# Boundary Integral Methods and Applications to Fluid Mechanics

G P Raja Sekhar
Professor, Department of Mathematics
Indian Institute of Technology Kharagpur
Kharagpur 721302, India

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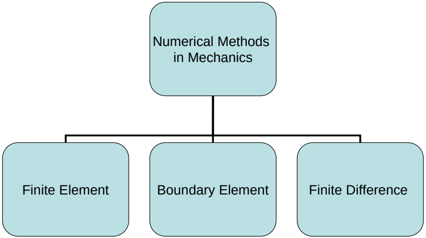
1. Difficulties

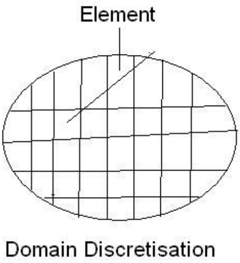
2. Numerical Finite part integration

3. Meth. of Kolm and Rokhlin
   - Numerical tests

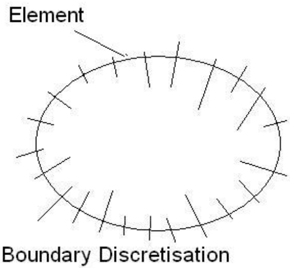
# Computational Techniques



# Discretisation of a Domain




based on Variational methods




based on Boundary integral methods

# Advantages of BEM

- Reduction of problem dimension by 1.
  - Less data preparation time.
  - Easier to change the applied mesh.
  - Useful for problems that require re-meshing.
- High Accuracy.
  - Stresses are accurate as there are no approximations imposed on the solution in interior domain points.
  - Suitable for modeling problems of rapidly changing stresses.
- Less computer time and storage.
  - For the same level of accuracy as other methods BEM uses less number of nodes and elements.
- Filter out unwanted information.
  - Internal points of the domain are optional.
  - Focus on particular internal region.
  - Further reduces computer time.

BEM is an attractive option.

# The Heart of Boundary Integral Formulation

**Fundamental solution of Laplacian:** Those functions $\phi$ that satisfy the Laplace equation $\nabla^2\phi(\mathbf{x}) = \delta(\mathbf{x})$ in a domain $D$ subject to homogeneous boundary conditions are known as Green's functions for the Laplace equation.

**1 dim:**:

$$\frac{d^2\phi^*}{dx^2} = \delta(x - x_0)$$

$$\Rightarrow \phi^*(x, x_0) = \begin{cases} -\frac{1}{2}(x - x_0), & x > x_0 \\ \frac{1}{2}(x - x_0), & x < x_0 \end{cases}$$

**2D Case:**

$$G(\mathbf{x}-\mathbf{x_0}) = \frac{1}{2\pi}\log r = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x_0}|$$

**3D Case:**

$$G(\mathbf{x}-\mathbf{x_0}) = \frac{1}{4\pi r}$$

# Two-dimensional Potential Problem

$$\nabla^2\phi = 0 \quad \text{in} \quad \Omega$$

Can guess easily that "suitable boundary data" is required!
Given a data, solution can be attempted!

# Divergence Theorem (flux conservation)

$$\int_S F_i n_i ds = \int_\Omega \frac{\partial F_i}{\partial x_i} d\Omega \quad (1)$$

$$\int_S \frac{\partial \phi}{\partial x_i} n_i ds = \int_S \frac{\partial \phi}{\partial n} d\mathbf{s} = \int_\Omega \nabla^2 \phi d\Omega \quad (2)$$

**Remark:** $\Omega \subset R^n$ : bounded, simply connected; $\Gamma$: sufficiently smooth (say, here $C^2$). Looking for classical solutions.

