Lecture

3.4 Non-homogeneous linear Systems

Assume in this section that $A(t)$ is an $n \times n$ matrix that is continuous on I. The system

$$x' = A(t)x + b(t), \quad t \in I, \tag{3.24}$$

is called a non-homogeneous linear system of order n. Here b is a continuous function defined on I and taking values in $\mathbb{R}^n$. An inspection shows that if $b(t) \equiv 0$, then (3.24) reduces to (3.14). The term $b(t)$ in (3.24) often goes by the name "forcing term" or "perturbation" for the system (3.14). The system (3.24) is a perturbed state of (3.14). The nature of the solution of (3.24) is quite closely connected with the solution of (3.14) and to some extent it is brought out in this section. Before proceeding further, it may be remarked here that the continuity of A and b ensures the existence and uniqueness of a solution for IVP on I for the system (3.24). The proof is postponed for the present and is dealt with in Module 4.

To express the solution (3.24) in term of (3.14) it becomes necessary to resort to the method of variation of parameters. Let $\Phi(t)$ be a fundamental matrix for the system (3.14) on I. Let $\Psi(t)$ be a solution of (3.24) such that for some $t_0 \in I, \psi(t_0) = 0$. Now let it be assumed that $\psi(t)$ is given by

$$\psi(t) = \Phi(t)u(t), \quad t \in I, \tag{3.25}$$

where $u(t)$ is an unknown vector function mapping I into $\mathbb{R}^n$ such that $u(t)$ is differentiable and $u(t_0) = 0$. The solution ψ is determined by finding $u(t)$ in terms of known quantities $\Phi(t)$ and $b(t)$. Substituting (3.25) in (3.24) notice that for $t \in I$,

$$\psi'(t) = \Phi'(t)u(t) + \Phi(t)u'(t) = A(t)\Phi(t)u(t) + \Phi(t)u'(t).$$

It is also seen that

$$\psi'(t) = A(t)\psi(t) + b(t) = A(t)\Phi(t)u(t) + b(t).$$

Equating the two expressions for $\psi'(t)$ it is concluded that $\Phi(t)u'(t) = b(t)$. Note that $\Phi(t)$, being a fundamental matrix, is non-singular on I and so

$$u'(t) = \Phi^{-1}(t).b(t)$$

or

$$u(t) = 0 + \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds, \quad t, t_0 \in I \tag{3.26}$$

Substituting the value of $u(t)$ in (3.25), we get,

$$\psi(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds, \quad t \in I \tag{3.27}$$

It can easily be verified that (3.27) is indeed a solution of (3.24). This discussion so far is now summed up in Theorem 3.4.1.

Theorem 3.4.1. *Let $\Phi(t)$ be a fundamental matrix for the system* (3.14) *for $t \in I$. Then ψ, defined by* (3.27)*, is a solution of the IVP*

$$x' = A(t)x + b(t), \ x(t_0) = 0. \tag{3.28}$$

Now let us assume that $x_h(t)$ is the solution of the IVP

$$x' = A(t)x,\ x(t_0) = c,\ \ t, t_0 \in I. \tag{3.29}$$

Then, a consequence of Theorem 3.4.1 is that

$$\psi(t) = x_h(t) + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds, \quad t \in I \tag{3.30}$$

is a solution of

$$x' = A(t)x + b(t);\ x(t_0) = c.$$

Thus with a prior knowledge of the solution of (3.29), the solution of (3.28) is computable from (3.30).

EXERCISES

1. Prove that the equation (3.27) can also be written as
   (i) $\Psi(t) = \Phi(t) \int_{t_0}^{t} \Psi^T(s)b(s)ds, \quad t \in I$ provided $\Psi^T(t)\Phi(t) = E$;
   (ii) $\Psi(t) = (\Psi^{-1})^T \int_{t_0}^{t} \Psi^T(s)b(s)ds, \quad t \in I$, where Ψ is a fundamental matrix for the adjoint system $x' = -A^T(t)x$. Assume that $A(t)$ is a real matrix.

2. Consider the system $x' = Ax + b(t)$, where
$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } b(t) = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}.$$
   Show that
$$\Phi(t) = \begin{bmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$
   is a fundamental matrix of $x' = Ax$. Compute the solution $y(t)$ of the non-homogeneous system for which $y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Consider the system $x' = Ax$ given that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.Show that a fundamental matrix is $\Phi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$. Let $b(t) = \begin{bmatrix} \sin at \\ \cos bt \end{bmatrix}$. Find the solution $\Psi(t)$ of the non-homogeneous equation $x' = Ax + b(t)$ for which $\Psi(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Lecture 17

3.5 Linear Systems with Constant Coefficients

In previous sections, the existence and uniqueness of solutions of linear systems of the type

$$x' = A(t)x,\ x(t_0) = x_0, \quad t, t_0 \in I, \tag{3.31}$$

has been proved. However, when trying to find the solution of such systems in an explicit form several difficulties are encountered. In fact, there are very few situation when the solution can be found explicitly. The aim of this article is to develop a method to find the solution of (3.31) with the assumption that $A(t)$ is a constant matrix. The method involves first finding the characteristic values of the matrix A. If the characteristic values of the matrix A are known then, in general, a solution can be obtained in an explicit form. Note that when the matrix $A(t)$ is variable, it is usually difficult to find solutions.

