). Since $\dot{V}(t,x) \le 0$, it is seen that

$$V(t,x(t;t_0,x_0)) \le V(t_0,x_0), \ t \in I. \tag{3.109}$$

Since V is a continuous function, given $\epsilon > 0$, a number $\delta = \delta(\epsilon) > 0$ can be found so that

$$V(t_0,x_0) < \phi(\epsilon), \tag{3.110}$$

whenever $|x_0| < \delta$. Now the inequalities (5.55) and (5.56) yield

$$0 \le \phi(|x(t;t_0,x_0)|) \le V(t,x(t;t_0,x_0)) \le V(t_0,x_0) < \phi(\epsilon).$$

Hence, $|x(t;t_0,x_0)| < \epsilon$ for $t \in I$, whenever $|x_0| < \delta$ which shows that the origin is stable. □

The next theorem provides us sufficient conditions for the asymptotic stability of the origin.

Theorem 3.18.5 (Theorem 6.35)**.** *Let the function $V(t,x)$ satisfying the hypotheses **(H*)** exists such that $V(t,x)$ is positive definite and decrescent, and $\dot{V}(t,x)$ is negative definite. Then, the system* (5.32) *is asymptotically stable.*

Proof. Let $x(t;t_0,x_0)$ be a solution of (5.32). Since the hypotheses of Theorem 6.35 include those of Theorem 6.34, the null solution of (5.32) is stable. Hence, given $\epsilon > 0$ assume that there exist two positive numbers and λ such that $0 < \lambda \le |x(t;t_0,x_0)| < \epsilon, \ t \ge t_0$, whenever $|x_0| < \delta$. By hypotheses, since $\dot{V}(t,x)$ is negative definite, there exists a function $\sigma \in \mathscr{K}$ such that

$$\dot{V}(t,x(t;t_0,x_0)) \le -\sigma(|x(t;t_0,x_0)|). \tag{3.111}$$

Further suppose that $|x(t;t_0,x_0)| \ge \lambda > 0$ for $t \ge t_0$. In view of (5.58) a number $\gamma > 0$ can be found out such that

$$\dot{V}(t,x(t;t_0,x_0)) \le -\gamma < 0, \quad t \ge t_0.$$

Integrating both sides of this inequality, we get

$$V(t, x(t; t_0, x_0)) \leq V(t_0, x_0) - \gamma(t - t_0). \tag{3.112}$$

For large value of t the right side of (5.59) becomes negative which contradicts the fact that V is positive definite. The assumption that $|x(t; t_0, x_0)| \geq \lambda > 0$ for $t \in I$ is false. No such λ exists. Since $V(t, x)$ is a positive definite and decrescent function, $V(t, x(t; t_0, x_0)) \to 0$ as $t \to \infty$ and therefore it follows that $|x(t; t_0, x_0)| \to 0$ as $t \to \infty$. Thus the origin is asymptotically stable. □

In some cases ρ may be infinite. Thus it is possible that the system is asymptotically stable for any choice of x_0. The following theorem is stated without proof which provides sufficient conditions for the asymptotic stability in the large.

Theorem 3.18.6 (Theorem 6.36)**.** *The equilibrium state of* (5.32) *is asymptotically stable in the large if there exists, a positive definite function $V(t, x)$ which is decrescent everywhere and such that $V(t, x) \to \infty$ as $|x| \to \infty$ for each $t \in I$ and such that $\dot{V}$ is negative definite.*

Example 3.18.7 (Example 6.37)**.** *Consider the system $x' = A(t)x$, where $A(t) = (a_{ij})$, $a_{ij} = -a_{ji}$, $i \neq j$ and $a_{ij} \leq 0$, for all values of $t \in I$ and $i, j = 1, 2, \cdots, n$. Let $V(x) = x_1^2 + x_2^2 + \cdots + x_n^2$. Obviously $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$. Further*

$$\dot{V}(x(t)) = 2\sum_{i=1}^{n} x_i(t)x_i'(t) = 2\sum_{i=1}^{n} x_i(t)\Big[\sum_{j=1}^{n} a_{ij}x_j(t)\Big]$$

$$= 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i(t)x_j(t) = 2\sum_{i=1}^{n} a_{ii}x_i^2(t) \leq 0.$$

The last step is obtained by using the assumption for the matrix $A(t)$. Now the conditions of the Theorem 6.34 *hold and so the origin is stable. If $a_{ii} < 0$ for all values of t then it is seen that $\dot{V}(x(t)) < 0$ which implies asymptotic stability of the origin of the given system.*

EXERCISES

1. (i) Show that $V(t, x_1, x_2) = t(x_1^2 + x_2^2) - 2x_1x_2 \cos t$ is positive definite for $n = 2$ and $t > 2$.

   (ii) Prove that $x_1^2(1+\sin^2 t)+x_2^2(1+\cos^2 t)$ is positive definite for all values of (t, x_1, x_2).

2. Show that

   (i) $(x_1^2 + x_2^2)\sin^2 t$ is decrescent.

   (ii) $x_1^2 + (1 + t)x_2^2$ is positive definite but not decrescent.

   (iii) $x_1^2 + \frac{1}{1+t^2}x_2^2$ is decrescent but not positive definite.

   (iv) $x_1^2 + e^{-2t}x_2^2$ is decrescent.

   (v) $(1 + e^{-2t})((x_1^2 + x_2^2))$ is positive definite and decrescent.

3. Prove that a function $V(t, x)$ which has bounded partial derivatives $\frac{\partial V}{\partial x_i}(i = 1, 2, \cdots, n)$ on S_ρ for $t \geq t_0 \geq 0$ is decrescent.

4. Consider the equation $x' = -x - \frac{x}{t}(1 - x^2t^2)$. For $y = tx$ it becomes $y' = y(y^2 - 1)$. Prove that the trivial solution is stable when, for a fixed t_0, $|x_0| \leq \frac{1}{t_0}$.

5. For the system

$$\begin{aligned} x_1' &= e^t x_2 - (t^2+1)x_1(x_1^2 + x_2^2) \\ x_2' &= -e^t x_1 - (t^2+1)x_2(x_1^2 + x_2^2), \end{aligned}$$

   show that the origin is asymptotically stable.

6. Prove that the trivial solution of the system

$$\begin{aligned} x_1' &= a(t)x_2 + b(t)x_1(x_1^2 + x_2^2) \\ x_1' &= -a(t)x_1 + b(t)x_2(x_1^2 + x_2^2) \end{aligned}$$

   is stable if $b(t) \leq 0$, asymptotically stable if $b(t) \leq q < 0$ and unstable if $b(t) > 0$.

3.19 A Particular Lyapunov Function

Consider a linear system

$$x' = Ax, \quad x \in \mathbb{R}^n, \tag{3.113}$$

where $A = (a_{ij})$ is an $n \times n$ constant matrix. The aim is to study the stability of (5.60) by Lyapunov's direct method. The stability is determined by the nature of the characteristic roots of the matrix A. Let $V(x)$ represent a quadratic form

$$V(x) = x^T R\, x, \tag{3.114}$$

where $R = (r_{ij})$ is an $n \times n$ constant, positive definite, symmetric matrix. The time derivative of $V(x)$ along the solution of (5.60) is given by

$$\begin{aligned} \dot{V}(x) = x'^T Rx + x^T Rx' &= x^T A^T Rx + x^T RAx \\ &= x^T(A^T R + RA)x = -x^T Qx, \end{aligned}$$

where

$$Q = -(A^T R + RA). \tag{3.115}$$

Here $Q = (q_{ij})$ is $n \times n$ constant symmetric matrix. For the asymptotic stability of (5.60) the time derivative of $V(x)$ needs to be negative definite. Hence, the matrix Q given by (5.62) must be positive definite. If we start with an arbitrary matrix R then, the matrix Q need not be positive definite. Hence, in order that $V(x)$ given by (5.61) be a Lyapunov function, the matrix R needs to be selected properly. One way out for this is that Q is assumed as an arbitrary positive definite matrix and the equation (5.62) is solved for the matrix R. The positive definiteness of the matrices R and Q is a necessary and sufficient condition for the asymptotic stability of the linear system (5.60).

The sufficiency of this condition is obvious. The function $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite. The conditions of Theorem 6.26 are satisfied and hence the system (5.60) is discussed below. The matrix Q is assumed to be positive definite and the equation (5.62) is solved for R. The question is :

Under what conditions the equation (5.62) gives rise to a unique solution?

It is remarked that the equation (5.62) unaffected if the system (5.60) is transformed from x-system to y-system by the relation $x = Py$, where P is a non-singular constant matrix. The system (5.60) is then transformed into $y' = (P^{-1}AP)y$. Thus, the matrix A is transformed to $P^{-1}AP$. Now choose the matrix P such that $P^{-1}AP$ is a triangular matrix. Such a transformation is always possible by Jordan normal form. Hence, there is no loss of generality by assuming in (5.60) that, the matrix A is such that its main diagonal consists of eigenvalues of A and for $i < j$, $a_{ij} = 0$. The triangular matrix A has the following form:

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda_2 & 0 & \cdots & 0 \\ a_{31} & a_{32} & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda_n \end{bmatrix}.$$

Now the equation (5.62) is

$$\begin{bmatrix} \lambda_1 & a_{21} & a_{31} & \cdots & a_{n1} \\ 0 & \lambda_2 & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & r_{nn} \end{bmatrix}$$
$$+ \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda_n \end{bmatrix}$$
$$= - \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & q_{n3} & \cdots & q_{nn} \end{bmatrix}.$$

Now the elements on both the side are equated. This result in a system of equations of the form $(\lambda_j + \lambda_k)r_{jk} = -q_{jk} + \delta_{jk}(\cdots, r_{hk}, \cdots)$, where δ_{jk} is a linear form in r_{hk}, $h+k > j+k$, with coefficients in a_{rs}. In the case of asymptotic stability all the characteristic roots of A have negative real parts. Hence, this system can be solved for r_{jk}. The solution of this linear system is unique if the determinant of the coefficients is non-zero. Obviously the determinant contains the product of the coefficients of the form $\lambda_j + \lambda_k$. Thus the matrix R is uniquely determined. If none of the characteristic roots λ_i is zero and further the sum of any two different roots is not zero.

The following example illustrates the procedure in determining the matrix R.

Example 3.19.1 (Example 6.38)**.** *Consider the system*

$$x_1' = -3x_1 + kx_2, \quad x_2' = -2x_1 - 4x_2.$$

In this case $A = \begin{bmatrix} -3 & k \\ -2 & 4 \end{bmatrix}$. *Let Q be an arbitrary positive definite matrix, say*

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now Eq. (5.62) *is*

$$\begin{bmatrix} -3 & -2 \\ k & -4 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} -3 & k \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Equating the terms on both sides, we get on solving the system of equations

$$r_{11} = \frac{16+k}{7(k+6)}, \quad r_{12} = r_{21} = \frac{-3+2k}{7(k+6)}, \quad r_{22} = \frac{21+2k+k^2}{14(k+6)}.$$

Thus

$$R = \frac{1}{14(k+6)} \begin{bmatrix} 32+2k & -6+4k \\ -6+4k & 21+2k+k^2 \end{bmatrix}.$$

Now R is positive definite if

(i) $\dfrac{32+2k}{14(k+6)} > 0,$

(ii) $\dfrac{(32+2k)(21+2k+k^2)-(4k-6)^2}{14(k+6)} > 0.$

Consequently, it is true if $k > -6$ or $k < -16$. Choose any k satisfying this condition to obtain the matrix R which is positive definite. Thus the given system is asymptotically stable for all x.

The stability of the system (5.60) is clear if the nature of the characteristic roots of A is known. The Lyapunov function (5.61) can also be used to study the stability behavior of certain nonlinear systems which are related to the system (5.60). Consider the following system of equation in a vector form

$$x' = g(x), \tag{3.116}$$

where $g(0) = 0$. Let the Taylor's expansion of $g(x)$ about the origin be

$$g(x) = g(0) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i}\right)_{x=0} x_i + \sum_{j=1}^{n}\sum_{i=1}^{n} \left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{x=0} x_i x_j + \text{ the terms of higher order.}$$

Let us denote $\frac{\partial g_i}{\partial x_j}$ by a_{ij}. Then equation (5.63) may be written as

$$x' = Ax + f(x), \tag{3.117}$$

where $f(x)$ contains terms of order two or more and $A = [a_{ij}]$. Now we consider the system (5.64) for stability behavior. The homogeneous part is the system (5.60). Let the Lyapunov function be $V(x) = x^T R x$, where R is the unique solution of the equation (5.62). We have already discussed the method to determine the matrix R. Thus, when A is a stable matrix the system (5.60) is asymptotically stable. It is to be remarked that the asymptotic stability property of (5.60) holds for all x. In such a case, the system (5.60) is said to be asymptotically stable in the large or that it is globally asymptotically stable.

For the asymptotic stability of the zero solution system (5.64), the function f has crucial role to play. It is natural that if f is small then there may be at least a small region containing the origin wherein the zero solution of the system (5.64) is asymptotically stable. With this short introduction let us employ the same Lyapunov function (5.61) to determine the stability of the origin of (5.64). Now the time derivative of $V(x)$ along a solution of (5.64) is

$$\begin{aligned}\dot{V}(x) &= x'^T Rx + x^T Rx' = (x^T A^T + f^T)Rx + x^T R(Ax + f) \\ &= x^T(A^T R + RA)x + f^T Rx + x^T Rf = -x^T Qx + 2x^T Rf, \qquad (3.118)\end{aligned}$$

because of (5.62) and (5.64). The second term on the right side of (5.65) contains terms of degree three or higher in x. The first one contains a term of degree two in x. The first term is negative whereas the sign of the second term depends on f. Whatever the second term is, at least a small region containing the origin can definitely be found such that the first term predominates the second term and thus, in this small region the sign of $\dot{V}(x)$ remains negative. This implies that the zero solution of nonlinear equation (5.64) is asymptotically stable. Obviously this stability is local since the negative definiteness of $\dot{V}(x)$ is only in a small region around origin.

The above discussion shows that the asymptotic stability of a system need not be the entire region. It may be a subset of $\mathbb{R}^n$. It is therefore interesting in each case to determine such a subset. At this juncture a Lyapunov function is very handy and useful.

Definition 3.19.2 (Definition 6.39)**.** *The region of stability for a differential equation* (5.64) *is the set of all initial points x_0 such that*

$$\lim_{t\to\infty} x(t, t_0, x_0) = 0.$$

If the stability region is the whole of $\mathbb{R}^n$ then we get the asymptotic stability in the large or global asymptotic stability. We give below a method of determining the stability region for the system (5.64).

Consider below a surface $V(x) = k$ (where k is a constant to be determined) lying entirely inside the surface $\dot{V}(x) = 0$. Now find k such that $V(x) = k$ is tangential to the surface $\dot{V}(x) = 0$. Then stability region for the system (5.64) is the set $\{x : V(x) \le k\}$.

Example 5.8.3 given below illustrates a procedure for finding the region of stability.