# Green's identities

**Green's first identity**

$$\int_S \phi \frac{\partial \psi}{\partial x_i} n_i ds = \int_\Omega \frac{\partial}{\partial x_i}\left(\phi \frac{\partial \psi}{\partial x_i}\right) d\Omega \quad (3)$$

$$= \int_\Omega \frac{\partial \phi}{\partial x_i}\frac{\partial \psi}{\partial x_i} d\Omega + \int_\Omega \phi \nabla^2 \psi d\Omega \quad (4)$$

**Green's second identity**

$$\int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) ds = \int_\Omega \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) d\Omega \quad (5)$$

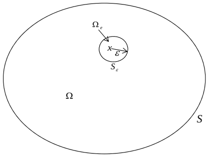
Consider the Green's identity

$$\int_S \left( \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds = \int_\Omega \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) d\Omega \qquad (6)$$

Choice: $\phi$, the harmonic function that needs to be obtained
$\phi^*$ the fundamental solution
$x = (x_1, x_2); y = (y_1, y_2)$

$$\begin{aligned} \int_S \left( \phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) ds &= \int_\Omega \left( \phi(y) \nabla^2 \phi^*(x, y) \right. \\ &\quad \left. - \phi^*(x, y) \nabla^2 \phi(y) \right) d\Omega_y \end{aligned} \qquad (7)$$

## Exclusion of internal source point



# Two - Dimensional Potential Problem contd.

$$\int_S \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}\right) ds + \int_{S_\epsilon} \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}\right) ds_\epsilon =$$

$$\int_{\Omega-\Omega_\epsilon} \left(\phi(y)\nabla^2\phi^*(x,y) - \phi^*(x,y)\nabla^2\phi(y)\right) d\Omega_y$$

- In $(\Omega - \Omega_\epsilon)$, $\nabla^2\phi = 0, \nabla^2\phi^* = 0 \Rightarrow RHS = 0.$

$$\int_S \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}\right) ds + \int_{S_\epsilon} \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}\right) ds_\epsilon = 0 \qquad (BI)$$

- over $S_\epsilon$ where $ds_\epsilon(y) = \epsilon d\theta$

$$\frac{\partial \phi^*(x,y)}{\partial n_y} = \frac{\partial \phi^*(x,y)}{\partial r} \frac{\partial r}{\partial n_y} = \frac{1}{2\pi\epsilon}$$

## Two - Dimensional Potential Problem contd.

Consider the second term on the LHS of (BI)

$$\int_{S_\epsilon}\left(\phi\frac{\partial\phi^*}{\partial n}-\phi^*\frac{\partial\phi}{\partial n}\right)ds_\epsilon(y) = \frac{1}{2\pi}\int_0^{\theta_x}\left(\phi(y)\frac{1}{\epsilon}+\ln\epsilon\frac{\partial\phi(x,y)}{\partial n_y}\right)\epsilon d\theta_x$$

$$= \frac{1}{2\pi}\int_0^{\theta_x}\left(\phi(y)+\epsilon\ln\epsilon\frac{\partial\phi(x,y)}{\partial n_y}\right)d\theta_x$$

$$\lim_{\epsilon\to 0}\int_{S_\epsilon}\left(\phi\frac{\partial\phi^*}{\partial n}-\phi^*\frac{\partial\phi}{\partial n}\right)ds_\epsilon(y) = \frac{\theta_x}{2\pi}\phi(x) \qquad (8)$$

The general integral representation is

$$\frac{\theta_x}{2\pi}\phi(x) = \int_S \phi^*(x,y)\frac{\partial \phi(y)}{\partial n_y} ds_y - \int_S \phi(y)\frac{\partial \phi^*(x,y)}{\partial n_y} ds_y$$

where

$$\theta_x = \begin{cases} 2\pi, & x \in \Omega \\ \pi, & x \in \partial\Omega \\ 0, & x \notin \Omega \end{cases}$$

analogy: general solution using separation of variables

H. Power, L. C. Wrobel, *Boundary Integral Methods in Fluid Mechanics*, WIT Press, Computational Mechanics Publications, Southampton, 1995.

# Layer Potentials

Consider the Integral Representation for $x \in \Omega$

$$\phi(x) = \int_S \left( \phi^*(x,y) \frac{\partial \phi(y)}{\partial n_y} - \phi(y) \frac{\partial \phi^*(x,y)}{\partial n_y} \right) ds_y$$

Here $\sigma = \frac{\partial \phi}{\partial n}, \tau = \phi$ are the Cauchy data on $\Gamma$.

$$V\sigma(x) = \int_S \phi^*(x,y)\sigma(y) ds_y \qquad (Single - Layer)$$

$$W\tau(x) = \int_S \frac{\partial \phi^*(x,y)}{\partial n_y} \tau(y) ds_y \qquad (Double - Layer)$$