Before proceeding further, recall the definition of the exponential of a given-matrix A. It is defined as follows:

$$\exp A = E + \sum_{p=1}^{\infty} \frac{A^p}{p!}$$

Also, if A and B are two matrices which commute then,

$$\exp(A + B) = \exp A . \exp B$$

For the present assume the proofs of the convergence of the sum through which $\exp A$ is defined and the result stated above. So by definition

$$\exp(tA) = E + \sum_{p=1}^{\infty} \frac{t^p A^p}{p!},\ t \in I$$

Here it is noted that the infinite series for $\exp(tA)$ converges uniformly on every compact interval of I.

Now consider a linear homogeneous system with a constant matrix, namely,

$$x' = Ax, \quad t \in I, \tag{3.32}$$

where I is an interval in $\mathbb{R}$. From Module 1 recall that the solution of (3.32), when A and x are scalars, is $x(t) = ce^{tA}$ for an arbitrary constant c. A similar situation prevails when we deal with (3.32). This leads to Theorem 3.5.1.

Theorem 3.5.1. *The general solution of the system* (3.32) *is* $x(t) = e^{tA}c$*, where* c *is an arbitrary constant column matrix. Further, the solution of* (3.32) *with the initial condition* $x(t_0) = x_0, t_0 \in I$*, is*

$$x(t) = e^{(t-t_0)A}x_0, \quad t \in I \tag{3.33}$$

Proof. Let $x(t)$ be any solution of (3.32). Define a vector $u(t)$ by, $u(t) = e^{-tA}x(t), \quad t \in I$. Then, it follows that

$$u'(t) = e^{-tA}(-Ax(t) + x'(t)), \quad t \in I.$$

Since x is a solution of (3.32) it is easy to observe that $u'(t) \equiv 0$, which means that $u(t) = c, t \in I$, where c is some constant vector. Substituting the value c for $u(t)$, it is seen that $x(t) = e^{tA}c$. Clearly $c = e^{-t_0A}x_0$, and so we have $x(t) = e^{tA}e^{-t_0A}x_0, t \in I$. Since A commutes with itself, it is seen that $e^{tA}e^{-t_0A} = e^{(t-t_0)A}$, and thus, (3.33) follows. This completes the proof. □

In particular, let us choose $t_0 = 0$ and n linearly independent vectors $e_j, j = 1, 2, \cdots, n$, the vector e_j being the vector with 1 at the jth component and zero elsewhere. In this case, we get n linearly independent solutions corresponding to the set of n vectors $(e_1, e_2, \cdots, e_n)$. Thus a fundamental matrix for (3.32) is

$$\Phi(t) = e^{tA}E = e^tA, \quad t \in I, \tag{3.34}$$

since the matrix with columns represented by $e_1, e_2, \cdots, e_n$ is the identity matrix E. Thus e^{tA} solves the matrix differential equation

$$X' = AX, \quad x(0) = E; \quad t \in I. \tag{3.35}$$

Example 3.5.2. Find a fundamental matrix for the system $x' = Ax$, where

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

where α_1,α_2 and α_3 are scalars.

The fundamental matrix is e^{tA}. It is very easy to verify that

$$A^k = \begin{bmatrix} \alpha_1^k & 0 & 0 \\ 0 & \alpha_2^k & 0 \\ 0 & 0 & \alpha_3^k \end{bmatrix}$$

Hence,

$$e^{tA} = \begin{bmatrix} \exp \alpha_1 t & 0 & 0 \\ 0 & \exp \alpha_2 t & 0 \\ 0 & 0 & \exp \alpha_3 t \end{bmatrix}.$$

Example 3.5.3. Consider a similar example to determine a fundamental matrix for $x' = Ax$, where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$. Notice that

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}.$$

By the remark which followed Theorem 3.5.1, it is known that the fundamental matrix in this case is given by

$$\exp(tA) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} t. \exp \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} t,$$

since $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$ commute. But

$$\exp \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} t = \exp \begin{bmatrix} 3t & 0 \\ 0 & 3t \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

It is left as an exercise to the readers to verify that

$$\exp \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} t = \frac{1}{2} \begin{bmatrix} e^{2t} + e^{-2t} & e^{-2t} - e^{2t} \\ e^{-2t} - e^{2t} & e^{2t} + e^{-2t} \end{bmatrix}.$$

Thus $e^{tA} = \frac{1}{2} \begin{bmatrix} e^{5t} + e^{t} & e^{t} - e^{5t} \\ e^{t} - e^{5t} & e^{5t} + e^{t} \end{bmatrix}$.