Example 3.19.3. *Consider a nonlinear system*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}.$$

Let $V(x) = x^T Rx$, where R is the solution of the equation

$$\begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} R + R \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} = Q$$

Choose $Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, so that $R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Thus

$$\begin{aligned}&V(x_1, x_2) = 2(x_1^2 + x_2^2) \\ &\dot{V}(x_1, x_2) = 4(x_1 x_1' + x_2 x_2') = 4[-x_1^2 - x_2^2(1 - x_2)]\end{aligned}$$

with respect to the given system. To find the region of asymptotic stability consider the surface $\dot{V}(x_1, x_2) = 4[-x_1^2 - x_2^2(1 - x_2)] = 0$. Clearly when $x_2 < 1$, $\dot{V}(x_1, x_2) < 0$ for all x_1. Hence, $V(x) = 2(x_1^2 + x_2^2) \le 1$ is the region which lies in the region $\dot{V}(x_1, x_2) < 0$. The size of the stability region thus obtained depends on the choice of a matrix Q.

EXERCISES

1. Prove that the stability properties of solutions the equation (5.62) remains unaffected by a transformation $x = Py$, where P is a non-singular matrix.

2. If R is a solution of the equation (5.62) then, prove that so is R^T and hence, $R^T = R$.

3. The matrices A and Q are given below. Find a matrix R satisfying the equation (5.62) for each of the following cases.

   (i) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$;

   (ii) $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$; and

   (iii) $A = \begin{bmatrix} -3 & -5 \\ -2 & -4 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

4. For the system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & p & 0 \\ 0 & -2 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

   Choose

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Determine the value/vaues of p for which the matrix R is positive definite.

5. For the system

$$x_1' = -x_1 + 2x_2, \; x_2' = -2x_1 + x_2 + x_2^2$$

   find the region of the asymptotic stability.

6. Prove that the zero solution of the system

$$(x_1, x_2)' = (-x_1 + 3x_2, -3x_1 - x_2 - x_2^3)$$

   is asymptotically stable.

Module 4

Oscillations and Boundary Value Problems

Lecture 22

4.1 Introduction

Qualitative properties of solutions of differential equations assume importance in the absence of closed form solutions. In case the solutions are not expressible in terms of the usual "known functions", an analysis of the equation is necessary to find the various facets of the solutions. One such qualitative property, which has wide applications, is the oscillation of solutions. We again stress that it is but natural to expect to know the solution in an explicit form which unfortunately is not always possible. A rewarding alternative is to resort to qualitative study. The point is asserted once again to justify the inclusion of qualitative theory to students who think that it is otherwise out of place.

Before proceeding further, some definitions and their consequences are looked into as a part of the ground work. Consider a second order equation

$$x'' = f(t, x, x'), \quad t \geq 0, \tag{4.1}$$

and let x be a solution of equation (4.1) existing on $[0, \infty)$. Unless or otherwise mentioned we understand (in this chapter) that a solution means a non-trivial solution.

Definition 4.1.1. A point $t = t^* \geq 0$ is called a zero of a solution x of the equation (4.1) if $x(t^*) = 0$.

Definition 4.1.2. (a) Equation (4.1) is called "non-oscillatory" if for every solution x there exists $t_0 > 0$ such that x does not have a zero in $[t_0, \infty)$

(b) Equation (4.1) is called "oscillatory" if (a) is false.

Example 4.1.3. Consider the linear equation

$$x'' - x = 0, t \geq 0.$$

It is an easy matter to show that the above equation is non-oscillatory once we recognize that the general solution is $Ae^t + Be^{-t}$ where A and B are constants.

Example 4.1.4. The equation

$$x'' + x = 0$$

is oscillatory. The general solution in this case is

$$x(t) = A\cos t + B\sin t, t \geq 0$$

and without loss of generality we assume that both A and B are non-zero constants; otherwise x is trivially oscillatory. It is easy to show that x has a zero at

$$n\pi + \tan^{-1}(A/B), n = 0, 1, 2, \cdots$$

and so the equation is oscillatory.

In this chapter we restrict our attention to only second order linear homogeneous equations. There are results concerning higher order equations. We conclude the introduction with a few basic results concerning linear equations.

$$x'' + a(t)x' + b(t)x = 0, \quad t \geq 0, \tag{4.2}$$

where a and b are real valued continuous functions defined on $[0, \infty)$

Theorem 4.1.5. *Assume that a' exists and is continuous for $t \geq 0$. Equation* (4.2) *is oscillatory if, and only if, the equation*

$$x'' + c(t)x = 0 \tag{4.3}$$

is oscillatory, where

$$c(t) = b(t) - \frac{1}{2}a^2(t) - \frac{a'(t)}{2}.$$

The equation (4.3) *is called the "normal" form of equation* (4.2).

Proof. Let x be any solution of (4.2). Consider a transformation

$$x(t) = v(t)y(t)$$

where v and y are twice differentiable functions. The computation of x', x'' and their substitution in (4.2) gives us

$$vy'' + (2v' + a(t)v)y' + (v'' + a(t)v' + b(t)v)y = 0.$$

Thus, equating the coefficients of y' to zero, it is seen that

$$v(t) = \exp(-\frac{1}{2}\int_0^t a(s)ds)$$

.

Therefore y satisfies a differential equation

$$y'' + c(t)y = 0, \quad t \geq 0$$

where $c(t) = b(t) - \frac{1}{2}a^2(t) - \frac{a'(t)}{2}$.Actually, if x is a solution of (4.2), then

$$y(t) = x(t)\exp(\frac{1}{2}\int_0^t a(s)ds)$$

is a solution of (4.3). Similarly if y is a solution of (4.3) then

$$x(t) = y(t)\exp(-\frac{1}{2}\int_0^t a(s)ds)$$

is a solution of (4.2). Thus, the theorem holds. □

Remark We note that (4.2) is oscillatory if and only if (4.3) is oscillatory. Although the proof of the Theorem 4.1.5 is elementary the conclusion simplifies subsequent work to a great extent.

The following two theorems are of interest in themselves.

Theorem 4.1.6. *Let x_1 and x_2 be two linearly independent solutions of* (4.2)*. Then, x_1 and x_2 do not admit common zeros.*

Proof. Suppose $t = a$ is a common zero of x_1 and x_2. Then, the Wronskian of x_1 and x_2 vanishes at $t = a$. Thus, it follows that x_1 and x_2 are linearly dependent which is a contradiction to the hypothesis or else x_1 and x_2 cannot have common zeros. □

Theorem 4.1.7. *The zeros of a solution of* (4.2) *are isolated.*

Proof. Let $t = a$ be a zero of a solution x of (4.2). Then ,$x(a) = 0$ and $x'(a) \neq 0$, otherwise $x \equiv 0$, which is not the case, since x is a non-trivial solution. There are two cases.
Case 1:

$$x'(a) > 0.$$

Since the derivative of x is continuous and positive at $t = a$ it follows that x is strictly increasing in some neighborhood of $t = a$ which means that $t = a$ is the only zero of x in that neighborhood. This shows that the zero $t = a$ of x is isolated.
Case 2:

$$x'(a) < 0.$$

The proof is similar to that of case 1 with minor changes. □

EXERCISES

1. Prove that the equation (4.2) is non-oscillatory if and only if the equation (4.3) is non-oscillatory.

2. If $t_1, t_2, \cdots, t_n, \cdots$ are zeros of a solution x of (4.2) in $(0, \infty)$, then show that $\lim t_n = \infty$ as $n \to \infty$.

3. Prove that any solution x of (4.2) has at most a countable number of zeros in $(0, \infty)$.

4. Show that the equation

$$x'' + a(t)x' + b(t)x = 0, \quad t \geq 0 \qquad (*)$$

transforms into an equation of the form

$$(p(t)x')' + q(t)x = 0, \quad t \geq 0 \qquad (**)$$

by multiplying (*) throughout by $\exp(\int_0^t a(s)ds)$, where a and b are continuous functions on $[0, \infty)$,

$$p(t) = \exp(\textstyle\int_0^t a(s)ds), q(t) = b(t)p(t).$$

State and prove a theorem similar to Theorem 4.1.5 for equation (*) and (**). Also show that if $a(t) \equiv 0$, then, (**) reduces to $x'' + q(t)x = 0, t \geq 0$.

Lecture 23

4.2 Sturm's Comparison Theorem

The phrase "comparison theorem" for differential equation is used in the sense stated below:

' If a solution of a differential equation has a certain known property P then the solution of a second differential equation have the same or some related property P under certain hypothesis.'

Sturm's comparison theorem is a result in this direction concerning zeros of solutions of a pair of linear homogeneous differential equations. Sturm's theorem has varied interesting implications in the theory of oscillations.

Theorem 4.2.1. *(Sturm's Comparison Theorem)*
Let p, r_1, r_2 and p be continuous functions on (a, b) and $p > 0$. Assume that x and y are real solutions of

$$(px')' + r_1 x = 0, \tag{4.4}$$

$$(py')' + r_1 y = 0 \tag{4.5}$$

respectively on (a, b). If $r_2(t) \geq r_1(t)$ for $t \in (a, b)$ then between any two consecutive zeros t_1, t_2 of x in (a, b) there exists at least *one zero of y (unless $r_1 \equiv r_2$) in $[t_1, t_2]$. Moreover, when $r_1 \equiv r_2$ in $[t_1, t_2]$ the conclusion still holds if x and y are linearly independent .*

Proof. If possible, let y be positive in (t_1, t_2). Without loss of generality let us assume that $x(t) > 0$ on (t_1, t_2) . Multiplying (4.4) and (4.5) by y and x respectively and subtraction leads to

$$(px')'y - (py')'x - (r_2 - r_1)xy = 0,$$

which, on integration gives us

$$\int_{t_1}^{t_2} [(px')'y - (py')'x] \; dt = \int_{t_1}^{t_2} (r_2 - r_1)xy \; dt.$$

If $r_2 \neq r_1$ on (t_1, t_2), then, $r_2(t) > r_1(t)$ in a small interval of (t_1, t_2) and therefore

$$\int_{t_1}^{t_2} [(px')'y - (py')'x] > 0. \tag{4.6}$$

Using the identity

$$\frac{d}{dt}[p(x'y - xy')] = (px')'y - (py')'x,$$

now the inequality (4.6) implies

$$p(t_2)x'(t_2)y(t_2) - p(t_1)x'(t_1)y(t_1) > 0, \tag{4.7}$$

since $x(t_1) = x(t_2) = 0$. However, $x'(t_1) > 0$ and $x'(t_2) < 0$ as x is a non-trivial solution which is positive in (t_1, t_2). As py is positive at t_1 as well as at t_2, (4.7) leads to a contradiction.

Again, if $r_1 \equiv r_2$ on $[t_1, t_2]$, then in place of (4.7), we have

$$p(t_2)y(t_2)x'(t_2) - p(t_1)y(t_1)x'(t_1) \geq 0.$$

which again leads to a contradiction as above unless y is a multiple of x. This completes the proof. □

Remark : What Sturm's comparison theorem asserts is that the solution y has *at least* one zero between two successive zeros t_1 and t_2 of x. Many times y may vanish more than once between t_1 and t_2. As a special case of Theorem 4.3,we have

Theorem 4.2.2. *Let r_1 and r_2 be two continuous functions such that $r_2 \geq r_1$ on (a, b). Let x and y be solutions of equations*

$$x'' + r_1(t)x = 0 \tag{4.8}$$

and

$$y'' + r_2(t)y = 0 \tag{4.9}$$

on the interval (a, b). Then y has at least a zero between any two successive zeros t_1 and t_2 of x in (a, b) unless $r_1 \equiv r_2$ on $[t_1, t_2]$. Moreover, in this case the conclusion remains valid if the solutions y and x are linearly independent.

Proof. the proof is immediate if we let $p \equiv 1$ in Theorem 4.3. Notice that the hypotheses of Theorem 4.3 are satisfied. □

The celebrated Sturm's separation theorem is an easy consequence of Sturm's comparison theorem as shown below.

Theorem 4.2.3. *(Sturm's Separation Theorem) Let x and y be two linearly independent real solutions of*

$$x'' + a(t)x' + b(t)x = 0, \ t \geq 0 \tag{4.10}$$

where a, b are real valued continuous functions on $(0, \infty)$. Then, the zeros of x and y separate each other, i.e. between any two consecutive zeros of x there is one and only one zero of y. (Note that the roles of x and y are interchangeable.)

Proof. First we note that all the hypotheses of Theorem 4.3 are satisfied by letting

$$r_1(t) \equiv r_2(t) = b(t) \exp\big(\int_0^t a(s)ds\big)$$
$$p(t) = \exp\big(\int_0^t a(s)ds\big)$$

So between any two consecutive zeros of x, there is at least one zero of y. By repeating the argument with x in place of y, it is clear that between any two consecutive zeros of y there is a zero of x which completes the proof. □

By setting $a \equiv 0$ in Theorem 4.2.3 gives us the following result.

Corollary 4.2.4. *Let r be a continuous function on $(0, \infty)$ and let x and y be two linearly independent solutions of*

$$x'' + r(t)x = 0.$$

Then, the zeros of x and y separate each other.

A few comments are warranted on the hypotheses of Theorem 4.3. Example shows that Theorem 4.3 fails if the condition $r_2 \geq r_1$ is dropped.

Example 4.2.5. Consider the equations

(i) $x'' + x = 0, r_1(t) \equiv +1, t \geq 0,$

(ii) $x'' - x = 0, r_2(t) \equiv -1, t \geq 0.$

All the conditions of Theorem 4.3 are satisfied except that r_2 is not greater than r_1. We note that between any consecutive zeros of a solution x (of (i), any solution y of (ii) does not admit a zero. Thus, Theorem 4.3 may not hold true if the condition $r_2 \geq r_1$ is dropped.

Assuming the hypotheses of Theorem 4.3, let us pose a question: is it true that between any two zeros of a solution y of equation (4.5) there is a zero of a solution x of equation (4.4)? The answer to this question is in the negative as is clear from Example .

Example 4.2.6. Consider

$$x'' + x = 0, r_1(t) \equiv 1$$
$$y'' + 4y = 0, r_2(t) \equiv 4.$$

Note that $r_2 \geq r_1$ and also that the remaining conditions of Theorem 4.8 are satisfied. $x(t) = \sin t$ is a solution of the first equation and $y(t) = \sin(2t)$ is a solution of the second equation which has zero at $t_1 = 0$ and $t_2 = \pi/2$. It is obvious that $x(t) = \sin t$ does not vanish at any point in $(0, \pi/2)$. This clearly shows that, under the hypotheses of Theorem 4.3, between two successive zeros of y there need not exist a zero of x.

EXERCISES

1. Let r be a positive continuous function and let m be a real number. Show that the equation
$$x'' + (m^2 + r(t))x = 0, t \geq 0$$
is oscillatory.

2. Assume that the equation
$$x'' + r(t)x = 0, t \geq 0$$
is oscillatory. Prove that the equation
$$x'' + (r(t) + s(t))x = 0, t \geq 0$$
is oscillatory, given that r, s are continuous functions and $s(t) \geq 0$.

3. Let r be a continuous function (for $t \geq 0$) such that $r(t) > m^2 > 0$, where m is an integer. For a solution y of

$$y'' + r(t)y = 0, t \geq 0$$

prove that y vanish in any interval of length π/m.

4. Show that the normal form of Bessel's equation

$$t^2x'' + tx' + (t^2 - p^2)x = 0 \qquad (*)$$

is given by

$$y'' + (1 + \tfrac{1-4p^2}{4t^2})y = 0 \qquad (**)$$

(a) Show that the solution J_p of (*) and Y_p of (**) have common zeros for $t > 0$.

(b) (i) If $0 \leq p < \frac{1}{2}$, show that every interval of length π contains at least one zero of $J_p(t)$;
(ii) If $p = \frac{1}{2}$ then prove that every zero of $J_p(t)$ is at a distance of π from its successive zero.