# Boundary potentials

for $x \in \Gamma$

$$
\begin{aligned}
V\sigma(x) :=& \quad V\sigma(x), \\
K\tau(x) := W\tau(x) + \frac{1}{2}\tau(x) \\
K^*\sigma(x) \quad :=& \quad grad_x V\sigma(x) \cdot \mathbf{n}_x - \frac{1}{2}\sigma(x) \\
D\tau(x) \quad :=& \quad -grad_x W\tau(z) \cdot \mathbf{n}_x;
\end{aligned}
$$

Consider the Integral Representation, when $\xi \in \partial\Omega$

$$\frac{1}{2}\phi(\xi) = \int_S \phi^*(\xi, y)\frac{\partial \phi(y)}{\partial n_y} ds_y - \int_S^{pv} \phi(y)\frac{\partial \phi^*(\xi, y)}{\partial n_y} ds_y$$

**Dirichlet Problem:** If the boundary value $\phi$ over $S$ is given, the missing data is $\sigma = \frac{\partial \phi}{\partial n}|_\Gamma$

$$\begin{aligned} V\sigma(x) &= \frac{1}{2}\phi(\xi) + K\tau(x) \\ \int_S^{pv} \phi^*(x, y)\sigma(y) ds_y &= f(x)(Fredholm \quad I \quad kind) \end{aligned}$$

# Neumann Problem:

If the boundary value $\frac{\partial \phi}{\partial n}$ over $S$ is given, the missing data is $\phi\mid_\Gamma$

$$\begin{aligned} \frac{1}{2}\sigma(x) - K^*(\sigma(x) &= D\tau(x) \\ \sigma(x) - 2\int_S^{pv} \frac{\partial \phi^*(\xi, y)}{\partial n_y}\sigma(y) ds_y &= g(x) \\ (Fredholm \quad II \quad kind) \end{aligned}$$

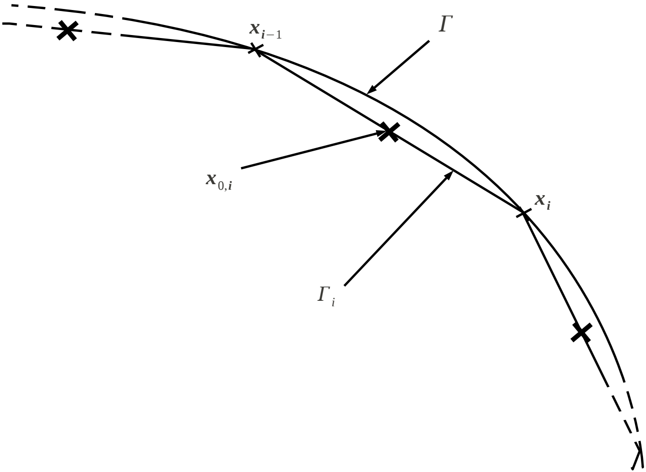
# The challenges:

Discretizing the surface integrals (3D) or line integrals (2D)
Handling "Singular Integrals"

# BEM Implementations with constant Elements - Kamal C Das

- approximate the boundary $\Gamma$ by line segments $\Gamma_i$ with endpoints $\mathbf{x}_{i-1}$,$\mathbf{x}_i$, with a nodal point at the midpoint of $\Gamma_i$.
- values at the mid-points of each element $\Gamma_i$ are assumed equal to their values over the whole element.
- $i \in 1, ..., N$ where $N$ is the no. of nodal points.

# Discretization



W T Ang, A beginner's course in BEMs, Universals-Publishers, 2007

Prem K Kythe, An introduction to BEMs, CRC Press, 1995.

T W Wu, Boundary Element Acoustics: Fundamentals and Computer Codes, WIT, 2001

$$\frac{1}{2}\phi(x) \;=\; \int_S \phi^*(x,y)\frac{\partial \phi(y)}{\partial n_y}ds_y - \int_S \phi(y)\frac{\partial \phi^*(x,y)}{\partial n_y}ds_y, \quad x \in \partial\Omega$$

$$\frac{1}{2}\phi^i = -\sum_{j=1}^{N}\int_{\Gamma_j} \phi^*(x_i,y)\frac{\partial \phi(y)}{\partial n_y}ds_y + \sum_{j=1}^{N}\int_{\Gamma_j} \phi(y)\frac{\partial \phi^*(x_i,y)}{\partial n_y}ds_y$$