From Theorem 3.5.1 we know that the general solution of the system (3.32) is $e^{tA}c$ but we have still not computed e^{tA}. Once e^{tA} determined, the solution of (3.32) is completely determined.

In order to be able to do this the procedure given below is followed. Choose a solution of (3.32) in the form

$$x(t) = e^{\lambda t} c, \tag{3.36}$$

where c is a constant vector and λ is a scalar. x is determined if λ and c are known. Substituting (3.36) in (3.32), we get

$$(\lambda E - A)c = 0. \tag{3.37}$$

which is a system of algebraic homogeneous linear equations for the unknown c. The system (3.37) has a non-trivial solution c if and only if λ satisfies $\det(\lambda E - A) = 0$. Let

$$P(\lambda) = \det(\lambda E - A).$$

Actually $P(\lambda)$ is a polynomial of degree n normally called the "characteristic polynomial" of the matrix A and the equation

$$P(\lambda) = 0 \tag{3.38}$$

is called the "characteristic equation" for A. Since (3.38) is an algebraic equation, it admits n roots which may be distinct, repeated or complex. The roots of (3.38) are called the "eigenvalues" or the "characteristic values" of A. Let λ_1 be an eigenvalue of A and corresponding to this eigen value, let c_1 be the non-trivial solution of (3.37). The vector c_1 is called an "eigenvector" of A corresponding to the eigenvalue λ_1. Note that any nonzero constant multiple of c_1 is also an eigenvector corresponding to λ_1. Thus, if c_1 is an eigenvector corresponding to an eigenvalue λ_1 of the matrix A then,

$$x_1(t) = e^{\lambda_1 t} c_1$$

is a solution of the system (3.32). Let the eigenvalues of A be $\lambda_1, \lambda_2, \cdots, \lambda_n$(not necessarily distinct) and let $c_1, c_2, \cdots, c_n$ be linearly independent eigenvectors corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively. Then, it is clear that

$$x_k(t) = e^{\lambda_k t} c_k (k = 1, 2, \cdots, n),$$

are n linearly independent solutions of the system (3.32). Here we stress that the eigenvectors corresponding to the eigenvalues are linearly independent. Thus, $\{x_k\}, k = 1, 2, \cdots, n$ is a set of n linearly independent solutions of (3.32). So by the principle of superposition the general solution of the linear system is

$$x(t) = \sum_{k=1}^{n} e^{\lambda_k t} c_k. \tag{3.39}$$

Now let Φ be a matrix whose columns are the vectors

$$e^{\lambda_1 t}c_1, e^{\lambda_2 t}c_2, \cdots, e^{\lambda_n t}c_n$$

So by construction Φ has n linearly independent columns which are solutions of (3.32) and hence, Φ is a fundamental matrix. Since e^{tA} is also a fundamental matrix, from Theorem 3.4, we therefore have

$$e^{tA} = \Phi(t)D,$$

where D is some non-singular constant matrix. A word of caution is warranted namely that the above discussion is based on the assumption that the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are linearly independent.

Example 3.5.4. Let

$$x' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} x.$$

The characteristic equation is given by

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

whose roots are

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3.$$

Also the corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix},$$

respectively. Thus, the general solution of the system is

$$x(t) = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} e^{2t} + \alpha_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t}$$

where α_1, α_2 and α_3 are arbitrary constants. Also a fundamental matrix is

$$\begin{bmatrix} \alpha_1 e^t & 2\alpha_2 e^{2t} & \alpha_3 e^{3t} \\ \alpha_1 e^t & 4\alpha_2 e^{2t} & 3\alpha_3 e^{3t} \\ \alpha_1 e^t & 8\alpha_2 e^{2t} & 9\alpha_3 e^{3t} \end{bmatrix}.$$

Lecture 18

When the eigenvalues of A are not distinct, the problem of finding a fundamental matrix is not that easy. The next step is to find the nature of the fundamental matrix in the case of repeated eigenvalues of A. Let $\lambda_1, \lambda_2, \cdots, \lambda_n (m < n)$ be the distinct eigenvalues of A with multiplicities $n_1, n_2, \cdots, n_m$, respectively, where $n_1 + n_2 + \cdots + n_m = n$. Consider the system of equations, for an eigenvalue λ_i with multiplicity n_i,

$$(\lambda_i E - A)^{n_i} x = 0, \quad i = 1, 2, \cdots, m. \tag{3.40}$$

Let X_i be the subspace of $\mathbb{R}^n$ generated by the solutions of the system (3.40) for each $\lambda_i,\ i = 1, 2, \cdots, m$. From linear algebra it is known that for any $x \in \mathbb{R}^n$, there exist unique vectors $y_1, y_2, \cdots, y_m$, where $y_i \in X_i, (i = 1, 2, \cdots, m)$, such that

$$x = y_1 + y_2 + \cdots + y_m. \tag{3.41}$$

It is common in linear algebra to speak of $\mathbb{R}^n$ as a "direct sum" of the subspaces $X_1, X_2, \cdots, X_m$.