(c) Suppose t_1 and t_2 are two consecutive zeros of $J_p(t), 0 \leq p < \frac{1}{2}$. Show that $t_2 - t_1 < \pi$ and that $t_2 - t_1$ approaches π in the limit as $t_1 \to \infty$. What is your comment when $p = \frac{1}{2}$ in this case ?

Lecture 24

4.3 Elementary Linear Oscillations

Presently we restrict our discussion to a class of second order equation of the type

$$x'' + a(t)x = 0, t \geq 0, \tag{4.11}$$

where a is a real valued continuous function defined for $t \geq 0$. A very interesting implication of Sturm's separation theorem is

Theorem 4.3.1. *(a) The equation* (4.11) *is non-oscillatory if, and only if, it has no solution with finite number of zeros in* $[0, \infty)$.

(b) Equation (4.11) *is either oscillatory or non-oscillatory but cannot be both.*

Proof. (a) *Necessity* It has an immediate consequence of the definition.
Sufficiency Let z be the given solution which does not vanish on (t^*, ∞) where $t^* \geq 0$. Then any non-trivial solution $x(t)$ of (4.11) can vanish utmost once in (t^*, ∞), i.e, there exists $t_0 (> t^*)$ such that $x(t)$ does not have a zero in $[t_0, \infty)$.
The proof of (b) is obvious. □

We conclude this section with two elementary results.

Theorem 4.3.2. *Let x be a solution of* (4.11) *existing on* $(0, \infty)$. *If $a < 0$ on* $(0, \infty)$, *then x has utmost one zero.*

Proof. Let t_0 be a zero of x. It is clear that $x'(t_0) \neq 0$ for $x(t) \not\equiv 0$. Without loss of generality let us assume that $x'(t_0) > 0$ so that x is positive in some interval to the right of t_0. Now $a < 0$ implies that x'' is positive on the same interval which in turn implies that x' is an increasing function, and so, x does not vanish to the right of t_0. A similar argument shows that x has no zero to the left of t_0. Thus, x has utmost one zero. $\square$

Remark Theorem is also a corollary of Sturm's comparison theorem. For the equation

$$y'' = 0$$

any non-zero constant function $y \equiv k$ is a solution. Thus, if this equation is compared with the equation (4.11) (observe that all the hypotheses of Theorem are satisfied) then, x vanishes utmost once, for otherwise if x vanishes twice then y necessarily vanishes at least once by Theorem ,which is not true. So x cannot have more than one zero.

From Theorem the question arises: If a is continuous and $a(t) > 0$ on $(0, \infty)$, is the equation (4.11) oscillatory ? A partial answer is given in the following theorem.

Theorem 4.3.3. *Let a be continuous and positive on $(0, \infty)$ with*

$$\int_1^\infty a(s)ds = \infty. \tag{4.12}$$

Also assume that x is any (non-zero) solution of (4.11) existing for $t \geq 0$. Then, x has infinite zeros in $(0, \infty)$.

Proof. Assume, on the contrary, that x has only a finite number of zeros in $(0, \infty)$. Then, there exist a point $t_0 > 1$ such that x does not vanish on $[t_0, \infty)$. Without loss of generality we assume that $x(t) > 0$ for all $t \geq t_0$. Thus

$$v(t) = \frac{x'(t)}{x(t)}, \ t \geq t_0$$

is well defined. It now follows that

$$v'(t) = -a(t) - v^2(t).$$

Integration on the above leads to

$$v(t) - v(t_0) = -\int_{t_0}^t a(s)ds - \int_{t_0}^t v^2(s)ds.$$

The condition (4.12) now implies that there exist two constants A and T such that $v(t) < A(< 0)$ if $t \geq T$ since $v^2(t)$ is always non-negative and

$$v(t) \leq v(t_0) - \int_{t_0}^t a(s)ds.$$

This means that x' is negative for large t. Let $T(\geq t_0)$ be so large that $x'(T) < 0$. Then, on $[T, \infty)$ notice that $x > 0, x' < 0$ and $x'' < 0$. But

$$\int_T^t x''(s)ds = x'(t) - x'(T) \leq 0$$

Now integrating once again we have

$$x(t) - x(T) \leq x'(T)(t - T), t \geq T \geq t_0. \tag{4.13}$$

Since $x'(T)$ is negative, the right hand side of (4.13) tends to $-\infty$ as $t \to \infty$ while the left hand side of (4.13) either tends to a finite limit (because $x(T)$ is finite) or tends to $+\infty$ (in case $x(t) \to \infty$ as $t \to \infty$). Thus, in either case we have a contradiction . So the assumption that x has a finite number of zeros in $(0, \infty)$ is false. Hence, x has infinite number of zeros in $(0, \infty)$, which completes the proof. □

It is not possible to do away with the condition (4.12) as shown by the following example.

Example 4.3.4. $x(t) = t^{1/3}$ is a solution of the Euler's equation

$$x'' + \frac{2}{9t^2}x = 0.$$

which does not vanish anywhere in $(0, \infty)$ and so the equation is non-oscillatory. Also in this case

$$a(t) = \frac{2}{9t^2} > 0; \int_1^\infty \frac{2}{9t^2}dt = \frac{2}{9} < \infty$$

.

Thus, all the conditions of Theorem are satisfied except the condition (4.12).

EXERCISES

1. Prove (b) part of Theorem .

2. Suppose a is a continuous function on $(0, \infty)$ such that $a(t) < 0$ for $t \geq \alpha, \alpha$ is a finite real number. Show that

$$x'' + a(t)x = 0$$

is non-oscillatory.

3. Check for the oscillations or non-oscillations of:

   (i) $x'' - (t - \sin t)x = 0, \quad t \geq 0$

   (ii) $x'' + e^t x = 0, \quad t \geq 0$

   (iii) $x'' - e^t x = 0, \quad t \geq 0$

   (iv) $x'' - \frac{t}{\log t}x = 0, \quad t \geq 1$

   (v) $x'' + (t + e^{-2t})x = 0, \quad t \geq 0$

4. Prove that Euler's equation $\quad x'' + \frac{k}{t^2}x = 0$

   (a) is oscillatory if $k > \frac{1}{4}$

   (b) is non-oscillatory if $k \leq \frac{1}{4}$

5. The normal form of Bessel's equation $\quad t^2x'' + tx' + (t^2 - p^2)x = 0, t \geq 0$, is

$$x'' + (1 + \tfrac{1-4p^2}{4t^2})x = 0, t \geq 0. \qquad (*)$$

(i) Show that Bessel's equation is oscillatory for all values of p.

(ii) If $p > \frac{1}{2}$ show that $t_2 - t_1 > \pi$ and approaches π as $t_1 \to \infty$, where t_1, t_2(with $t_1 < t_2$) are two successive zeros of Bessel's function J_p.
(Hint: Show that J_p and the solution Y_p of (*) have common zeros. Then compare (*) with $x'' + x = 0$, successive zeros of which are at a distance of π.)
(Exercise 4 of sec. 2 and Exercise 5 above justifies the assumption of the existence of zeros of Bessel's functions (which was taken for granted in Theorem 4.9 in Chapter 4.)

6. Decide whether the following equations are oscillatory or non-oscillatory:

(i) $(tx')' + x/t = 0$,

(ii) $x'' + x'/t + x = 0$,

(iii) $tx'' + (1-t)x' + nx = 0, n$ is a constant(Laguerre's equation),

(iv) $x'' - 2tx' + 2nx = 0, n$ is a constant(Hermite's equation),

(v) $tx'' + (2n+1)x' + tx = 0, n$ is a constant,

(vi) $t^2x + ktx' + nx = 0, k, n$ are constants.

Lecture 25

4.4 Boundary Value Problems

Boundary value problems (BVPs) appear in various branches of sciences and engineering. Many problems in calculus of variation leads to a BVPs. Solutions to the problems of vibrating strings and membranes are the outcome of solutions of certain class of BVPs. Thus, the importance of the study of BVP, both in mathematics and in the applied sciences, needs no emphasis.

Speaking in general, BVPs pose many difficulties in comparison with IVPs. The problem of existence, both for linear and nonlinear equations with boundary conditions, requires discussions which are quite intricate. Needless to say the study of nonlinear BVPs are far tougher to solve than linear BVPs.

In this module attention is focused on some aspects of the regular BVP of the second order. Picard's theorem on the existence of a unique solution to a nonlinear BVP is also dealt with in the last section.

Consider a second order linear equation

$$L(x) = a(t)x'' + b(t)x' + c(t)x = 0, \quad A \leq t \leq B. \qquad (4.14)$$

It is tacitly assumed throughout this chapter that a, b, c are continuous real valued functions defined on $[A, B]$. L is a differential operator defined twice continuously differentiable functions on $[A, B]$.

To proceed further we need the concepts of linear forms. Let x_1, x_2, x_3, x_4 be four variables. Then, for any scalars a_1, a_2, a_3, a_4

$$V(x_1, x_2, x_3, x_4) = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$$

is called a "linear form" in the variables x_1, x_2, x_3, x_4. $V(x_1, x_2, x_3, x_4)$ is denoted in short by V. Two linear forms V_1 and V_2 are said to be linearly dependent if there exists a scalar K such that $V_1 = KV_2$ for all x_1, x_2, x_3, x_4. V_1 and V_2 are called linearly independent if V_1 and V_2 are not linearly dependent.

Definition 4.4.1. *(Linear Homogeneous BVP)* Consider an equation of type (4.14). Let V_1 and V_2 be two linearly independent linear forms in the variables $x(A), x(B), x'(A)$ and $x'(B)$. A linear homogeneous BVP is the problem of finding a function x defined on $[A, B]$ which satisfies

$$L(x) = 0, \quad t \in (A.B) \quad \text{and}$$

$$V_i(x(A), x(B), x'(A), x'(B)) = 0, \quad i = 1, 2 \tag{4.15}$$

simultaneously. The condition 4.15 is called a "linear homogeneous boundary condition" stated at $t = A$ and $t = B$.

Definition 4.4.2. *(Linear Non-homogeneous BVP)* Let $d : [a, B] \to \mathbb{R}$ be a given continuous function . A linear non-homogeneous BVP is the problem of finding a function x defined on $[A, B]$ satisfying

$$L(x) = d(t), \quad t \in (A.B) \text{ and}$$

$$V_i(x(A), x(B), x'(A), x'(B)) = 0, \quad i = 1, 2 \tag{4.16}$$

where V_i are two given linear forms and the operator L is defined by equation (4.14).

Example 4.4.3. (i) Consider

$$L(x) = x'' + x' + x = 0 \text{ and}$$

$$V_1(x(A), x'(A), x(B), x'(B)) = x(A)$$
$$V_2(x(A), x'(A), x(B), x'(B)) = x(B).$$

Then, any solution x of

$$L(x) = 0, A < t < B$$

which satisfies $x(A) = x(B) = 0$ is a solution of the given BVP. In this example it is no way implied that whether such a solution exists or not.

(ii) An example of a linear homogeneous BVP is

$$L(x) = x'' + e^t x' + 2x = 0, 0 < t < 1,$$

with boundary conditions $x(0) = x(1)$ and $x'(0) = x'(1)$. In this case

$$V_1(x(0), x'(0), x(1), x'(1)) = x(0) - x(1)$$
$$V_2(x(0), x'(0), x(1), x'(1)) = x'(0) - x'(1).$$

Also

$$L(x) = \sin 2\pi t, \quad 0 < t < 1,$$

along with boundary conditions $x(0) = x(1)$ and $x'(0) = x'(1)$ is another example of linear non-homogeneous BVP.

Definition 4.4.4. *(Periodic Boundary Conditions)* The boundary conditions

$$x(A) = x(B) \text{ and } x'(A) = x'(B)$$

are usually known as periodic boundary conditions stated at $t = A$ and $t = B$.

Definition 4.4.5. *(Regular Linear BVP)* A linear BVP, homogeneous or non-homogeneous, is called a regular BVP if A and B are finite and in addition to that $a(t) \neq 0$ for all t in (A, B).

Definition 4.4.6. *(Singular Linear BVP)* A linear BVP which is not regular is called a singular linear BVP.

Lemma 4.4.7. *A linear BVP* (4.14) *and* (4.15) *(or* (4.16) *and* (4.15)*) is singular if and only if one of the following conditions holds:*

(a) Either $A = -\infty$ or $B = \infty$.

(b) Both $A = -\infty$ and $B = \infty$.

(c) $a(t) = 0$ for at least one point t in (A, B).

The proof is obvious.

In this chapter, the discussions are confined to only regular BVPs. The definitions listed so far lead to the definition of a nonlinear BVP.

Definition 4.4.8. A BVP which is not a linear BVP is called a nonlinear BVP.

A careful analysis of the above definition shows that the nonlinearity in a BVP may be introduced because

(i) the differential equation may be nonlinear;

(ii) the given differential equation may be linear but the boundary conditions may not be linear homogeneous.

The assertion made in (i) and (ii) above is further clarified in the following example .

Example 4.4.9. (i) The BVP

$$x'' + |x| = 0, \quad 0 < t < \pi$$

with boundary conditions $x(0) = x(\pi) = 0$ is not linear due of the presence of $|x|$.

(ii) The BVP

$$x'' - 4x = e^t, \ 0 < t < 1$$

with boundary conditions

$$x(0).x(1) = x'(0), \quad x'(1) = 0$$

is a nonlinear BVP since one of the boundary conditions is not linear homogeneous.

EXERCISES

1. State with reasons whether the following BVPs are linear homogeneous, linear non-homogeneous or non-linear.
   (i) $x'' + \sin x = 0, \quad x(0) = x(2\pi) = 0.$
   (ii) $x'' + x = 0, \quad x(0) = x(\pi), \quad x'(0) = x'(\pi).$
   (iii) $x'' + x = \sin 2t, \quad x(0) = x(\pi) = 0.$
   (iv) $x'' + x = \cos 2t, \quad x^2(0) = 0, \quad x^2(\pi) = x'(0).$

2. Are the following BVPs regular ?
   (i) $2tx'' + x' + x = 0, \quad x(-1) = 1, \quad x(1) = 1.$
   (ii) $2x'' - 3x' + 4x = 0, \quad x(-\infty) = 0, \quad x(0) = 1.$
   (iii) $x'' - 9x = 0, \quad x(0) = 1, \quad x(\infty) = 0.$

3. Find a solution of
   (i) BVP (ii) of Exercise 2;
   (ii) BVP (iii) of Exercise 2.

Lecture 26

4.5 Sturm-Liouville Problem

The Sturm-Liouville problems represents a class of linear BVPs which have wide applications. The importance of these problems lies in the fact that they generate sets of orthogonal functions (sometimes complete sets of orthogonal functions). The sets of orthogonal functions are useful in the expansion for a certain class of functions. Few examples of such sets of functions are the Legendre and Bessel functions. In all of what follows, we consider a differential equation of the form

$$(px')' + qx + \lambda rx = 0, \quad A \le t \le B \tag{4.17}$$

where p', q and r are real valued continuous functions on $[A, B]$ and λ is a real parameter. We focus our attention on second order equations with a special kind of boundary condition. Let us consider two sets of boundary conditions, namely

$$m_1 x(A) + m_2 x'(A) = 0, \tag{4.18}$$

$$m_3 x(B) + m_4 x'(B) = 0, \tag{4.19}$$

$$x(A) = x(B), \quad x'(A) = x'(B), \quad p(A) = p(B), \tag{4.20}$$

where at least one of m_1 and m_2 and at least one of m_3 and m_4 are non-zero. A glance at the boundary conditions (4.18) and (4.19) shows that the two conditions are separately stated at $x = A$ and $x = B$. Relation (4.20) is the periodic boundary condition at $x = A$ and $x = B$.