$\phi = \phi^j; \;\; \frac{\partial \phi}{\partial n} = \partial \phi^j$ on the $j^{\text{th}}$ element

$$-\frac{1}{2}\phi^{i} + \sum_{j=1}^{N}\left(\int_{\Gamma_j}\frac{\partial\phi^*}{\partial n}dS\right)\phi^{j} = \sum_{j=1}^{N}\left(\int_{\Gamma_j}\phi^* dS\right)\partial\phi^{j}$$

$$\widehat{H}_{ij} = \int_{\Gamma_j}\frac{\partial\phi^*(x_i,y)}{\partial n_y}dS, \quad G^*_{ij} = \int_{\Gamma_j}\phi^*(x_i,y)dS$$

$$-\frac{1}{2}\phi^{*i} + \sum_{j=1}^{N} \widehat{H}_{ij}\phi^{*j} = \sum_{j=1}^{N} G_{ij}\partial\phi^{*j}$$

Let, $H_{ij} = \widehat{H}_{ij} - \frac{1}{2}\delta_{ij}$, we have

$$\sum_{j=1}^{N} H_{ij}\phi^{*j} = \sum_{j=1}^{N} G_{ij}\partial\phi^{*j}$$

$$[H]_{N\times N}{\phi^{*}}_{N\times 1} = [G]_{N\times N}\partial{\phi^{*}}_{N\times 1}$$

# Example:

$$\nabla^2\phi = 0, \quad 0 < x < 1,\ 0 < y < 1$$

$$\phi(0, y) = 0; \phi(1, y) = \cos \pi y$$

$$\frac{\partial \phi}{\partial n}(x, 0) = 0; \frac{\partial \phi}{\partial n}(x, 1) = 0$$

$$\phi(x, y) = \frac{\sinh \pi x \cos \pi x}{\sinh \pi}$$

# Laplacian

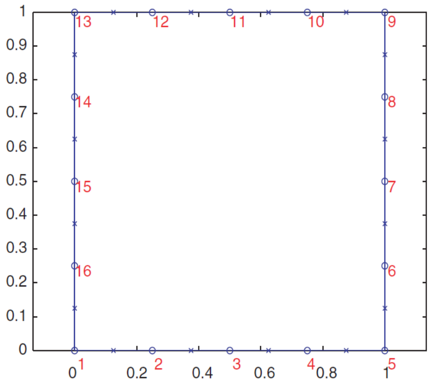



Figure: Boundary nodes (o) and midpoints of boundary elements (x).

# Laplacian

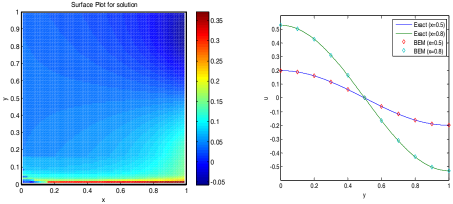



Figure: (a) Surface plot of solution $\phi$, (b) BEM solution Vs exact solution of along $x$=0.5 and $x$=0.8.

# Helmholtz equation-application to Acoustics

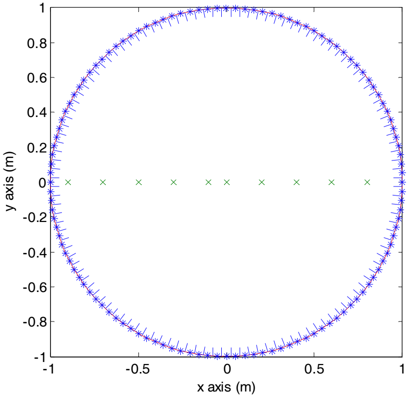



Figure: The geometry of the circular cylinder

# Helmholtz

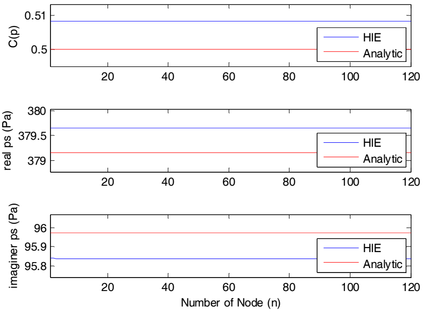



Figure: Distribution of solid angles $C(P)$ and surface pressure (real and imaginary part) along the boundary nodes