Consider the problem of determining e^{tA} discussed earlier. Let x be a solution of (3.32) with $x(0) = \alpha$. By the result which was quoted, unique vectors $\alpha_1, \alpha_2, \cdots, \alpha_m$ are obtained, such that

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m.$$

It is also known from Theorem 3.5.1 that the solution $x(t)$ of (3.32) with $x(0) = \alpha$ is

$$x(t) = e^{tA}\alpha = \sum_{i=1}^{m} e^{tA}\alpha_i$$

But,

$$e^{tA}\alpha_i = \exp(\lambda_i t)\exp[t(A - \lambda_i E)]\alpha_i$$

By the definition of the exponential function, we get

$$e^{tA}\alpha_i = \exp(\lambda_i t)[E + t(A - \lambda_i E) + \cdots + \frac{t^{n_i - 1}}{(n_i - 1)!}(A - \lambda_i E)^{n_i - 1} + \cdots]\alpha_i.$$

It is to be noted here that the terms of form

$$(A - \lambda_i E)^k \alpha_i = 0 \ \text{ if } k \geq n_i,$$

because recall that the subspace X_i is generated by the vectors, which are solutions of $(A - \lambda_i E)^{n_i} x = 0$, and that $\alpha_i \in X_i, i = 1, 2, \cdots, m$. Thus,

$$x(t) = e^{tA}\sum_{i=1}^{m}\alpha_i = \sum_{i=1}^{m}\exp(\lambda_i t)\Big[\sum_{j=0}^{n_i - 1}\frac{t^j}{j!}(A - \lambda_j E)^j\Big]\alpha_j, \quad t \in I. \tag{3.42}$$

Indeed one might wonder whether (3.42) is the desired solution. To start with we were aiming at $\exp(tA)$ but all we have in (3.42) is $\exp(tA).\alpha$, where α is an arbitrary vector. But a simple consequence of (3.42) is the deduction of $\exp(tA)$ which is done as follows. Note that

$$\begin{aligned}\exp(tA) &= \exp(tA)E \\ &= [\exp(tA)e_1, \exp(tA)e_2, \cdots, \exp(tA)e_n].\end{aligned}$$

$\exp(tA)e_i$ can be obtained from (3.42), $i = 1, 2, \cdots, n$ and hence $\exp(tA)$ is determined. It is important to note that (3.42) is useful provided all the eigenvalues are known along with their multiplicities.

Example 3.5.5. Let $x' = Ax$ where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic equation is given by

$$\lambda^3 = 0.$$

whose roots are

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

Since the rank of the co-efficient matrix A is 2, there is only one eigenvector namely

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The other two generalized eigenvectors are determined by the solution of

$$A^2 x = 0 \quad \text{and} \quad A^3 x = 0.$$

The other two generalized eigenvectors are

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Since

$$A^3 = 0,$$

$$e^{At} = I + At + \frac{A^2 t^2}{2}$$

or

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & t & 0 \end{bmatrix}.$$

We leave it as exercice to find the e^{At} given

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Lecture 20
Phase Portraits in $\mathbb{R}^2$

In this part, we undertake an elementary study of the Phase Portraits in $\mathbb{R}^2$ for a system of two linear ordinary differential equations, viz,

$$\dot{x} = Ax \tag{3.49}$$

Here A is a 2×2 real matrix (i.e. an element of $M_2(\mathbb{R})$) and $x \in \mathbb{R}^2$ is a column vector. The tuple $(x_1(t), x_2(t))$ for $t \in \mathbb{R}^2$ represents a curve C in $\mathbb{R}^2$ in a parametric form; the curve C is called the phase portrait of . It is easier to draw the curve when A is in its canonical form. However, in its original form (i.e. when A is not in the canonical form) these portraits have similar (but distorted) diagrams. The following example clarifies the same ideas.

Example : Let $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$. The canonical form B is $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, i.e., $A = P^{-1}BP$. The equation (3.49) with $y = Px$, is

$$y' = By \tag{3.50}$$

Equation (3.50) is sometimes is referred to (3.49), when A is in its canonical form. The phase Portrait for (3.50) is (fig2)

Figure 3.2: