Finite differences - Wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \,,$$

central

i.c : $u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

b.c suitable (depends on the domain)

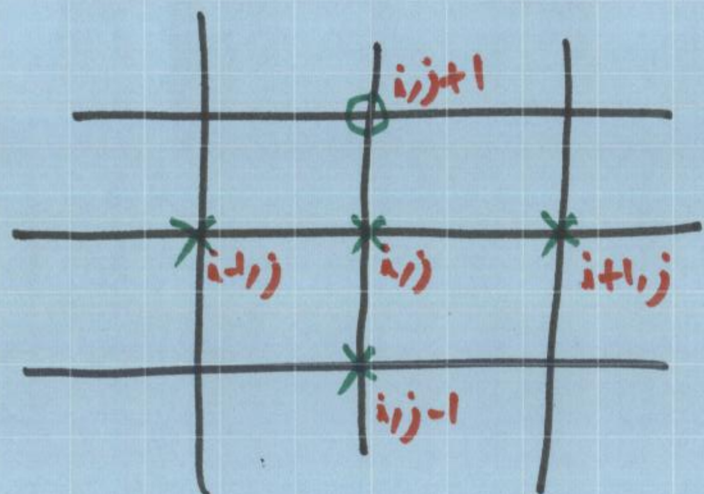
. fixed at both ends $u(a,t) = u_a$

$u(b,t) = u_b$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{1}{c^2}\,\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{K^2} \quad \text{—} \; (*)$$

$$\Rightarrow \quad u_{i,j+1} = 2(1-\lambda^2)\,u_{i,j} + \lambda^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

$$\lambda = \frac{kc}{h}$$



if the data at levels j and j-1 are available, one can compute the values at level (j+1).

"Explicit" three level

$$u_{i,j+1} = 2(1-\lambda^2)\, u_{i,j} + \lambda^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{— (A)}$$

- in order to start the computation, one need data at 2 past time levels.

$$\frac{\partial u}{\partial t}(x,0) = g(x) \Rightarrow \frac{u_{i,j+1} - u_{i,j-1}}{2K} = g_i + O(k^2)$$

at $(x,0)$, $j=0$ $\Rightarrow$

$$\frac{u_{i,1} - u_{i,-1}}{2K} = g_i$$

$$\Rightarrow u_{i,-1} = u_{i,1} - 2k g_i \quad \text{— (B).}$$

(A) at $j=0$

$$u_{i,1} = 2(1-\lambda^2)\, u_{i,0} + \lambda^2 (u_{i-1,0} + u_{i+1,0}) - u_{i,-1}$$

$$= 2(1-\lambda^2)\, u_{i,0} + \lambda^2 (u_{i-1,0} + u_{i+1,0}) - u_{i,1} + 2k g_i$$

$$u_{i,1} = \frac{\lambda^2}{2} u_{i-1,0} + (1-\lambda^2) u_{i,0} + \frac{\lambda^2}{2} u_{i+1,0} + k g_i$$

at first time step

$$u_{i,j+1} = \lambda^2 u_{i-1,j} + 2(1-\lambda^2) u_{i,j} + \lambda^2 u_{i+1,j} - u_{i,j-1}$$

at other time levels

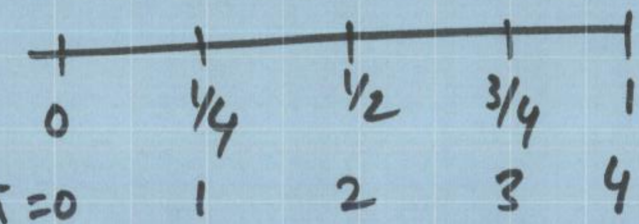
Explicit three level scheme.

example

$$u_{tt} = u_{xx} \quad , \; 0 \le x \le 1$$

i.c
$$u(x,0) = x \; ; \quad \underbrace{u(1,t) = 0 \; ; \; u(0,t) = 0}_{\text{b.c}}$$
$$\frac{\partial u}{\partial t}(x,0) = x = g(x)$$

$$h = 1/4 \; ; \; \lambda = 1/2 \; , \; k = 1/8$$

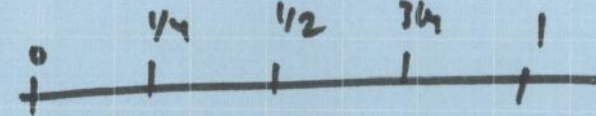


b.c $\quad u(0,t) = 0 \Rightarrow u_{0,j} = 0$

$\quad u(1,t) = 0 \Rightarrow u_{4,j} = 0$

i.c $\quad u(x,0) = x \Rightarrow u_{i,0} = x_i$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = x$$