# Helmholtz

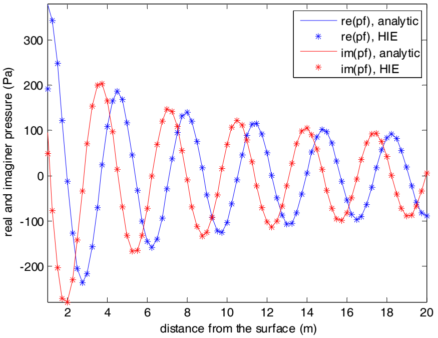



Figure: Distribution of field pressure $p_f$ away from the surface of the cylinder

# Stokes flow-Example

$$-\nabla p + \nabla^2 \mathbf{v} = \mathbf{0} \quad \text{in } \Omega, \tag{9}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \tag{10}$$

$$\begin{aligned} v_\theta(a,\theta) &= f(\theta) = 4a^3 \cos 2\theta - 3a^2(1-\sin\theta)\cos\theta \\ p(a,\theta) &= g(\theta) = -2a(1-12\sin\theta)\cos\theta \end{aligned}$$

A. Zeb, L. Elliott, D.B. Ingham, D. Lesnic, The boundary element method for the solution of Stokes equations in two-dimensional domains, Engineering Analysis with Boundary Elements, 22 (1998), 317326

# Stokes flow

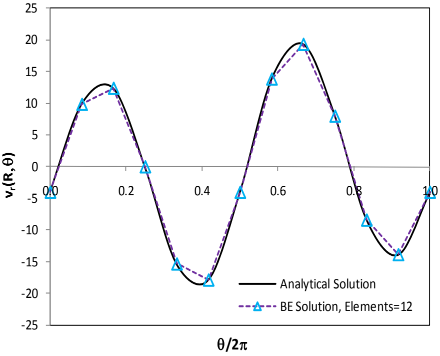



Figure: The normal components of velocity along the boundary with 12 constant elements

# Stokes flow

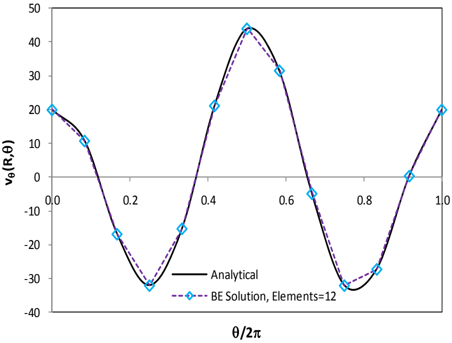



Figure: tangential components of velocity along the boundary with 12 constant elements

# Difficulties as per M.Sc. project by Aftab Yusuf Patel

- off diagonal coefficients are regular. Standard integration methods used.
- diagonal coefficients involve singular, hypersingular integrals, for which singular terms cancel(see [Kohr, Raja Sekhar, 2008]). Standard methods do not work due to complexity of kernels(esp. corresponding to Brinkman eq.).
- standard method of taking distorted paths of integration around singularities is too slow to be of practical utility. Program written in MATHEMATICA proved to be too inefficient to run on available hardware.

# Principal value and finite part integrals

- Attacking the subproblems

$$\int_a^b \frac{\phi(x)}{x-y}\,dx \tag{11}$$

where $y \in (a,b)$ do not exist in the classical Riemann or Lebesgue sense.

- Cauchy Principal Value:

$$\lim_{\epsilon\to 0}\left(\int_a^{y-\epsilon} \frac{\phi(x)}{x-y}\,dx + \int_{y+\epsilon}^b \frac{\phi(x)}{x-y}\,dx\right) \tag{12}$$

# Strongly singular or hypersingular

$$\int_a^b \frac{\phi(x)}{(x-y)^2} dx \tag{13}$$

with $y \in (a, b)$ are divergent in the classical sense.