A BVP consisting of equation (4.17) with (4.18) and (4.19) or equation (4.17) with (4.20) is called a Sturm-Liouville boundary value problem. It is trivial to show that the identically zero functions on $[A, B]$ is always a solution of Sturm-Liouville problem. It is of interest to examine the existence of a non-trivial solution and its properties.

Suppose that for a value of λ, x_λ is a non-trivial solution of (4.17) with (4.18) and (4.19) or (4.17) with (4.20) . Then λ is called an "eigenvalue" and x_λ is called an "eigenfunction" (corresponding to λ) of the Sturm-Liouville problem of (4.17) with (4.18) and (4.19) or with (4.20) respectively. The following theorem is of fundamental importance whose proof is beyond the scope of this book.

Theorem 4.5.1. *Assume that*

(i) A, B *are finite real numbers;*

(ii) the functions p', q *and* r *are real valued continuous functions on* $[A, B]$*; and*

(iii) m_1, m_2, m_3 *and* m_4 *are real numbers.*

Then, the Sturm-Liouville problem (4.17) *with* (4.18) *and* (4.19) *or* (4.17) *with* (4.20) *has countably many eigenvalues with no finite limit point. (consequently corresponding to each eigenvalue there exists an eigenfunction.)*

Theorem 4.5.3 just guarantees the existence of solutions. Such a class of such eigenfunctions are useful in a series expansion of a few functions . These expansions are a consequence of the orthogonal property of the eigenfunctions.

Definition 4.5.2. Two functions x and y (smooth enough), defined and continuous on $[A, B]$ are said to be orthogonal with respect to a continuous weight function r if

$$\int_A^B r(s)x(s)y(s)ds = 0. \tag{4.21}$$

By smoothness of x and y we mean the integral in Definition 4.5.2 exists. We are now ready to state and prove the orthogonality of the eigenfunctions.

Theorem 4.5.3. *Let all the assumptions of Theorem hold. For the parameters* λ, $\mu(\lambda \neq \mu)$ *let* x *and* y *be the corresponding solutions of* (4.17) *such that*

$$\Big[pW(x, y)\Big]_A^B = 0,$$

where $W(x, y)$ *is the Wronskian of* x *and* y *and* $\Big[Z\Big]_A^B$ *means* $Z(B) - Z(A)$*. Then,*

$$\int_A^B r(s)x(s)y(s)ds = 0$$

.

Proof. From the hypotheses we have

$$(px')' + qx + \lambda rx = 0,$$
$$(py')' + qy + \mu ry = 0.$$

which imply

$$(\lambda - \mu)rxy = (py')'x - (px')'y,$$

that is

$$(\lambda - \mu)rxy = \frac{d}{dt}[(py')x - (px')y]. \tag{4.22}$$

Now integration of Equation (4.22) leads to

$$(\lambda - \mu)\int_A^B r(s)x(s)y(s)ds = \Big[(py')x - (px')y\Big]_A^B = \Big[pW(x,y)\Big]_A^B$$

.

Since $\lambda \neq \mu$ (by assumptions) it readily follows that

$$\int_A^B r(s)x(s)y(s)ds = 0$$

which completes the proof. □

From Theorem it is clear that if we have any conditions which imply

$$\Big[pW(x,y)\Big]_A^B = 0,$$

then, the desired orthogonal property is folows. Now the boundary conditions (4.18) and (4.19) or (4.20) play a central ole in the desired orthogonality of the eigenfunction.. In fact (4.18) and (4.19) or (4.20) implies $\Big[pW(x,y)\Big]_A^B = 0$.

Theorem 4.5.4. *Let the hypotheses of Theorem 4.5.3 be satisfied. In addition let x_m and x_n be two eigenfunctions of the BVP* (4.17) *and* (4.18) *and* (4.19) *corresponding to two distinct eigenvalues λ_m and λ_n. Then*

$$\Big[pW(x_m,x_n)\Big]_A^B = 0. \tag{4.23}$$

If $p(A) = 0$ then (4.23) *holds without the use of* (4.18) *. If $p(B) = 0$, then* (4.23) *holds with* (4.19) *deleted.*

Proof. Let $p(A) \neq 0, p(B) \neq 0$. From (4.18) we note

$$m_1x_n(A) + m_2x_n'(A) = 0, \qquad m_1x_m(A) + m_2x_m'(A) = 0.$$

Without loss of generality, let us assume that $m_1 \neq 0$. Elimination of m_2 from the above two equation leads to

$$m_1[x_n(A)x_m'(A) - x_m(A)x_m'(A)] = 0.$$

Since $m_1 \neq 0$, we have

$$x_n(A)x_m'(A) - x_m(A)x_n'(A) = 0. \tag{4.24}$$

Similarly if $m_4 \neq 0$ (or $m_3 \neq 0$) in (4.19) , it is seen that

$$x_n(B)x_m'(B) - x_n'(B)x_m(B) = 0. \tag{4.25}$$

From the relations (4.24) and (4.25) it is obvious that (4.23) is satisfied.

If $p(A) = 0$, then the relation (4.23) holds since

$$\Big[pW(x_m,x_n)\Big]_A^B = p(B)[x_n(B)x_m'(B) - x_n'(B)x_m(B)] = 0,$$

in view of the equation (4.25) . Similar is the case when $p(B) = 0$. This completes the proof. $\square$

Lecture 27

The following theorem deals with periodic boundary conditions given in (4.20) .

Theorem 4.5.5. *Let the assumptions of theorem 4.5.3 be true. Suppose x_m and x_n are eigenfunctions of BVP (4.17) and (4.20) corresponding to the distinct eigenvalues λ_m and λ_n respectively. Then, x_m and x_n are orthogonal with respect to the weight function $r(t)$.*

Proof. In this case

$$\Big[pW(x_n, x_m)\Big]_A^B = p(B)[x_n(B)x_m'(B) - x_n'(B)x_m(B) - x_n(A)x_m'(A) + x_n'(A)x_m(A)].$$

The expression inside the brackets is zero once we use the periodic boundary condition (4.20) . $\square$

The following theorem ensures that the eigenvalues of (4.17), (4.18) or (4.17), (4.19) are real if $r > 0$ (or $r(t0)$ on (A, B) and r is continuous on $[a, B]$.

Theorem 4.5.6. *Let the hypotheses of Theorem 4.5.3 hold. Suppose that r is positive on (A, B) or r is negative on (A, B) and r is continuous on $[a, B]$. Then, all the eigenvalues of BVP (4.17), (4.18) or (4.17), (4.19) are real.*

Proof. Let $\lambda = a + ib$ be an eigenvalue and let $x(t) = m(t) + in(t)$ be a corresponding eigenfunction. We have $a, b, m(t)$ and $n(t)$ are real and so,

$$(pm' + pin')' + q(m + in) + (a + ib)r(m + in) = 0.$$

Equating the real and imaginary parts, we have

$$(pm')' + (q + ar)m - brn = 0$$

and

$$(pn')' + (q + ar)n + brm = 0.$$

Elimination of $(q + ar)$ in the above two equations implies

$$-b(m^2 + n^2)r = m(pn')' - n(pm')' = \tfrac{d}{dt}[(pn')m - (pm')n].$$

Thus, by integrating, we get

$$-b\int_A^B (m^2(s) + n^2(s))r(s)ds = \Big[(pn')m - (pm')n\Big]_A^B. \tag{4.26}$$

Since m and n satisfy one of the boundary conditions (4.18) and (4.19) or (4.20) , we have, as shown earlier,

$$\Big[p(n'm - m'n)\Big]_A^B = \Big[pW(m, n)\Big]_A^B = 0. \tag{4.27}$$

Also

$$\int_A^B [m^2(s) + n^2(s)]r(s)ds \neq 0$$

by the assumptions. Hence, from (4.26) and (4.27) it follows that $b = 0$, which means that λ is real which completes the proof. $\square$

An important application of the previous discussion is Theorem 4.5.7.

Theorem 4.5.7. *(Eigenfunction expansion) Let g be a piecewise continuous function defined on $[A, B]$ satisfying the boundary conditions* (4.18) *and* (4.19) *or* (4.20) *. Let $x_1, x_2, \cdots, x_n, \cdots$ be the set of eigenfunctions of the Sturm-Liouville problem* (4.17) *and* (4.18) *and* (4.19) *or* (4.17) *and* (4.20) *. Then*

$$g(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + \cdots \tag{4.28}$$

where c_n's are given by

$$c_n \int_A^B r(s) x_n^2(s) ds = \int_A^B r(s) g(s) x_n(s) ds, \quad n = 1, 2, \cdots \tag{4.29}$$

note that

$$r(s) x_n^2(s) > 0 \text{ on } [A, B]$$

so that c_n's in (4.29) *are well defined.*

Example 4.5.8. (i) Consider the BVP

$$x'' + \lambda x = 0, \; x(0) = 0, x'(1) = 0.$$

Note that this BVP is a Sturm-Liouville problem with

$$p \equiv 1, q \equiv 0, r \equiv 1; A = 0, \text{ and } B = 1.$$

Hence, by Theorem 4.5 the eigenfunctions are pairwise orthogonal. It is easy to show that the eigenfunctions are

$$x_n(t) = \sin \frac{(2n+1)}{2} \pi t, \quad n = 0, 1, 2, \cdots; 0 \leq t \leq 1. \tag{4.30}$$

Thus, if g is any function such that $g(0) = 0$ and $g'(1) = 0$, then there exist constants $c_1, c_2, \cdots$ such that

$$g(t) = c_0 x_0(t) + c_1 x_1(t) + \cdots + c_n x_n(t) + \cdots \tag{4.31}$$

where c_n's are determined by the relation (4.29) .

(ii) Let the Legendre polynomials $P_n(t)$ be the solutions of the Legendre equation

$$\tfrac{d}{dt}[(1 - t^2)x'] + \lambda x = 0, \lambda = n(n+1), -1 \leq t \leq 1.$$

The polynomials P_n form an orthogonal set of functions on $[-1, 1]$. In this case $p(t) = (1 - t^2), q \equiv 0, r \equiv 1$. Also note that

$$p(1) = p(-1) = 0$$

so that the boundary conditions are not needed for establishing the orthogonality of P_n. Hence, if g is any piece-wise continuous function, then the eigenfunction expansion of g is

$$g(t) = c_0 p_0(t) + c_1 p_1(t) + \cdots + c_n p_n(t) + \cdots,$$

where

$$c_n = \tfrac{2n+1}{2} \int_{-1}^{1} g(s) P_n(s) ds, n = 0, 1, 2, \cdots$$

since

$$\int_{-1}^{1} P_n^2(s) ds = \tfrac{2}{2n+1}, n = 0, 1, 2, \cdots$$

EXERCISES

1. Show that corresponding to an eigenvalue the Sturm-Liouville problem (4.17), (4.18) or (4.17), (4.19) has a unique eigenfunction.

2. Show that the eigenvalues for the BVP

$$x'' + \lambda x = 0, x(0) = 0 \text{ and } x(\pi) + x'(\pi) = 0$$

satisfies the equation

$$\sqrt{\lambda} = -\tan \pi \sqrt{\lambda}.$$

Prove that the corresponding eigenfunctions are

$$\sin(t\sqrt{\lambda_n})$$

where λ_n is an eigenvalue.

3. Consider the equation

$$x'' + \lambda x = 0, 0 < t \le \pi.$$

Find the eigenvalues and eigenfunctions for the following cases:

(i) $x'(0) = x'(\pi) = 0$;

(ii) $x(0) = 0, x'(\pi) = 0$;

(iii) $x(0) = x(\pi) = 0$;

(iv) $x'(0) = x(\pi) = 0$.

Lecture 28

4.6 Green's Functions

The aim of this article is to construct what is known as Green's Function and then use it to solve a non-homogeneous BVP. We start with

$$L(x) + f(t) = 0, \quad a \le t \le b \tag{4.32}$$

where L is a differential operator defined by $L(x) = (px')' + qx$. Here p, p' and q are given real valued continuous functions defined on $[a, b]$ such that $p(t)$ is non-zero on $[a, b]$. Equation (4.32) is considered with separated boundary conditions

$$m_1 x(a) + m_2 x'(a) = 0 \tag{4.33}$$

$$m_3 x(b) + m_4 x'(b) = 0 \tag{4.34}$$

with the usual assumptions that at least one of m_1 and m_2 and one of m_3 and m_4 are non-zero.

Definition 4.6.1. A function $G(t, s)$ defined on $[a, b] \times [a, b]$ is called Green's function for $L(x) = 0$ if, for a given $s, G(t, s) = G_1(t, s)$ if $t < s$ and $G(t, s) = G_2(t, s)$ for $t > s$ where G_1 and G_2 are such that:

(i) G_1 satisfies the boundary condition (4.33) at $t = a$ and $L(G_1) = 0$ for $t < s$;

(ii) G_2 satisfies the boundary condition (4.34) at $t = b$ and $L(G_2) = 0$ for $t > s$;

(iii) The function $G(t, s)$ is continuous at $t = s$;

(iv) The derivative of G with respect to t has a jump discontinuity at $t = s$ and

$$\left[\tfrac{\partial G_2}{\partial t} - \tfrac{\partial G_1}{\partial t}\right]_{t=s} = -\tfrac{1}{p(s)}.$$

With this definition, the Green's function for (4.32) with conditions (4.33) and (4.34) is constructed. Let $y(t)$ be a non-trivial solution of $L(x) = 0$ satisfying the boundary condition (4.33). Also let $z(t)$ be a non-trivial solution of $L(x) = 0$ which satisfies the boundary condition (4.34).

Assumption Let y and z be linearly independent solutions of $L(x) = 0$ on (a, b). For some constants c_1 and c_2 define $G_1 = c_1 y(t)$ and $G_2 = c_2 z(t)$. Let

$$G(t, s) = \begin{cases} c_1 y(t) & \text{if } t \leq s, \\ c_2 z(t) & \text{if } t \geq s. \end{cases} \tag{4.35}$$

Choose c_1 and c_2 such that

$$\begin{aligned} c_2 z(s) - c_1 y(s) &= 0 \\ c_2 z'(s) - c_1 y'(s) &= -1/p(s). \end{aligned} \tag{4.36}$$

With this choice of c_1 and $c_2, G(t, s)$ defined by the relation (4.35) has all the properties of the Green's function. Since y and z satisfy $L(x) = 0$ it follows that

$$y(pz')' - z(py')' \equiv \frac{d}{dt}[p(yz' - y'z)] = 0. \tag{4.37}$$

Hence

$$p(t)[y(t)z'(t) - y'(t)z(t)] = A \text{ for all } t \text{ in } [a, b]$$

where A is a non-zero constant (because y and z are linearly independent solutions of $L(x) = 0$). In particular it is seen that

$$y(s)z'(s) - y'(s)z(s)] = A/p(s), A \neq 0 \tag{4.38}$$

From equation (4.36) and (4.38) it is seen that

$$c_1 = -z(s)/A, c_2 = -y(s)/A.$$

Hence the Green's function is

$$G(t, s) = \begin{cases} -y(t)z(s)/A & \text{if } t \leq s, \\ -y(s)z(t)/A & \text{if } t \geq s. \end{cases} \tag{4.39}$$

The main result of this article is Theorem .