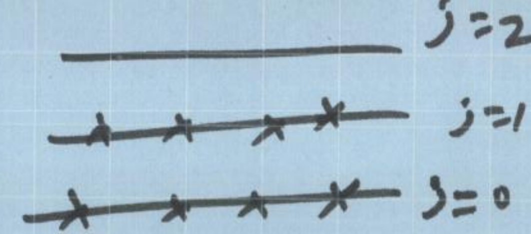


$$u_{i,1} = \frac{\lambda^2}{2} u_{i-1,0} + (1-\lambda^2) u_{i,0} + \frac{\lambda^2}{2} u_{i+1,0} + \frac{1}{8} x_i$$

$$u_{i,0} = x_i$$

$$u_{1,1} = \frac{1}{8} u_{0,0} + \frac{3}{4} u_{1,0} + \frac{1}{8} u_{2,0} + \frac{1}{8}\cdot\frac{1}{4}$$

$$= \frac{1}{8}\cdot 0 + \frac{3}{4}\cdot\frac{1}{4} + \frac{1}{8}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{4} = \frac{1}{16}\cdot\frac{9}{2}$$



$$u_{2,1} = \frac{1}{8} u_{1,0} + \frac{3}{4} u_{2,0} + \frac{1}{8} u_{3,0} + \frac{1}{8}\cdot\frac{1}{2}$$

$$= \frac{1}{8}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{3}{4} + \frac{1}{8}\cdot\frac{1}{2}$$

$$= \frac{1}{8}\cdot\frac{9}{2}$$

$$u_{3,1} = \frac{1}{8} \cdot u_{2,0} + \frac{3}{4} u_{3,0} + \frac{1}{8} u_{4,0} + \frac{1}{8} \cdot \frac{3}{4}$$

$$= \frac{1}{8} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{3}{4} + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot \frac{3}{4} = \frac{1}{8} \cdot \frac{27}{4}$$

<u>at higher levels</u>

$$u_{i,j+1} = \lambda^2 u_{i-1,j} + 2(1-\lambda^2) u_{i,j} + \lambda^2 u_{i+1,j} - u_{i,j-1}$$

<u>$j=1$</u>

$$u_{i,2} = \frac{1}{4} u_{i-1,1} + \frac{3}{2} u_{i,1} + \frac{1}{4} u_{i+1,1} - u_{i,0}$$

<u>$i=1$</u>

$$u_{1,2} = \frac{1}{4} u_{0,1} + \frac{3}{2} u_{1,1} + \frac{1}{4} u_{2,1} - u_{1,0}$$

$$= \frac{1}{4}(0) + \frac{3}{2} \cdot \frac{1}{16} \cdot \frac{9}{2} + \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{9}{12} - \frac{1}{4} = \frac{5}{16}$$

$\underline{i=2}$

$$u_{2,2} = \frac{1}{4} u_{1,1} + \frac{3}{2} u_{2,1} + \frac{1}{4} u_{3,1} - u_{2,0}$$

$$= \frac{1}{4} \frac{1}{16} \cdot \frac{9}{2} + \frac{3}{2} \cdot \frac{1}{8} \cdot \frac{9}{2} + \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{27}{4} - \frac{1}{2} = 5/8 .$$