- Hadamard Finite Part interpretation:

$$\lim_{\epsilon \to 0} \left( \int_a^{y-\epsilon} \frac{\phi(x)}{(x-y)^2} dx + \int_{y+\epsilon}^b \frac{\phi(x)}{(x-y)^2} dx - \frac{2\phi(y)}{\epsilon} \right) \tag{14}$$

# Method Of Kolm and Rokhlin, 2001

In [Kolm, Rokhlin, 2001] a method was developed to numerically integrate functions of the form,

$$f(x) = A(x) + B(x)log|x| + \frac{C(x)}{x} + \frac{D(x)}{x^2} \tag{15}$$

• 0 belongs to the interior of the interval of integration
• the functions $A, B, C, D$ are not available separately
• Method was implemented in C for maximum efficiency and is the first such publicly available implementation to the best of our knowledge

# Numerical tests

The implementation of the method was tested on the test cases:

$$\int_{-1}^{1} \frac{(sin(x+\frac{\pi}{3})+cos(x))}{x}dx$$

has a singularity of the $1/x$ type.

$$\int_{-1}^{1} K_2(|x|)dx$$

has singularities of $log|x|$ and $1/x^2$ type. The method was found to be accurate to 10 decimal places. The degree $M$ was taken to be $M=[N/4]+1$, $[\cdot]$ being the greatest integer function.

# Test case 1

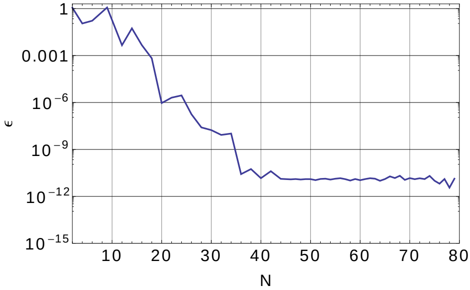



Figure: Error in computing integral $\int_{-1}^{1}((sin(x+\frac{\pi}{3})+cos(x))/x)dx$ from $-1$ to 1 with $M=[N/4]+1$, vs. number of quadrature points N.

# Test case 2

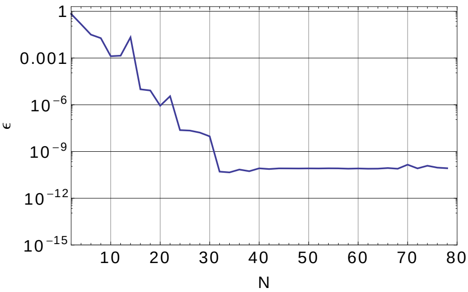



Figure: Error in computing integral $\int_{-1}^{1} K_2(|x|)dx$ from $-1$ to 1 with $M = [N/4] + 1$, vs. number of quadrature points N.

# Summary

try to avoid handling the entire domain via boundary integrals
Green's function is required
friendly to linear PDEs
Boundary and Domain integral methods for Non-linear PDEs possible
Rich theory of existence and uniqueness for the Integral Operators: Lipschitz domains, Sobolev spaces
Numerical methods: Boundary Elements
handling singular integrals is the bottle neck
when we succeed, the expenses are going to be very less compared to domain methods

- H. Power, L. C. Wrobel, *Boundary Integral Methods in Fluid Mechanics*, WIT Press, Computational Mechanics Publications, Southampton, 1995.
- M. Kohr, G. P. Raja Sekhar. Existence and uniqueness result for two-dimensional porous media flows with porous inclusions based on Brinkman equation, Engineering Analysis Boundary Elements pp 604-613, No. 7, Vol. 31, July 2007.
- M. Kohr, G. P. Raja Sekhar. Boundary integral equations for 2D Brinkman flow problem past a void, Proceedings of 53rd conference of ISTAM, September 2008.
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- P. Kolm, V. Rokhlin. Numerical Quadratures for Singular and Hypersingular Integrals, Computers and Mathematics with Applications, Vol. 41, pp. 327-352, 2001.
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- Mirela Kohr, Wolfgang L Wendland, G P Raja Sekhar, Boundary integral equations for two-dimensional low Reynolds number flow past a porous body, Math. Meth. Appl. Sci. 2009; 32:922962.
- Mirela Kohr, G P Raja Sekhar, Wolfgang L Wendland, Boundary integral equations for a three-dimensional Stokes-Brinkman cell model, Mathematical Methods and Models in Applied Sciences, Vol. 18, No. 12 (2008) 20552085

ThAnKs FoR yOuR aTtEnTiOn !