Theorem 4.6.2. *Let $G(t,s)$ be given by the relation* (4.39) *then $x(t)$ is a solution of* (4.32)
,
(4.33) *and* (4.34) *if and only if*

$$x(t) = \int_a^b G(t,s)f(s)ds. \tag{4.40}$$

Proof. Let the relation (4.40) hold. Then

$$x(t) = -\Big[\int_a^t z(t)y(s)f(s)ds + \int_t^b y(t)z(s)f(s)ds\Big]\Big/A. \tag{4.41}$$

Differentiating (4.41) with respect to t yields

$$x'(t) = -\Big[\int_a^t z'(t)y(s)f(s)ds + \int_t^b y'(t)z(s)f(s)ds\Big]\Big/A. \tag{4.42}$$

Next on computing $(px')'$ from (4.42) and adding to qx in view of y and z being solutions of $L(x) = 0$ it follows that

$$L(x(t)) = -f(t) \tag{4.43}$$

Further, from the relations (4.41) and (4.42), it is seen that

$$\begin{cases} Ax(a) = -y(a)\int_a^b z(s)f(s)ds, \\ Ax'(a) = -y'(a)\int_a^b z(s)f(s)ds. \end{cases} \tag{4.44}$$

Since $y(t)$ satisfies the boundary condition given in (4.33), it follows from (4.44) that $x(t)$ also satisfies the boundary condition (4.33). Similarly $x(t)$ satisfies the boundary condition (4.34). This proves that $x(t)$ satisfies (4.32) and (4.33) and (4.34).

Conversely, let $x(t)$ satisfy (4.32) and (4.33) and (4.34). Then from (4.32) it is clear that

$$-\int_a^b G(t,s)L(x(s))ds = \int_a^b G(t,s)f(s)ds \tag{4.45}$$

The left side of (4.45) is

$$-\int_a^t G_1(t,s)L(x(s))ds - \int_t^b G_2(t,s)L(x(s))ds. \tag{4.46}$$

Now a well-known result is used that if u and v are two functions which admit continuous derivatives in $[t_1, t_2]$, then

$$\int_{t_1}^{t_2} u(s)L(v(s))ds = \int_{t_1}^{t_2} v(s)L(u(s))ds + \Big[p(s)(u(s)v'(s) - u'(s)v(s))\Big]_{t_1}^{t_2} \tag{4.47}$$

Applying the identity (4.47) in (4.46) and using the properties of $G_1(t,s)$ and $G_2(t,s)$ the left side of (4.45) becomes

$$-p(t)\Big\{\Big[G_1(t,t)x'(t) - \frac{\partial G_1(t,s)}{\partial t}\Big|_{s=t}x(t)\Big] - \Big[G_2(t,t)x'(t) - \frac{\partial G_2(t,s)}{\partial t}\Big|_{s=t}x(t)\Big]\Big\} \tag{4.48}$$

The first and third term in (4.48) cancel each other because of continuity of $G(t,s)$ at $t = s$. The condition (iv) in the definition of Green's function now shows that the value of the expression (4.48) is $x(t)$. But (4.48) is the left side of (4.45) which means $x(t) = \int_a^b G(t,s)f(s)ds$. This completes the proof. □

Example 4.6.3. Consider the BVP

$$x'' = f(t); x(0) = x(1) = 0. \tag{4.49}$$

It is easy to verify that the Green's function $G(t,s)$ is given by

$$G(t,s) = \begin{cases} t(1-s) & \text{if } t \le s, \\ s(1-t) & \text{if } t \ge s. \end{cases} \tag{4.50}$$

Thus the solution of (4.49) is given by $x(t) = -\int_0^1 G(t,s)f(s)ds$.

EXERCISES

1. In theorem establish the relations (4.41), (4.45) and (4.48). Also show that if x satisfies (4.40), then x also satisfies the boundary conditions (4.33) and (4.34).

2. Prove that the Green's function defined by (4.39) is symmetric, that is $G(t,s) = G(s,t)$.

3. Show that the Green's function for $L(x) = x'' = 0, x(1) = 0; x'(0) + x'(1) = 0$ is

$$G(t,s) = \begin{cases} 1-s & \text{if } t \le s, \\ 1-t & \text{if } t \ge s. \end{cases}$$

   Hence solve the BVP

$$x'' = f(t), x(0) + x(1) = 0, \quad x'(0) + x'(1) = 0$$

   where

   (i) $f(t) = \sin \pi t$;

   (ii) $f(t) = e^t$; $\quad 0 \le t \le 1$

   (iii) $f(t) = t$.

4. Consider the BVP $x'' + f(t, x, x') = 0, x(a) = 0, x(b) = 0$. Show that $x(t)$ is a solution of the above BVP if and only if

$$x(t) = \int_a^b G(t,s)f(s, x(s), x'(s))ds,$$

   where $G(t,s)$ is the Green's function given by

$$(b-a)G(t,s) = \begin{cases} (b-t)(s-a) & \text{if } a \le s \le t \le b, \\ (b-s)(t-a) & \text{if } a \le t \le s \le b. \end{cases}$$

   Also establish that

   (i) $0 \le G(t,s) \le \frac{b-a}{4}$

   (ii) $\int_a^b G(t,s)ds = \frac{(b-t)(t-a)}{2}$

   (iii) $\int_a^b G(t,s)ds \le \frac{(b-a)^2}{8}$

   (iv) $G(t,s)$ is symmetric.

5. Consider the BVP $x'' + f(t, x, x') = 0, x(a) = 0, x'(b) = 0$. Show that x is a solution of this BVP if, and only if, x satisfies

$$x(s) = \int_a^b H(t,s) f(s, x(s), x'(s)) ds, \quad a \le t \le b$$

where $H(t,s)$ is the Green's function defined by

$$H(t,s) = \begin{cases} s - a & \text{if } a \le s \le t \le b, \\ t - a & \text{if } a \le t \le s \le b. \end{cases}$$

Module 5

Asymptotic behavior and Stability Theory

Lecture 29

5.1 Introduction

Once the existence of a solution for a differential equation is established, the next question is :
How does a solution grow with time?
It is all the more necessary to investigate such a behavior of solutions in the absence of an explicit solution. One of the way out is to find suitable criteria, in terms of the known quantities, to establish the asymptotic behavior. A few such criteria are studied below. More or less we have adequate information for the asymptotic behavior of linear systems .

5.2 Linear Systems with Constant Coefficients

Consider a linear system

$$x' = Ax, \quad 0 \le t < \infty, \tag{5.1}$$

where A is an $n \times n$ constant matrix. The *priori* knowledge of eigenvalues of the matrix A completely determines the solutions of (5.1). So much so, the eigenvalues determine the behavior of solutions as $t \to \infty$. A suitable upper bound for the solutions of (5.1) are very useful and we have one such result in the ensuing result.

Theorem 5.2.1. *Let $\lambda_1, \lambda_1, \cdots, \lambda_m$ $(m \le n)$ be the distinct eigenvalues of the matrix A and λ_j be repeated n_j times $(n_1 + n_2 + \cdots + n_m = n)$. Let*

$$\lambda_j = \alpha_j + i\beta_j \ \ (i = \sqrt{-1}, j = 1, 2, \cdots, m), \tag{5.2}$$

and $\eta \in \mathbb{R}$ be a number such that

$$\alpha_j > \eta, \quad (j = 1, 2, \cdots, m). \tag{5.3}$$

Then, there exists a real constant $M > 0$ such that

$$|e^{At}| \le Me^{\eta t}, \quad 0 \le t < \infty. \tag{5.4}$$

Proof. Let e_j be the n-vector with 1 in the j-th place and zero elsewhere. Then,

$$\varphi_j(t) = e^{At}e_j, \tag{5.5}$$

denotes the j-th column of the matrix e^{At}. From the previous module on systems of equations, we know that

$$e^{At}e_j = \sum_{r=1}^{m}(c_{r1} + c_{r2}t + \cdots + c_{rn_r}t^{n_r-1})e^{\lambda_r t}, \tag{5.6}$$

where $c_{r1}, c_{r2}, \cdots, c_{rn_r}$ are constant vectors. From (5.5) and (5.6) we have

$$|\varphi_j(t)| \leq \sum_{r=1}^{m}(|c_{r1}| + |c_{r2}|t + \cdots + |c_{rn_r}|t^{n_r-1})|\exp(\alpha_r + i\beta_r)t| = \sum_{r=1}^{m} P_r(t)e^{\alpha_r t} \tag{5.7}$$

where P_r is a polynomial in t. By (5.3),

$$t^k e^{\alpha_r t} < e^{\eta t}, \tag{5.8}$$

for sufficiently large values of t. In view of (5.7) and (5.8) there exists $M_j > 0$ such that

$$|\varphi_j(t)| \leq M_j e^{\eta t},\ 0 \leq t < \infty\ ; (j = 1, 2, \cdots, n).$$

Now

$$|e^{At}| \leq \sum_{j=1}^{n} |\varphi_j(t)| \leq (M_1 + M_2 + \cdots + M_n)e^{\eta t} = Me^{\eta t} \quad (0 \leq t < \infty),$$

where $M = M_1 + M_2 + \cdots + M_n$ which proves the inequality (5.4). □

Actually we have estimated an upper bound for the fundamental matrix e^{At} for the equation (5.1) in terms of an exponential function through the inequality (5.4). Theorem 5.2.2 proved subsequently is a direct consequence of Theorem 5.2.1 . It tells us about a necessary and sufficient conditions for the solutions of (5.1) decaying to zero as $t \to \infty$. In other words, it characterizes a certain asymptotic behavior of solutions of (5.1) It is quite easy to sketch the proof and so the details are omitted.

Theorem 5.2.2. *Every solution of the equation* (5.1) *tends to zero as* $t \to +\infty$ *if and only if the real parts of all the eigenvalues of* A *are negative.*

Obviously, if the real part of an eigenvalue is positive and if φ is a solution corresponding to this eigenvalue then,

$$|\varphi(t)| \to +\infty,\ \text{as} t \to \infty.$$

Stability of Nonlinear Systems

Introduction

Presently we study the stability of stationary solutions of systems described by ordinary differential equations. The definitions of stability stated below is due to Lyapunov . Among the methods known today, to study the stability properties, the direct or the second method due to Lyapunov is important and useful. This method rests on the construction of a scalar function satisfying certain conceivable conditions. Further it does not depend on the knowledge of solutions in a closed form. These results are known in the literature as energy methods. Analysis plays an important role for obtaining proper estimates on energy functions.

Stability Definitions

We again recall here that in many of the problems the main interest revolves round the stability behavior of solutions of nonlinear differential equations which describes the problem. Such a study turns out to be difficult due to the lack of closed form for their solutions . The study is more or less concerned with the family of motions described through a differential equation (or through a systems of equation). The following notations are used:

$$I = [t_0, \infty), \quad \text{for } \rho > 0, \quad S_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}. \tag{5.9}$$

$$x' = f(t, x), \quad t \geq t_0 \geq 0, \tag{5.10}$$

where x and f are n-vectors. where f in the equation (5.32) is defined and in continuous on $I \times S_\rho$. Let the IVP (5.32) posses a unique solution $x(t; t_0, x_0)$ in S_ρ passing through a point (t_0, x_0) on I and let it continuously depend on (t_0, x_0). For simplicity, the solution $x(t; t_0, x_0)$ is denoted by $x(t)$ or x. We are basically interested in studying the stability of x . In the physical problems x is called an equilibrium position of an object, the motion of which is described by the equation (5.32). tacitly we are assume the existence of a unique solution of the IVP (5.32). The concept is stability is dealt below.

Definition 5.2.3.

(i) A solution x is said to be *stable* if for each $\epsilon > 0 (\epsilon < \rho)$ there exists a positive number $\delta = \delta(\epsilon)$ such that any solution y (ie $y(t) = y(t, t_0, y_0)$) of (5.32) existing on I satisfies

$$|y(t) - x(t)| \ < \ \epsilon, t \geq t_0 \ \text{ whenever } \ |y(t_0) - x(t_0)\ | < \ \delta.$$

(ii) A solution x is said to be *asymptotically stable* if it is stable and if there exists a number $\delta_0 > 0$ such that any other solution y of (5.32) existing on I is such that

$$|y(t) - x(t)| \to 0 \text{ as } t \to \infty \ \text{ whenever } \ |y(t_0) - x(t_0)| \ < \ \delta_0.$$

(iii) A solution x is said to be *unstable* if it is not stable.

We emphasize that in the above definitions, the existence of a solution x of (5.32) is taken for granted. In general, there is no loss of generality if we let x to be the zero solution. Such an assumption is at once clear if we look at the transformation

$$z(t) = y(t) - x(t), \tag{5.11}$$

where y is any solution of (5.32). Since y satisfies (5.32), we have

$$y'(t) = z'(t) + x'(t) = f(t, z(t) + x(t))$$

or else ,

$$z'(t) = f(t, z(t) + x(t)) - x'(t).$$

By setting

$$\tilde{f}(t, z(t)) = f(t, z(t) + x(t)) - x'(t)$$

we have

$$z'(t) = \tilde{f}(t, z(t)). \tag{5.12}$$

Clearly, (5.32) implies that

$$\tilde{f}(t, 0) = f(t, x(t)) - x'(t) \equiv 0$$

Thus, the resulting system (5.34) possesses a trivial solution or a zero solution. It is important to note that the transformation (5.33) does not change the character of the stability of a solution of (5.32). In subsequent discussions we assume that (5.32) admits a trivial or a null solution or zero solution which i fact is a state of equilibrium.

Lecture 33

Let us go through the following examples for illustration. .

Example 5.2.4. For an arbitrary constant c $y(t) = c$ is a solution of $x' = 0$. . Let the solution $x \equiv 0$ be the unperturbed state. For a given $\epsilon > 0$, for stability it is necessary to have

$$|y(t) - x(t)| = |y(t) - 0| = |c| < \epsilon$$

for $t \geq t_0$ whenever $|y(t_0) - x(t_0)| = |c - 0| = |c| < \delta$. By choosing $\delta < \epsilon$, then, the criterion for stability is trivially satisfied. Also $x \equiv 0$ is not asymptotically stable.

Example 5.2.5. $y(t) = ce^{-(t-t_0)}$ is a solution of $x' = -x$. Let $\epsilon > 0$ be given. For the stability of the origin we need to verify

$$|y(t) - x(t)| = |ce^{-(t-t_0)}| < \epsilon \text{ for } \quad t \geq t_0.$$

whenever $|y(t_0) - x(t_0)| = |c| < \delta$. By choosing $\delta < \epsilon$. now it is obvious that $x \equiv 0$ is stable. Further, for any $\delta_0 > 0$, then $|c| < \delta_0$ implies

$$|ce^{-(t-t_0)}| \to 0 \text{ as } \quad t \to \infty$$

or in other words $x \equiv 0$ is asymptotically stable.

Example 5.2.6. Any solution of the equation $x' = x$ through (t_0, η) is $y(t) = \eta \exp(t - t_0)$. Choose any $\eta > 0$. Clearly as $t \to \infty$ (ie increases indefinitely) y escapes out of any neighborhood of the origin or else the origin, in this case, is unstable. The details of a proof is left to the readers.

EXERCISES

1. Show that the system

$$x' = y, \; y' = -x$$

is stable but not asymptotically stable.

2. Prove that the system
$$x' = -x,\ y' = -y$$
is asymptotically stable; however, the system
$$x' = x,\ y' = y$$
is unstable.

3. Is the origin stabile in the following cases:
   (i) $x''' + 6x'' + 11x' + 6x = 0$,
   (ii) $x''' - 6x'' + 11x' - 6x = 0$,
   (iii) $x''' + ax'' + bx' + cx = 0$, for all possible values of a, b and c.

4. Consider the system
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$
Show that no non-trivial solution of this system tends to zero as $t \to \infty$. Is every solution bounded ? Is every solution periodic ?

5. Prove that for $1 < \alpha < \sqrt{2},\ x' = (\sin\log t + \cos\log t - \alpha)x$ is asymptotically stable.

6. Consider the equation
$$x' = a(t)x.$$
Show that the origin is asymptotically stable if and only if
$$\int_0^\infty a(s)ds = -\infty.$$
Under what condition the zero solution is stable ?

5.3 Stability of Linear and Quasi-linear Systems

In this section the stability of linear and a class of quasilinear systems are discussed with more focus on linear stems.Needless to stress the importance of these topics as these have wide applications. Many physical problems have a representing through an the (5.32) which may be written in a more useful form
$$x' = A(t)x + f(t, x). \tag{5.13}$$
The equation (5.35) simplifies the work since it is closely related with the system
$$x' = A(t)x. \tag{5.14}$$
The equation (5.35) is perturbed form of (5.35). Many properties of (5.36) have already been discussed. Under some restrictions on A and f, stability properties of (5.35) are very similar to those of (5.36). We assumed ,to proceed further,

(i) Let us recall : $I = [t_0, \infty)$, for $\rho > 0$, $S_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$.

(ii) the matrix $A(t)$ is an $n \times n$ matrix which is continuous on I;

(iii) $f : I \times S_\alpha \to \mathbb{R}^n$ is a continuous function with $f(t, 0) \equiv 0,\ t \in I$.

These two conditions guarantee the existence of local solutions of (5.35) on some interval. The solutions may not be unique. However, for stability we assume that solutions of (5.35) uniquely exist on I. Let $\Phi(t)$ denote a fundamental matrix of (5.36) such that $\Phi(t_0) = E$, where E is the identity matrix. As a first step, we obtain necessary and sufficient conditions for the stability of the linear system (5.36). Note that $x \equiv 0$, on I satisfies (5.36) or in other words $x \equiv 0$ or the zero solution or or the null the origin is an equilibrium state of (5.36).

Theorem 5.3.1. *The zero solution of equation* (5.36) *is stable if and only if a positive constant k exists such that*

$$|\Phi(t)| \leq k, \quad t \geq t_0. \tag{5.15}$$

Proof. The solution y of (5.36) which takes the value c at $t_0 \in I$ (or $y(t_0) = c$) is given by

$$y(t) = \Phi(t)c \quad (\Phi(t_0) = E).$$

Suppose that the inequality (5.37) hold. Then, for $t \in I$

$$|y(t)| = |\Phi(t)c| \leq k|c| < \epsilon$$

, if $|c| < \epsilon/k$. The origin is thus stable.

Conversely, let

$$|y(t)| = |\Phi(t)c| < \epsilon,\ t \geq t_0, \text{ for all } c \text{ such that } |c| < \delta.$$

Then, $|\Phi(t)| < \epsilon/\delta$. By Choosing $k = \epsilon/\delta$ the inequality (5.37) follows and hence the proof. □

The result stated below concerns about the asymptomatic stability of the zero solution of the system (sometimes we call it an an equation)(5.36).

Theorem 5.3.2. *The null solution of the system* (5.36) *is asymptotically stable if and only if*

$$|\Phi(t)| \to 0 \quad as \quad t \to \infty. \tag{5.16}$$

Proof. Firstly we note that (5.37) is a consequence of (5.38) and so the origin is obviously stable. Since

$$|\Phi(t)| \to 0\text{as} \quad t \to \infty$$

in view of (5.38) we have $|y(t)| \to 0$ as $t \to \infty$ or in other words the zero solution is The asymptotic stabile. □

The stability of (5.36) has already been considered when $A(t) = A$ is a constant matrix. We have seen earlier that if the characteristic roots of the matrix A have negative real parts then every solution of (5.36) tends to zero as $t \to \infty$. In fact, this is asymptotic stability. We already are familiar with the fundamental matrix $\Phi(t)$ which is given by

$$\Phi(t) = e^{(t-t_0)A}, \quad t_0, t \in I. \tag{5.17}$$

When the characteristic roots of the matrix A have negative real parts then,there exist two positive constants M and ρ such that

$$|e^{(t-t_0)A}| \leq Me^{-\rho(t-t_0)}, \quad t_0, t \in I. \tag{5.18}$$

Let the function f satisfy the condition

$$|f(t,x)| = o(|x|) \tag{5.19}$$

uniformly in t for $t \in I$. This implies that for x in a sufficiently small neighborhood of the origin, $\dfrac{|f(t,x)|}{|x|}$ can be made arbitrary small. The proof of the following result depends on the use of Gronwall's inequality.

Theorem 5.3.3. *In equation* (5.35)*, let* $A(t)$ *be a constant matrix* A *and let all the characteristic roots of* A *have negative real parts. Assume further that* f *satisfies the condition* (5.41)*. Then, the origin for the system* (5.35) *is asymptotically stable.*

Proof. By the variation of parameters formula, the solution y of the equation (5.35) passing through (t_0, y_0) satisfies the integral equation

$$y(t) = e^{(t-t_0)A}y_0 + \int_{t_0}^{t} e^{(t-s)A} f(s, y(s))ds. \tag{5.20}$$

The inequality (5.40) together with (5.42) yields

$$|y(t)| \leq M|y_0|e^{-\rho(t-t_0)} + M\int_{t_0}^{t} e^{-\rho(t-s)}|f(s,y(s))|ds. \tag{5.21}$$

which takes the form

$$|y(t)|e^{\rho t} \leq M|y_0|e^{\rho t_0} + M\int_{t_0}^{t} e^{\rho s}|f(s,y(s))|ds.$$

Let $|y_0| < \alpha$. Then, the relation (5.42) is true in any interval $[t_0, t_1)$ for which $|y(t)| < \alpha$. In view of the condition (5.41), for a given $\epsilon > 0$ we can find a positive number δ such that

$$|f(t,x)| \leq \epsilon|x|, \quad t \in I, for |x| < \delta. \tag{5.22}$$

Let us assume that $|y_0| < \delta$. Then, there exists a number T such that $|y(t)| < \delta$ for $t \in [t_0, T]$. Using (5.44) in (5.43), we obtain

$$e^{\rho t}|y(t)| \leq M|y_0|e^{\rho t_0} + M\epsilon\int_{t_0}^{t} e^{\rho s}|y(s)|ds, \tag{5.23}$$

for $t_0 \leq t < T$. an application of Gronwall's inequality to (5.45), yields

$$e^{\rho t}|y(t)| \leq M|y_0|e^{\rho t_0}.e^{M\,\epsilon(t-t_0)} \tag{5.24}$$

or for $t_0 \leq t < T$, we obtain

$$|y(t)| \leq M|y_0|e^{(M\,\epsilon-\rho)(t-t_0)}. \tag{5.25}$$

Choose $M\epsilon < \rho$ and $y(t_0) = y_0$. If $|y_0| < \delta/M$, then, (5.47) yields

$$|y(t)| < \delta, \quad t_0 \le t < T.$$

The solution y of the equation (5.35) exists locally at each point (t, y), $t \ge t_0$, $|y| < \alpha$. Since the function f is defined on $I \times S_\alpha$, we extend the solution y interval by interval by preserving its bound by δ. So given any solution $y(t) = y(t; t_0, y_0)$ with $|y_0| < \delta/M$, y exists on $t_0 \le t < \infty$ and satisfies $|y(t)| < \delta$. In the above discussion, δ can be made arbitrarily small. Hence, $y \equiv 0$ is asymptotically stable when $M\epsilon < \rho$. □

When the matrix A is a function of t still the stability properties solutions of (5.35) and (5.36) are shared but now the fundamental matrix needs to satisfy some stronger conditions. Let the function f be continuous and satisfy the inequality

$$|f(t,x)| \le r(t)|x|, (t,x) \in I \times S_\alpha, \tag{5.26}$$

where r is a non-negative continuous function such that

$$\int_{t_0}^{\infty} r(s)ds < +\infty.$$

The condition (5.48) guarantees the existence of a null solution of (5.35). Now the following is a result on asymptotic stability of the zero solution of (5.35).

Theorem 5.3.4. *Let the fundamental matrix $\Phi(t)$ satisfy the condition*

$$|\Phi(t)\Phi^{-1}(s)| \le K, \tag{5.27}$$

where K is a positive constant and $t_0 \le s \le t < \infty$. Let f satisfy the hypotheses given by (5.48). *Then, a positive constant M can be found such that if $t_1 \ge t_0$, any solution y of* (5.35) *is defined and satisfies*

$$|y(t)| \le M|y(t_1)|, \ t \ge t_1 \ \textit{whenever}\ |y(t_1)| < \alpha/M.$$

Moreover, if $|\Phi(t)| \to 0$ as $t \to \infty$ then

$$|y(t)| \to 0 \ \textit{as} \ \ t \to \infty.$$

Proof. Let $t_1 \ge t_0$ and y be any solution of (5.35) such that $|y(t_1)| < \alpha$. Then y satisfies the integral equation

$$y(t) = \Phi(t)\Phi^{-1}(t_1)y(t_1) + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s)f(s, y(s))ds. \tag{5.28}$$

for $t_1 \le t < T$, where $|y(t)| < \alpha$ for $t_1 \le t < T$. By hypotheses (5.48) and (5.49) we obtain

$$|y(t)| \le K|y(t_1)| + K\int_{t_1}^{t} r(s)|y(s)|ds$$

The Gronwall's inequality now yields

$$|y(t)| \le K|y(t_1)| \exp\Big(K\int_{t_1}^{t} r(s)ds\Big). \tag{5.29}$$

By the condition (5.48) the integral on the right side is bounded. Let

$$M = K \exp\Big(K \int_{t_1}^{\infty} r(s) ds\Big).$$

Then,

$$|y(t)| \leq M|y(t_1)|. \tag{5.30}$$

Clearly this inequality holds if $|y(t_1)| < \alpha/M$. By following the lines of proof of in Theorem we extend the solution for all $t \geq t_1$. Hence, the inequality (5.52) holds for $t \geq t_1$.

The general solution $y(t)$ of (5.35) also satisfies the integral equation

$$\begin{aligned} y(t) &= \Phi(t)\Phi^{-1}(t_0)y(t_0) + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)f(s, y(s))ds \\ &= \Phi(t)y(t_0) + \int_{t_0}^{t_1} \Phi(t)\Phi^{-1}(s)f(s, y(s))ds + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s)f(s, y(s))ds. \end{aligned}$$

Note that $\Phi(t_0) = E$. By using the conditions (5.48), (5.49) and (5.52), we obtain

$$\begin{aligned} |y(t)| &\leq |\Phi(t)||y(t_0)| + |\Phi(t)| \int_{t_0}^{t_1} |\Phi^{-1}(s)||f(s, y(s))|ds + K \int_{t_1}^{\infty} r(s)|y(s)|ds \\ &\leq |\Phi(t)||y(t_0)| + |\Phi(t)| \int_{t_0}^{t_1} |\Phi^{-1}(s)||f(s, y(s))|ds + KM|y(t_1)| \int_{t_1}^{\infty} r(s)ds. \end{aligned} \tag{5.31}$$

The last term of the right side of the inequality (5.53) can be made less than (arbitrary) $\epsilon/2$ by choosing t_1 sufficiently large. By hypotheses $\Phi(t) \to 0$ as $t \to \infty$. The first two terms on the right side contain the term $|\Phi(t)|$. Hence, their sum together can be made arbitrarily small by choosing t large enough, say less than $\epsilon/2$. Thus, $|y(t)| < \epsilon$ for large t. This proves that $|y(t)| \to 0$ as $t \to \infty$. □

The inequality (5.52) shows that the origin is stable for $t \geq t_1$. But note that $t_1 \geq t_0$ is any arbitrary number. Here, condition (5.52) holds for any $t_1 \geq t_0$. Thus, we have established a stronger than the stability of the origin .In literature such a property is called uniform stability. We do not propose to go into the detailed study of such types of stability behaviors.

EXERCISES

1. Prove that all solutions of the system (5.36) are stable if and only if they are bounded.

2. Let $b : I \to \mathbb{R}^n$ be a continuous function. Prove that a solution x of linear nonhomogeneous system

$$x' A(t)x + b(t)$$

is stable, asymptotically stable, unstable, if the same holds for the null solution of the corresponding homogeneous system (5.36).

3. Prove that if the characteristic polynomial of the matrix A is stable, the matrix $C(t)$ is continuous on $0 \leq t < \infty$ and $\int_0^\infty |C(t)|dt < \infty$, then all solutions of

$$x' = (A + C(t))x$$

are asymptotically stable.

4. Prove that the system (5.36) is unstable if

$$Re\Big(\int_{t_0}^{t} tr\ A(s)ds\Big) \to \infty, \quad \text{as} \quad t \to \infty.$$

5. Define the norm of a matrix $A(t)$ by $\mu(A(t)) = \lim\limits_{h \to 0} \dfrac{|E + hA(t)| - 1}{h}$, where E is the $n \times n$ identity matrix.

   (i) Prove that μ is a continuous function of t.

   (ii) For any solution y of (5.36) prove that

   $$|y(t_0)| \exp\Big(-\int_{t_0}^{t} \mu(-A(s))ds\Big) \leq |y(t)| \leq |y(t_0)| \exp \int_{t_0}^{t} \mu(A(s))ds.$$

   $\Big[$Hint : Let $r(t) = |y(t)|$. Then

   $$r'_{+}(t) = \lim_{h \to 0^+} \frac{|y(t) + hy'(t)| - |y(t)|}{h}.$$

   Show that $r'_{+}(t) \leq \mu(A(t))r(t).\Big]$

   (iii) When $A(t) = A$ a constant matrix, show that $|\exp(tA)| \leq \exp[t\mu(A)]$.

   (iv) Prove that the trivial solution is stable if $\limsup\limits_{t \to \infty} \displaystyle\int_{t_0}^{t} \mu(A(s))ds < \infty$.

   (v) Show that the trivial solution is asymptotically stable if

   $$\int_{t_0}^{t} \mu(A(s))ds \to -\infty \quad \text{as} \quad t \to \infty.$$

   (vi) Establish that the solution is unstable if $\liminf\limits_{t \to \infty} \displaystyle\int_{t_0}^{t} \mu(-A(s))ds = -\infty$.

Lecture 35

5.4 Stability of Autonomous Systems

Many a times the time (variable) t does not appear explicitly in he equations which describes the physical problem . For example, the equation

$$x' = k\,x$$

(where k is a constant) represents a simple model for the growth of population where t does not appear explicitly. Let us recall : In general such equations assumes a form

$$x' = g(x) \tag{5.32}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$.Let us assume that the function g together with its first partial derivatives with respect to $x_1, x_2, \cdots, x_n$ are continuous in S_ρ. A system described by (5.54) is called an autonomous system. Let $g(0) = 0$ so that (5.54) admits the trivial or the zero solution. Presently,our aim is to study the stability of the zero of solution of (5.54) on I.

Lyapunov's direct method revolves round the construction of a scalar function satisfying certain properties which has close resemblance to he energy function. In fact, this method is the generalization of the energy method in classical mechanics. It is well known that a mechanical system is stable if its energy(kinetic energy+ potential energy) continuously decreases. The energy is always positive quantities and is zero when the system is completely at rest. Lyapunov generalized energy function which is known in the literature as the 'Lyapunov function'. This function is generally denoted by V. A function

$$V : S_\rho \to \mathbb{R}$$

is said to be positive definite if the following conditions hold:

(i) V and $\frac{\partial V}{\partial x_j}$ $(j = 1, 2, \cdots, n)$ be continuous on S_ρ.

(ii) $V(0) = 0$.

(iii) V is positive for all $x \in S_\rho$ and $x \neq 0$.

V is called negative definite $-V$ is positive definite. The function V attains the minimum value at the origin. Further the origin is the only point in S_ρ at which the minimum value is attained. Since V has continuous first order partial derivatives, the chain rule may be used to obtain $\frac{dV(x)}{dt}$ as

$$\frac{dV(x)}{dt} = V(x) = \frac{\partial V(x)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V(x)}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial V(x)}{\partial x_n}\frac{dx_n}{dt}$$

$$= \sum_{j=1}^{n} \frac{\partial V(x)}{\partial x_j} x_j' = grad\ V(x).g(x).$$

along a solution x of (5.54). The last step is a consequence of (5.54). We also that the derivative of V with respect to t along a solution of (5.54) is now known to us, although we do not have the explicit form of a solution. The conditions on the V function are not very stringent and it is not difficult to construct several functions which satisfy these conditions. For instance

$$V(x) = x^2, (x \in \mathbb{R}) \text{ or } \; V(x_1, x_2) = x_1^4 + x_2^4, (x_1, x_2) \in \mathbb{R}^2$$

are some simple examples of positive definite functions while is not a positive definite function ,since $V(0) = 1 \neq 0$. The function

$$V(x_1, x_2) = x_1^2 - x_2^2, (x_1, x_2) \in \mathbb{R}^2$$

is not a positive definite since $V(x, x) = 0$ even if $x \neq 0$. In general, let A be a $n \times n$ positive definite real matrix then V defined by

$$V(x) = x^T A\ x, \ \textit{where } x \in \mathbb{R}^n$$

is a positive definite function. Let us assume that a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ given by

$$V(x) = V(x_1, x_2, \cdots, x_n)$$

is positive definite. Geometrically,when $n = 3$, we may visualize V in three dimensional space. For example let us consider a simple function

$$V(x_1, x_2) = x_1^2 + x_2^2;$$

clearly all the conditions (i),(ii) and (iii) hold. Let

$$z = x_1^2 + x_2^2.$$

Since $z \geq 0$ for all (x_1, x_2) the surface will always lie in the upper part of the plane OX_1X_2. Further $z = 0$ when $x_1 = x_2 = 0$. Thus, the surface passes through the origin. Such a surface is like a parabolic mirror pointing upwards.

Now consider a section of this cup-like surface by a plane parallel to the plane OX_1X_2. This section is a curve

$$x_1^2 + x_2^2 = k, z = k.$$

Its projection on the X_1X_2 plane is

$$x_1^2 + x_2^2 = k, z = 0.$$

Clearly these are circles with radius k, and the center at the origin. In a general , instead of circles, we have closed curves around the origin . The geometrical picture for any Lyapunov function in three dimensional, in a small neighborhood of the origin, is more or less is of this character. In higher dimensions larger than three, the above discussion helps us to visualize of such functions.

We state below 3 results concerning the stability behavior of the zero solution of the system (5.54). The geometrical explanation given below for these results shows a line of the proof. But they are not proofs in a strict mathematical sense. The detailed mathematical proofs are given in the next section We also Note that these are only sufficient.

Theorem 5.4.1. *Let there exists a positive definite function V defined on such that $\dot{V} \leq 0$ then, the origin of the equation/system* (5.54) *is stable.*

Theorem 5.4.2. *If in S_ρ there exists a positive definite function V such that $-\dot{V}$ is also positive definite, then, the origin of the equation* (5.54) *is asymptotically stable.*

Theorem 5.4.3. *[(Cetav)] Let V be given function and N a region in S_ρ such that*

(i) V has continuous first partial derivatives on N;

(ii) at the boundary points of N(inside S_ρ), $V(x) = 0$;

(iii) the origin is on the boundary of N;

(iv) V and $\dot{V}$ are positive on N.

Then, the origin of (5.54) *is unstable.*

Example 5.4.4. Consider the system

$$x_1' = -x_2, \ x_2' = x_1.$$

The system is autonomous and possesses a trivial solution. The function V defined by

$$V(x_1, x_2) = x_1^2 + x_2^2.$$

is positive definite. The derivative $\dot{V}$ along the solution is

$$\dot{V}(x_1, x_2) = 2[x_1(-x_2) + x_2(x_1)] = 0.$$

So the hypotheses of Theorem 5.6.1 holds and hence the zero solution or origin is stable.

Example 5.4.5. Consider the system

$$\begin{aligned} x_1' &= (x_1 - bx_2)(\alpha x_1^2 + \beta x_2^2 - 1) \\ x_2' &= (ax_1 + x_2)(\alpha x_1^2 + \beta x_2^2 - 1). \end{aligned}$$

Let

$$V(x_1, x_2) = ax_1^2 + bx_2^2.$$

When $a > 0, \ b > 0, \ V(x_1, x_2)$ is positive definite. Also

$$\dot{V}(x_1, x_2) = 2(ax_1^2 + bx_2^2)(\alpha x_1^2 + \beta x_2^2 - 1).$$

Let $\alpha > 0, \ \beta > 0$. If $\alpha x_1^2 + \beta x_2^2 < 1$ then, $\dot{V}(x_1, x_2)$ is negative definite and by Theorem 5.6.2 the trivial solution is asymptotically stable .

Example 5.4.6. Consider the system

$$\begin{aligned} x_1' &= x_2 - x_1 f(x_1, x_2) \\ x_2' &= -x_1 - x_2 f(x_1, x_2), \end{aligned}$$

where f is represented by a convergent power series in x_1, x_2 and $f(0,0) = 0$. By letting

$$V = \frac{1}{2}(x_1^2 + x_2^2)$$

we have

$$\dot{V}(x_1, x_2) = -(x_1^2 + x_2^2) f(x_1, x_2).$$

Obviously, if $f(x_1, x_2) \geq 0$ arbitrarily near the origin, the origin is stable. If f is positive definite in some neighborhood of the origin, the origin is asymptotically stable. If $f(x_1, x_2) < 0$ arbitrarily near the origin, the origin is unstable.

Some more examples:

1. We claim that the zero solution of a scalar equation
$$x' = x(x-1)$$
is asymptotically stable. For
$$V(x) = x^2, |x| < 1$$
is positive definite and its derivative $\dot{V}$ along the solution is negative definite.

2. again we claim that the zero solution of a scalar equation
$$x' = x(1-x)$$
is unstable. For
$$V(x) = x^2, |x| < 1$$
is positive definite and its derivative $\dot{V}$ along the solution is positive.

EXERCISES

1. Determine whether the following functions are positive definite or negative definite:
   (i) $4x_1^2 + 3x_1x_2 + 2x_2^2$,
   (ii) $-3x_1^2 - 4x_1x_2 - x_2^2$,
   (iii) $10x_1^2 + 6x_1x_2 + 9x_2^2$,
   (iv) $-x_1^2 - 4x_1x_2 - 10x_2^2$.

2. Prove that
$$ax_1^2 + bx_1x_2 + cx_2^2$$
is positive definite if $a < 0$ and $b^2 - 4ac < 0$ and negative definite if $a < 0$ and $b^2 - 4ac > 0$.

3. Consider the quadratic form $Q = x^T Rx$ where x is a n-column-vector and $R = [r_{ij}]$ is an $n \times n$ symmetric real matrix. Prove that Q is positive definite if and only if
$$r_{11} > 0,\ r_{11}r_{22} - r_{21}r_{12} > 0, \text{ and } \quad \det[r_{ij}] > 0,\ i = 1, 2, \cdots; m = 3, 4, \cdots, n.$$

4. Find a condition on a, b, c under which the following matrices are positive definite:
   (i) $\frac{1}{ab-c}\begin{bmatrix} ac & c & 0 \\ c & a^2+b & a \\ 0 & a & 1 \end{bmatrix}$
   (ii) $\frac{1}{9-a}\begin{bmatrix} \frac{6a+27}{a} & a+2a & 9-a \\ 9+2a & a(a+3) & 3a \\ 9-a & 3a & 3a \end{bmatrix}$.

5. Let
$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} f(s)ds$$
where f is such that $f(0) = 0$, and $xf(x) > 0$ for $x \neq 0$. Show that V is positive definite.

6. Show that the trivial solution of the equation

$$x'' + f(x) = 0,$$

where f is a continuous function on $|x| < \rho$, $f(0) = 0$ and $xf(x) > 0$ is stable.

7. Show that the following systems are asymptotically stable:

   (i) $x_1' = -x_2 - x_1^3, \quad x_2' = x_1 - x_2^3.$

   (ii) $x_1' = -x_1^3 - x_1 x_2^3, \quad x_2' = x_1^4 - x_2^3.$

   (iii) $x_1' = -x_1^3 - 3x_2, \quad x_2' = 3x_1 - 5x_2^3.$

8. Show that the zero solution or origin for the system

$$\begin{aligned} x_1' &= -x_1 + 2x_1(x_1 + x_2)^2 \\ x_2' &= -x_2^3 + 2x_2^3(x_1 + x_2)^2 \end{aligned}$$

is asymptotically stable if $|x_1| + |x_2| < 1/\sqrt{2}$.

Lecture 37

5.5 Stability of Non-autonomous Systems

The study of the stability properties of non-autonomous systems have some inherent difficulties. Systems of this kind are given by (5.32). For this purpose a Lyapunov function $V(t,x)$ is needed which depends on t and x. Let f in (5.32) be such that $f(t,0) \equiv 0,\ t \in I$. Let f together with its first partial derivative be continuous on $I \times S_\rho$. This condition guarantees the existence and the uniqueness of solutions. For stability it is assumed that solutions of (5.32) exist on the entire time interval I and that the trivial solution is the equilibrium or the steady state.

Definition 5.5.1. A real valued function ϕ is said to belong to the class $\mathscr{K}$ if

(i) ϕ is defined and continuous on $0 \le r < \infty$,

(ii) ϕ is strictly increasing on $0 \le r < \infty$,

(iii) $\phi(0) = 0$ and $\phi(r) \to \infty$ as $r \to \infty$.

Example: The function $\phi(r) = \alpha r^2,\ \alpha > 0$, is of class $\mathscr{K}$.

Definition 5.5.2. A real valued function V defined on $I \times S_\rho$ is said to be positive definite if $V(t,0) \equiv 0$ and there exists a function $\phi \in \mathscr{K}$ such that

$$V(t,x) \ge \phi(|x|), (t,x) \in I \times S_\rho.$$

It is negative definite if

$$V(t,x) \le -\phi(|x|), t,x) \in I \times S_\rho.$$

Many times real valued positive definite function V is also known as energy function or Lyapunov function. Example: The function

$$V(t,x) := (t^2+1)x^4$$

is positive definite since $V(t,0) \equiv 0$ and there exists a $\phi \in \mathscr{K}$ such that $V(t,x) \ge \phi(|x|)$.

Definition 5.5.3. A real valued function V defined on $I \times S_\rho$ is said to be decrescent if there exists a function $\psi \in \mathscr{K}$ such that in a neighborhood of the origin and for all

$$t \ge t_0,\ V(t,x) \le \psi(|x|).$$

Examples : The function

$$V(t,x_1,x_2) = \frac{1}{t^2+1}(x_1^2+x_2^2),\ (t,x) \in I \times \mathbb{R}^2,$$

is decrescent. In this case, we may choose $\Psi(r) = r^2$. The function

$$V(t,x_1,x_2) = (1+e^{-t})(x_1^2+x_2^2)$$

is both positive definite and decrescent since

$$x_1^2+x_2^2 \le (1+e^{-t})(x_1^2+x_2^2) \le 2(x_1^2+x_2^2)$$

for the choice

$$\phi(r) = r^2, \ \psi(r) = 2r^2.$$

We are now set to prove the fundamental theorems on the stability of the equilibrium of the system (5.32). We need the energy function in these results. In order to avoid repetitions ,we need the following hypotheses **(H*)** :

(H*) Let $V : I \times S_\rho \to \mathbb{R}$ be a bounded C^1 function such that $V(t, 0) \equiv 0$ and with bounded first order partial derivatives.

By using the chain rule the derivative $\dot{V}(t, x)$ is

$$\dot{V}(t, x) = \frac{dV(t, x)}{dt} = \frac{\partial V(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{dx_i}{dt}.$$

Our interest is in the derivative of V along a solution x of the system (5.32). Indeed, we have

$$\dot{V}(t, x(t)) = \frac{\partial V(t, x(t))}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, x(t))}{\partial x_i} f_i(t, x(t)).$$

Theorem 5.5.4. *Let V be a positive definite function satisfying the hypotheses **(H*)** such that $\dot{V}(t, x) \le 0$; then the zero solution of the system* (5.32) *is stable.*

Proof. The positive definiteness of V tells us that there exists a function $\phi \in \mathscr{K}$ such that

$$0 \le \phi(|x|) \le V(t, x), \ |x| < \rho, \ t \in I. \tag{5.33}$$

Let $x(t) = x(t; t_0, x_0)$ be a solution of (5.32). Since $\dot{V}(t, x) \le 0$, we have

$$V(t, x(t; t_0, x_0)) \le V(t_0, x_0), \ t \in I. \tag{5.34}$$

By the continuity of V , given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ so that

$$V(t_0, x_0) < \phi(\epsilon), \tag{5.35}$$

whenever $|x_0| < \delta$. Now the inequalities (5.55) and (5.56) yield

$$0 \le \phi(|x(t; t_0, x_0)|) \le V(t, x(t; t_0, x_0)) \le V(t_0, x_0) < \phi(\epsilon).$$

Hence,

$$|x(t; t_0, x_0)| < \epsilon, \ for \ t \in I$$

whenever $|x_0| < \delta$ which shows that the origin or the zero solution is stable. □

The ensuing result provides us sufficient conditions for the asymptotic stability of the origin.

Theorem 5.5.5. *Let V be a positive definite decrescent function satisfying the hypotheses **(H*)** such that $\dot{V}(t, x) \le 0$ and , and $\dot{V}$ is negative definite. Then, the zero solution of the system* (5.32) *is asymptotically stable.*

Proof. Let $x(t; t_0, x_0)$ be a solution of (5.32). Since the hypotheses of Theorem 5.7.7 the null or the zero solution of (5.32) is stable. In other words, given $\epsilon > 0$ there exists $|x_0| < \delta$ such that

$$0 < |x(t; t_0, x_0)| < \epsilon, \ t \geq t_0, \quad \text{whenever } |x_0| < \delta.$$

Let $\delta_0 = \delta(\epsilon)$. Suppose that for some $\lambda > 0$

$$V(x(t; t_0, x_0)) \geq \lambda > 0, \quad \text{for } t \geq t_0.$$

By hypotheses, since $\dot{V}$ is negative definite,so there exists a function $\sigma \in \mathscr{K}$ such that

$$\dot{V}(t, x(t; t_0, x_0)) \leq -\sigma(|x(t; t_0, x_0)|). \tag{5.36}$$

In the light of (5.58) we have a number $\gamma > 0$ such that

$$\dot{V}(t, x(t; t_0, x_0)) \leq -\gamma < 0, \quad t \geq t_0.$$

Integrating both sides of this inequality, we get

$$V(t, x(t; t_0, x_0)) \leq V(t_0, x_0) - \gamma(t - t_0). \tag{5.37}$$

For large value of t the right side of (5.59) becomes negative which contradicts the fact that V is positive definite. So the assumption that that for some $\lambda > 0$

$$V(x(t_n; t_0, x_0)) \geq \lambda > 0, \quad \text{for } t \geq t_0.$$

is false. No such λ exists. Since V is a positive definite and decrescent function,

$$V(t, x(t; t_0, x_0)) \to 0 \ as \ t \to \infty$$

and therefore it follows that

$$|x(t; t_0, x_0)| \to 0 \ as \ t \to \infty.$$

Thus, the origin or the zero solution is asymptotically stable. □

In some cases ρ may be infinite. Thus it is possible that the system is asymptotically stable for any choice of x_0. The following theorem is stated without proof which provides sufficient conditions for the asymptotic stability in the large.

Theorem 5.5.6. *The equilibrium state of* (5.32) *is asymptotically stable in the large if there exists, a positive definite function $V(t, x)$ which is decrescent everywhere and such that $V(t, x) \to \infty$ as $|x| \to \infty$ for each $t \in I$ and such that $\dot{V}$ is negative definite.*

Example 5.5.7. Consider the system $x' = A(t)x$, where $A(t) = (a_{ij})$, $a_{ij} = -a_{ji}$, $i \neq j$ and $a_{ij} \leq 0$, for all values of $t \in I$ and $i, j = 1, 2, \cdots, n$. Let $V(x) = x_1^2 + x_2^2 + \cdots + x_n^2$. Obviously $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$. Further

$$\begin{aligned}\dot{V}(x(t)) &= 2\sum_{i=1}^{n} x_i(t)x_i'(t) = 2\sum_{i=1}^{n} x_i(t)\Big[\sum_{j=1}^{n} a_{ij}x_j(t)\Big] \\ &= 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i(t)x_j(t) = 2\sum_{i=1}^{n} a_{ii}x_i^2(t) \leq 0.\end{aligned}$$

The last step is obtained by using the assumption for the matrix $A(t)$. Now the conditions of the Theorem hold and so the origin is stable. If $a_{ii} < 0$ for all values of t then it is seen that $\dot{V}(x(t)) < 0$ which implies asymptotic stability of the origin of the given system.

EXERCISES

1. (i) Show that
$$V(t, x_1, x_2) = t(x_1^2 + x_2^2) - 2x_1x_2 \cos t$$
is positive definite for $n = 2$ and $t > 2$.

   (ii) Prove that
$$x_1^2(1 + \sin^2 t) + x_2^2(1 + \cos^2 t)$$
is positive definite for all values of (t, x_1, x_2).

2. Show that

   (i) $(x_1^2 + x_2^2)\sin^2 t$ is decrescent.

   (ii) $x_1^2 + (1 + t)x_2^2$ is positive definite but not decrescent.

   (iii) $x_1^2 + (\frac{1}{1+t^2})x_2^2$ is decrescent but not positive definite.

   (iv) $x_1^2 + e^{-2t}\, x_2^2$ is decrescent.

   (v) $(1 + e^{-2t})\ ((x_1^2 + x_2^2))$ is positive definite and decrescent.

3. Prove that a function V which has bounded partial derivatives $\frac{\partial V}{\partial x_i}(i = 1, 2, \cdots, n)$ on $I \times S_\rho$ for $t \geq t_0 \geq 0$ is decrescent.

4. Consider the equation $x' = -x - \frac{x}{t}(1 - x^2t^2)$. For $y = tx$ it becomes $y' = y(y^2 - 1)$. Prove that the trivial solution is stable when, for a fixed t_0, $|x_0| \leq \frac{1}{t_0}$.

5. For the system
$$\begin{aligned} x_1' &= e^t x_2 - (t^2 + 1)x_1(x_1^2 + x_2^2) \\ x_2' &= -e^t x_1 - (t^2 + 1)x_2(x_1^2 + x_2^2), \end{aligned}$$
show that the origin is asymptotically stable.

6. Prove that the trivial solution of the system
$$\begin{aligned} x_1' &= a(t)x_2 + b(t)x_1(x_1^2 + x_2^2) \\ x_1' &= -a(t)x_1 + b(t)x_2(x_1^2 + x_2^2) \end{aligned}$$
is stable if $b \leq 0$, asymptotically stable if $b \leq q < 0$ and unstable if $b > 0$.

Lecture 38

5.6 A Particular Lyapunov Function

The results stated earlier depends on the existence of an energy or a Lyapunov function. Let us study such a construction method for a linear equation and we also exploit it for studying stability of zero solution of a nonlinear systems close enough to the corresponding linear system.At the moment let us consider a linear system

$$x' = Ax, \quad x \in \mathbb{R}^n, \tag{5.38}$$

where $A = (a_{ij})$ is an $n \times n$ constant matrix. The aim is to study the stability of the zero solution of (5.60) by Lyapunov's direct method. The stability is determined by the nature of the characteristic roots of the matrix A. Let V represent a quadratic form

$$V(x) = x^T R x, \tag{5.39}$$

where $R = (r_{ij})$ is an $n \times n$ constant, positive definite, symmetric matrix. The time derivative of V along the solution of (5.60) is given by

$$\begin{aligned} \dot{V}(x) = x'^T Rx + x^T Rx' &= x^T A^T Rx + x^T RAx \\ &= x^T (A^T R + RA)x = -x^T Qx, \end{aligned}$$

where

$$(A^T R + RA) = -Q. \tag{5.40}$$

Here $Q = (q_{ij})$ is $n \times n$ constant symmetric matrix. For the asymptotic stability of (5.60) we need the negative definiteness of the time derivative of V.On the overhand if we start with an arbitrary matrix R then, the matrix Q may not be positive definite. probably one way out is to choose Q (an arbitrary) positive definite matrix and try to solve the equation (5.62) for R. We again stress that the positive definiteness of the matrices R and Q is a sufficient condition for the asymptotic stability of the zero solution of the linear system (5.60). The sufficiency is obvious since V is positive definite and $\dot{V}$ is negative definite by the Theorem 5.6.2 the zero solution of of the system (5.60) is asymptotically stable. So let us assume the matrix Q to be positive definite and solve the equation (5.62) R. The question is :

We again stress that the positive definiteness of the matrices R and Q is a sufficient condition for the asymptotic stability of the zero solution of the linear system (5.60).

Under what conditions the equation (5.62) gives rise to a unique solution? The answer lies in the following result whose proof is given here. A square matrix R is called a Stable matrix if all the eigen values of R have stich negative real parts.

Proposition: Let A be a real matrix. Then, the equation (5.62) namely,

$$(A^T R + RA) = -Q$$

has a positive definite solution R for every for every positive definite matrix Q if and only if A is a stable matrix.

A consequence :
In the light of the above proposition we again repeat that that the positive definiteness of

the matrices R and Q is a necessary and sufficient condition for the asymptotic stability of the zero solution of the linear system (5.60).

Remark: The stability properties of zero solution of the equation (5.62). unaffected if the system (5.60) is transformed by the relation $x = Py$, where P is a non-singular constant matrix. The system (5.60) is then transforms to

$$y' = (P^{-1}AP)y.$$

Now choose the matrix P such that

$$P^{-1}AP$$

is a triangular matrix. Such a transformation is always possible by Jordan normal form. So there is no loss of generality by assuming in (5.60) that, the matrix A is such that its main diagonal consists of eigenvalues of A and for $i < j,\ a_{ij} = 0$. In other words the matrix A is of the following form:

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda_2 & 0 & \cdots & 0 \\ a_{31} & a_{32} & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda_n \end{bmatrix}.$$

The equation (5.62) is

$$\begin{bmatrix} \lambda_1 & a_{21} & a_{31} & \cdots & a_{n1} \\ 0 & \lambda_2 & a_{32} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & r_{nn} \end{bmatrix}$$

$$+ \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & r_{22} & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda_n \end{bmatrix}$$

$$= - \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & q_{n3} & \cdots & q_{nn} \end{bmatrix}.$$

Equating the corresponding terms on both sides results in the following system of equations

$$(\lambda_j + \lambda_k)r_{jk} = -q_{jk} + \delta_{jk}(\cdots, r_{hk}, \cdots),$$

where δ_{jk} is a linear form in $r_{hk},\ h+k > j+k$, with coefficients in a_{rs}. Hopefully the above system determines r_{jk}. The solution of the linear system is unique if the determinant of the coefficients is non-zero. Obviously the determinant is the product of the coefficients of the form

$$\lambda_j + \lambda_k.$$

In such a case the matrix R is uniquely determined if none of the characteristic roots λ_i is zero and further the sum of any two different roots is not zero. The following example illustrates the procedure for the determination of R.

Example 5.6.1. Let us construct a Lyapunov function for the system

$$x_1' = -3x_1 + kx_2, \quad x_2' = -2x_1 - 4x_2$$

to find values of k which ensures the asymptotic stability of the zero solution. In this case $A = \begin{bmatrix} -3 & k \\ -2 & 4 \end{bmatrix}$. Let Q be an arbitrary positive definite matrix, say

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now Eq. (5.62) is

$$\begin{bmatrix} -3 & -2 \\ k & -4 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} -3 & k \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Consequently (equating the terms on both sides solving the system of equations) we have

$$r_{11} = \frac{16+k}{7(k+6)}, \quad r_{12} = r_{21} = \frac{-3+2k}{7(k+6)}, \quad r_{22} = \frac{21+2k+k^2}{14(k+6)}$$

orelse

$$R = \frac{1}{14(k+6)} \begin{bmatrix} 32+2k & -6+4k \\ -6+4k & 21+2k+k^2 \end{bmatrix}.$$

Now R is positive definite if

(i) $\dfrac{32+2k}{14(k+6)} > 0,$

(ii) $\dfrac{(32+2k)(21+2k+k^2) - (4k-6)^2}{14(k+6)} > 0.$

Consequently, it is true if $k > -6$ or $k < -16$. So for any k between $(-16, -6)$ the matrix R which is positive definite and therefore, the zero solution of the system is asymptotically stable.

Lecture 39

Let $g : S_\rho \to \mathbb{R}^n$ be a smooth function . Let us consider the following system of equation (in a vector form)

$$x' = g(x), \tag{5.41}$$

where $g(0) = 0$. Let us denote $\frac{\partial g_i}{\partial x_j}$ by a_{ij}. Then, equation (5.63) may be written as

$$x' = Ax + f(x), \tag{5.42}$$

where f contains terms of order two or more in $(x)$and $A = [a_{ij}]$. Now we study the stability of the zero solution of the system (5.64). The system (5.60) namely,

$$x' = Ax, \quad x \in \mathbb{R}^n,$$

is called the homogeneous part of the system (5.63) (which sometimes is also called the linearized part of the system (5.64).We know that the zero solution of the system (5.60) is asymptotically stable when A is a stable matrix. We now make use of the Lyapunov function given by (5.61) to study the stability behavior of certain nonlinear systems which are related to the linearized system (5.60). Let the Lyapunov function be

$$V(x) = x^T Rx,$$

where R is the unique solution of the equation (5.62). We have already discussed a method for the determination of a matrix R.

For the asymptotic stability of the zero solution system (5.64), the function f naturally has a role to play. We expect that if f is small then, the zero solution of the system (5.64) is asymptotically stable. With this short introduction let us employ the same Lyapunov function (5.61) to determine the stability of the origin of (5.64). Now the time derivative of V along a solution of (5.64) is

$$\begin{aligned} \dot{V}(x) &= x'^T Rx + x^T Rx' = (x^T A^T + f^T)Rx + x^T R(Ax + f) \\ &= x^T(A^T R + RA)x + f^T Rx + x^T Rf = -x^T Qx + 2x^T Rf, \end{aligned} \tag{5.43}$$

because of (5.62) and (5.64). The second term on the right side of (5.65) contains terms of degree three or higher in x. The first one contains a term of degree two in x. The first term is negative whereas the sign of the second term depends on f. Whatever the second term is, at least a small region containing the origin can definitely be found such that the first term predominates the second term and thus, in this small region the sign of $\dot{V}$ remains negative. This implies that the zero solution of nonlinear equation (5.64) is asymptotically stable. Obviously the negative definiteness of $\dot{V}$ is only in a small region around origin.

Definition 5.6.2. The region of stability for a differential equation (5.64) is the set of all initial points x_0 such that

$$\lim_{t\to\infty} x(t, t_0, x_0) = 0.$$

If the stability region is the whole of $\mathbb{R}^n$ then the we say the zero solution is asymptotic stability in the large or globally asymptotically stabile. We give below a method of determining the stability region for the system (5.64).

Consider below a surface $\{x : V(x) = k\}$ (where k is a constant to be determined) lying entirely inside the region $\{x : \dot{V}(x) \le 0\}$. Now find k such that $V(x) = k$ is tangential to the surface $\dot{V}(x) = 0$. Then, stability region for the system (5.64) is the set $\{x : V(x) \le k\}$.

Example 5.8.3 given below illustrates a procedure for finding the region of stability.

Example 5.6.3. Consider a nonlinear system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2^2 \end{bmatrix}.$$

Let $V(x) = x^T R x$, where R is the solution of the equation

$$\begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} R + R \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} = Q$$

Choose $Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, so that $R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Thus

$$\begin{aligned} &V(x_1, x_2) = 2(x_1^2 + x_2^2) \\ &\dot{V}(x_1, x_2) = 4(x_1 x_1' + x_2 x_2') = 4[-x_1^2 - x_2^2(1 - x_2)] \end{aligned}$$

with respect to the given system. To find the region of asymptotic stability consider the surface

$$(x_1, x_2) : \dot{V}(x_1, x_2) = 4[-x_1^2 - x_2^2(1 - x_2)] = 0.$$

When

$$x_2 < 1, \ \dot{V}(x_1, x_2) < 0 \text{ for all } x_1.$$

Hence,

$$(x_1, x_2) : V(x) = 2(x_1^2 + x_2^2) \le 1$$

is the region which lies in the region

$$\dot{V}(x_1, x_2) < 0.$$

The size of the stability region thus obtained depends on the choice of a matrix Q.

EXERCISES

1. Prove that the stability properties of solutions the equation (5.62) remains unaffected by a transformation $x = Py$, where P is a non-singular matrix.

2. If R is a solution of the equation (5.62) then, prove that so is R^T and hence, $R^T = R$.

3. The matrices A and Q are given below. Find a matrix R satisfying the equation (5.62) for each of the following cases.

   (i) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$;

   (ii) $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$; and

   (iii) $A = \begin{bmatrix} -3 & -5 \\ -2 & -4 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

4. For the system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & p & 0 \\ 0 & -2 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Choose

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Determine the value/vaues of p for which the matrix R is positive definite.

5. For the system
$$x_1' = -x_1 + 2x_2,\ x_2' = -2x_1 + x_2 + x_2^2$$
find the region of the asymptotic stability.

6. Prove that the zero solution of the system
$$(x_1, x_2)' = (-x_1 + 3x_2, -3x_1 - x_2 - x_2^3)$$
is asymptotically stable.