$\underline{i=3}$

$$u_{3,2} =$$

Hyperbolic Eqn — Explicit Method – Error

$$u_{i,j+1} - 2(1-\lambda^2)u_{i,j} - \lambda^2(u_{i-1,j} + u_{i+1,j}) + u_{i,j-1} \simeq 0$$

$$u_{i,j} + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{k^4}{24}\frac{\partial^4 u}{\partial t^4}\Big|_{(x_i, t_j)} + \cdots$$

$$- 2(1-\lambda^2)u_{i,j}$$

$$- \lambda^2\left(u_{i,j} - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} - \cdots\right)$$

$$- \lambda^2\left(u_{i,j} + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} \cdots\right)$$

$$+ u_{i,j} - k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{k^4}{24}\frac{\partial^4 u}{\partial t^4} - \cdots \simeq 0$$

$$\Rightarrow \quad k^2 \frac{\partial^2 u}{\partial t^2} + \frac{k^4}{12} \frac{\partial^4 u}{\partial t^4} + \cdots$$

$$- \lambda^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12} \frac{\partial^4 u}{\partial x^4} \right) \cdots$$

$$= k^2 \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right) + \frac{k^2}{12} \frac{\partial^4 u}{\partial t^4} - \frac{k^2 h^2}{12} \frac{\partial^4 u}{\partial t^4}$$

$$k^{-2} \; t.e \longrightarrow \underbrace{\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}}_{0} + \frac{k^2}{12} \left(\frac{\partial^4 u}{\partial t^4} - h^2 \frac{\partial^4 u}{\partial t^4} \right).$$

$$\sim O(k^2 + h^2)$$

Implicit Method for wave equation

Consider the explicit method

$$u_{i,j+1} = 2(1-\lambda^2)u_{i,j} + \lambda^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

replace in RHS, $u_{i,j} \simeq \frac{1}{2}(u_{i,j-1} + u_{i,j+1})$

$$u_{i,j+1} = 2(1-\lambda^2)\frac{1}{2}(u_{i,j-1} + u_{i,j+1}) + \frac{\lambda^2}{2}(u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1} + u_{i-1,j+1}) - u_{i,j-1}$$

$$\boxed{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} = -u_{i-1,j-1} + 2u_{i,j-1} - u_{i+1,j-1}} \quad \lambda = 1$$

Implicit Method

Consider $u_{i,j+1} = 2(1-\lambda^2)u_{i,j} + \lambda^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$

$$= \lambda^2 (u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + 2u_{i,j} - u_{i,j-1}$$

replace $u_{i,j}$ on RHS by $u_{i,j} \simeq \frac{1}{2}(u_{i,j+1} + u_{i,j-1})$ "data at jth level would be replaced"

$$u_{i,j+1} = \lambda^2 \left\{ \frac{u_{i-1,j+1} + u_{i-1,j-1}}{2} - \cancel{2}\,\frac{u_{i,j+1} + u_{i,j-1}}{\cancel{2}} + \frac{u_{i+1,j+1} + u_{i+1,j-1}}{2} \right\} + \cancel{2}\,\frac{u_{i,j+1} + u_{i,j-1}}{\cancel{2}} - \boxed{u_{i,j-1}}$$

$u_{i,j+1}(-\lambda^2 + 1)$

$u_{i,j-1}(-\lambda^2 + 1)$

$$\Rightarrow u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}$$

$$= -u_{i-1,j-1} + 2u_{i,j-1} - u_{i+1,j-1}$$

Implicit scheme

Special value of λ

Example Consider $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

If the equation is approximated by explicit finite diff. scheme $u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$

and the derivative is by a forward diff. scheme, estimate $e_{i,1}$.

Soln. $e = u - \bar{u}$, $h = k$; $\delta x = \delta k$

$u(x,0) = f(x)$
$\Rightarrow u_{i,0} = f_i$

$$\frac{\partial u}{\partial t} = g \Rightarrow \frac{u_{i,j+1} - u_{i,j}}{k} = g_i$$

at $t=0$, $j=0$ $\Rightarrow u_{i,1} - u_{i,0} = kg_i \Rightarrow \boxed{u_{i,1} = hg_i + f_i}$

Taylor series of $u_{i,1}$

$$u_{i,1} = u_{i,0} + h\frac{\partial u_{i,0}}{\partial t} + \frac{h^2}{2}\frac{\partial^2 u_{i,\theta}}{\partial t^2} \quad , \quad 0<\theta<1$$

$$u_{i,1} = hg_i + f_i$$

$$\Rightarrow \quad |e_{i,1}| \le \frac{h^2}{2} M_2 , \quad \text{— E} \qquad M_2 = \max_{0<\theta<1} \frac{\partial^2 u_{i,\theta}}{\partial t^2}$$

In order to see the behavior of $e_{i,j}$ as $h \to 0$

$$e_{i,j+1} = e_{i+1,j} + e_{i-1,j} - e_{i,j-1} + \underbrace{\boxed{\frac{h^4}{6} M_4 \cdot c}}_{T} \quad , \quad M_4 = \max_{c<1} \frac{\partial^4 u}{\partial t^4}$$

<u>j=1</u>

$$e_{i,2} = e_{i+1,1} + e_{i-1,1} - e_{i,0} + T$$
$$\le 2\frac{h^2}{2} M_2 + T$$

$$e_{i,3} = e_{i+1,2} + e_{i-1,2} - e_{i,1} + T$$

$$\leq 4\frac{h^2}{2}M_2 + \text{ same process}$$

$$e_{i,j+1} \leq (j+1)\frac{h^2}{2}M_2 + \frac{h^4}{12}j(j+1)M_4$$

$$\therefore |e_{i,j}| \leq \frac{jh^2}{2}M_2 + \frac{h^4}{12}j(j-1)M_4 \quad , jh = t$$

$$= \frac{th}{2}M_2 + \frac{1}{12}t^2h^2M_4 - \frac{1}{12}th^3M_4$$

$$\rightarrow 0 \text{ as } h \rightarrow 0.$$

Stability of Explicit Method

Consider $u_{i,j+1} = 2(1-\lambda^2)\, u_{i,j} + \lambda^2 (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$

let $u_{i,j} = u_{p,q} = A\,\xi^q\, e^{i\beta p h}$

$$\Rightarrow \quad \xi^{q+1} = 2(1-\lambda^2)\xi^q + \lambda^2 (e^{i\beta h} + e^{-i\beta h})\xi^q - \xi^{q-1}$$

$$\Rightarrow \quad \xi^2 - (2 - 4\lambda^2 \sin^2 \beta h/2)\xi + 1 = 0$$

Roots $\xi_{1,2} = (1 - 2\lambda^2 \sin^2 \beta h/2) \pm \sqrt{(1 - 2\lambda^2 \sin^2 \beta h/2)^2 - 1}$

$$\xi_{1,2} = \left(1 - 2\lambda^2 \sin^2 \beta h/2\right) \pm \sqrt{\left(1 - 2\lambda^2 \sin^2 \beta h/2\right)^2 - 1}$$

if $|1 - 2\lambda^2 \sin^2 \beta h/2| > 1$ then $|\xi_1| > 1 \Rightarrow$ unstable

if $|1 - 2\lambda^2 \sin^2 \beta h/2| < 1$, roots are complex conjugates whose magnitude is 1.

if $|1 - 2\lambda^2 \sin^2 \beta h/2| = 1$, $|\xi_{1,2}| = 1$.

$\therefore$ method is stable for $-1 \le 1 - 2\lambda^2 \sin^2 \beta h/2 \le 1$

$\Rightarrow \lambda \le 1$.

In general, if one replaces

$$Ⓐ — \quad u_{i,j} = \theta u_{i,j+1} + (1-2\theta) u_{i,j} + \theta u_{i,j-1}$$ in RHS of explicit method

$$u_{i,j+1} = \lambda^2 (u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + 2u_{i,j} - u_{i,j-1}$$

why?

replace approximation of the type Ⓐ

we get,

$$\boxed{\begin{aligned} D_t^2 u_{i,j} &= \lambda^2 D_x^2 \left[\theta u_{i,j+1} + (1-2\theta) u_{i,j} + \theta u_{i,j-1}\right] \\ \text{where} \quad D_x^2 &= u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \\ D_t^2 &= u_{i,j+1} - 2u_{i,j} + u_{i,j-1} \end{aligned}}$$

when $\theta = 1/2$,

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1}$$
$$= \frac{\lambda^2}{2}\Big(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}$$
$$+ u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1} \Big)$$

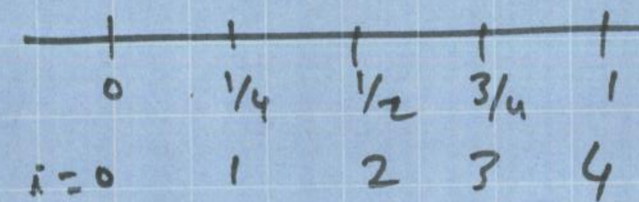
$$\Rightarrow \quad -\frac{\lambda^2}{2} u_{i-1,j+1} + (1+\lambda^2) u_{i,j+1} - \frac{\lambda^2}{2} u_{i+1,j+1}$$
$$= 2u_{i,j} + \frac{\lambda^2}{2} u_{i-1,j-1} - (1+\lambda^2) u_{i,j-1} + \frac{\lambda^2}{2} u_{i+1,j+1}$$

Example Solve $u_{tt} = u_{xx}$ $\quad 0 \le x \le 1$

$$u(x,0) = 2\sin \pi x$$
$$u(0,t) = 0$$
$$u(1,t) = 0$$
$$\frac{\partial u}{\partial t}(x,0) = 0$$

$\lambda = 3/2$ $\quad k = \lambda h = 3/8$ $\quad 1 + \frac{9}{4}$

$h = 1/4$



$\theta = 1/2$

$$-\frac{\lambda^2}{2} u_{i-1,j+1} + (1+\lambda^2) u_{i,j+1} - \frac{\lambda^2}{2} u_{i+1,j+1}$$

$$= 2u_{i,j} + \frac{\lambda^2}{2} u_{i-1,j-1} - (1+\lambda^2) u_{i,j-1} + \frac{\lambda^2}{2} u_{i+1,j+1}$$

$\lambda = 3/2$

$$\Rightarrow -\frac{9}{8} u_{i-1,j+1} + \frac{13}{4} u_{i,j+1} - \frac{9}{8} u_{i+1,j+1} = 2u_{i,j} + \frac{9}{8} u_{i-1,j-1} - \frac{13}{4} u_{i,j-1} + \frac{9}{8} u_{i+1,j-1} \quad - (*)$$

$\underline{j=0}$

$$-\frac{9}{8}u_{i-1,1} + \frac{13}{4}u_{i,1} - \frac{9}{8}u_{i+1,1} = 2u_{i,0} + \frac{9}{8}u_{i-1,-1} - \frac{13}{4}u_{i,-1} + \frac{9}{8}u_{i+1,-1}$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \implies \boxed{u_{i,1} = u_{i,-1}} \quad \text{— (B)}$$

using (B), we get

$$\boxed{-\frac{9}{4}u_{i-1,1} + \frac{13}{2}u_{i,1} - \frac{9}{4}u_{i+1,1} = 2u_{i,0}.}$$

at $j=0$ $\quad -\frac{9}{4} u_{i-1,1} + \frac{13}{2} u_{i,1} - \frac{9}{4} u_{i+1,1} = 2 u_{i,0}$

$i=1$ $\quad -\frac{9}{4} \underbrace{u_{0,1}}_{0} + \frac{13}{2} u_{1,1} - \frac{9}{4} u_{2,1} = 2u_{1,0} = 2\sin\frac{\pi}{4}$, $x_1 = 1/4$

$i=2$ $\quad -\frac{9}{4} u_{1,1} + \frac{13}{2} u_{2,1} - \frac{9}{4} u_{3,1} = 2u_{2,0} = 2\sin\frac{\pi}{2}$

$i=3$ $\quad -\frac{9}{4} u_{2,1} + \frac{13}{2} u_{3,1} - \frac{9}{4} \underbrace{u_{4,1}}_{0} = 2u_{3,0} = 2\sin\frac{3\pi}{4}$

$$\Rightarrow \begin{pmatrix} \frac{13}{2} & -\frac{9}{4} & 0 \\ -\frac{9}{4} & \frac{13}{2} & -\frac{9}{4} \\ 0 & -\frac{9}{4} & \frac{13}{2} \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix} = 2 \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix} \Rightarrow \text{solve } \begin{matrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{matrix}$$

Matrix stability analysis of Implicit method

Consider $D_t^2 u_{i,j} = \frac{\lambda^2}{2} D_x^2 (u_{i,j+1} + u_{i,j-1})$, $\theta = 1/2$

$$\Rightarrow \quad u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{\lambda^2}{2}\big(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1}\big)$$

$$\Rightarrow \quad u_{i,j+1} - \frac{\lambda^2}{2}\big(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\big)$$
$$= 2u_{i,j} - u_{i,j-1} + \frac{\lambda^2}{2}\big(u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1}\big)$$

$e_{i,j} = u_{i,j} - \bar{u}_{i,j}$

$$\Rightarrow (1+\lambda^2)\, u_{i,j+1} - \frac{\lambda^2}{2} u_{i-1,j+1} - \frac{\lambda^2}{2} u_{i+1,j+1} \quad \text{--- } (*)$$

$$= 2 u_{i,j} - (1+\lambda^2)\, u_{i,j-1} + \frac{\lambda^2}{2} u_{i-1,j-1} + \frac{\lambda^2}{2} u_{i+1,j-1}$$

$$A = \begin{bmatrix} (1+\lambda^2) & -\frac{\lambda^2}{2} & 0 & \\ -\frac{\lambda^2}{2} & (1+\lambda^2) & -\frac{\lambda^2}{2} & \\ & & - \; - \; \cdots & \end{bmatrix}$$

$u_{i,j} = \bar{u}_j$

(*) becomes

$$A\bar{u}_{j+1} = 2\bar{u}_j - A\bar{u}_{j-1} - b_j$$

$$A\bar{e}_{j+1} = 2\bar{e}_j - A\bar{e}_{j-1}$$

$$\Rightarrow \quad \bar{e}_{j+1} = 2A^{-1}\bar{e}_j - \bar{e}_{j-1}$$

$$\begin{pmatrix} \bar{e}_{j+1} \\ \bar{e}_j \end{pmatrix} = \begin{pmatrix} 2A^{-1} & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{e}_j \\ \bar{e}_{j-1} \end{pmatrix}$$

$$\Rightarrow \quad \bar{v}_{j+1} = P\bar{v}_j \quad , \quad P = \begin{pmatrix} 2A^{-1} & I \\ I & 0 \end{pmatrix}, \; \bar{v}_j = \begin{pmatrix} \bar{e}_j \\ \bar{e}_{j-1} \end{pmatrix}$$

the eigenvalues of P are $\begin{vmatrix} 2\eta_k^{-1}-\xi & -1 \\ 1 & -\xi \end{vmatrix} = 0$

$$\Rightarrow \eta_k = (1+\lambda^2) - 2\left(\frac{\lambda^2}{2}\cos\frac{k\pi}{N}\right), \quad k=1,\ldots N-1$$

$$\therefore \xi = \frac{\pm\sqrt{-1}\sqrt{\eta_k^2-1}}{\eta_k}$$

$$\eta_k = 1 + 2\lambda^2 \sin^2\frac{k\pi}{2N} > 1 \qquad \therefore \quad |\xi| = 1 \;\; \forall k$$

hence stable

Stability of Implicit Method

Consider $\quad D_t^2 u_{i,j} = \lambda^2 D_x^2 \left[\theta u_{i,j+1} + (1-2\theta) u_{i,j} + \theta u_{i,j-1} \right]$

$$u_{i,j} = u_{p,q} = A \xi^q e^{i\beta p h}$$

$$\Rightarrow \quad \xi^2 - 2R\xi + 1 = 0 \quad \text{where} \quad R = 1 - \frac{2\lambda^2 \sin \beta h/2}{1 + 4\theta\lambda^2 \sin^2 \beta h/2}$$

if $|R| > 1, \quad |\xi_1| > 1 \Rightarrow$ unstable

$|R| \leq 1, \quad |\xi| \leq 1 \Rightarrow$ stable

$$\therefore \quad \left| 1 - \frac{2\lambda^2 \sin^2 \beta h/2}{1 + 4\theta\lambda^2 \sin^2 \beta h/2} \right| \leq 1 \quad \Rightarrow \quad -1 \leq 1 - \frac{2\lambda^2}{1+4\theta\lambda^2} \leq 1$$

$$1 + \lambda^2 (1 - 4\theta) \geq 0$$

for $\theta \geq 1/4$, unconditionally stable

for $0 < \theta < 1/4$ stable for $0 < \lambda^2 < 1/(1-4\theta)$

Finite Difference Approximations to Elliptic PDEs - I

Potential flow (Irrotational) $\quad \nabla \times \bar{V} = 0 \Rightarrow \bar{V} = \nabla \phi$

$\nabla \cdot \bar{V} = 0 \Rightarrow \nabla^2 \phi = 0$

Laplace equation

Electricity / Magnetism : $\nabla^2 \bar{B} = 0$

$\nabla^2 \bar{E} = 0$ etc.

Laplace equation

$$\nabla^2 u = 0 \quad \text{in } \Omega$$

2D: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

3D: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

u - dependent

x, y, z : independent variables

The domain of integration of an elliptic equation is :

in 2D $\rightarrow$ area bounded by a closed curve C

in 3D $\rightarrow$ volume bounded by a closed surface S

Dirichlet problem

solve $\nabla^2 u = 0$ in Ω

$u = f$ on $\partial\Omega$

the missing data is $\frac{\partial u}{\partial n}$

Neumann problem

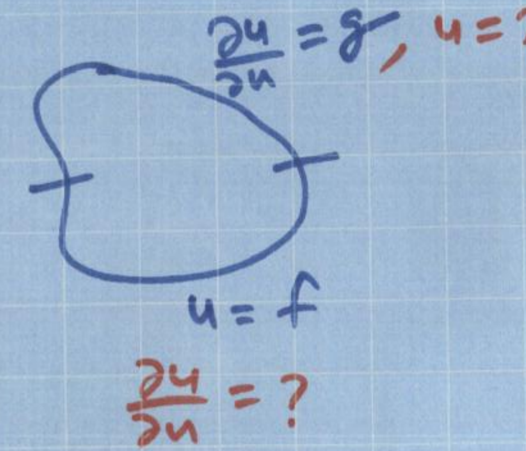
solve $\nabla^2 u = 0$ in Ω

$\frac{\partial u}{\partial n} = g$ on $\partial\Omega$

the missing data is u

Cauchy data

$\left(u, \frac{\partial u}{\partial n}\right)$



Boundary value problems

i) Solve $\nabla^2 u = 0$ in Ω

$u = f$ on $\partial\Omega$ (boundary of Ω)

known

"Dirichlet problem"

ii) Solve $\nabla^2 u = 0$ in Ω

$\frac{\partial u}{\partial n} = g$ on $\partial\Omega$

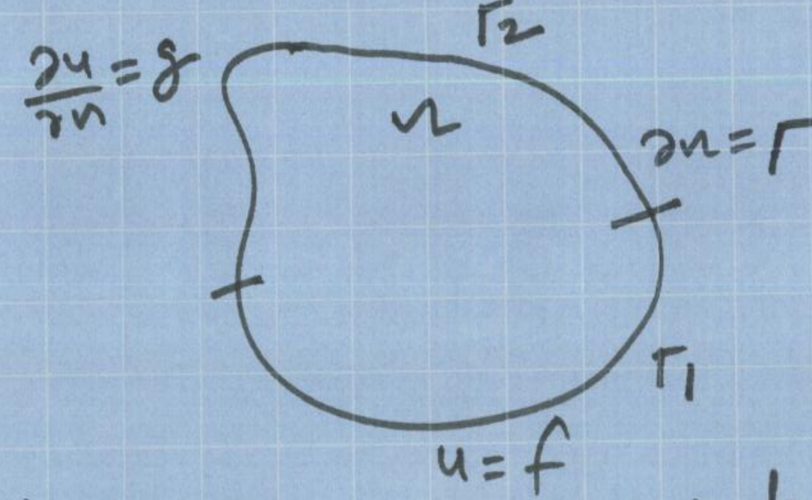
known

" Neumann Problem"

iii) Solve $\nabla^2 u = 0$ in Ω

$$\alpha u + \beta \frac{\partial u}{\partial n} = \text{given on } \partial\Omega$$

"Robin problem"



<u>Note:</u>

① $u = f$ on Γ_1

$\frac{\partial u}{\partial n} = g$ on Γ_2

$$\Gamma = \Gamma_1 \cup \Gamma_2$$

② prescribing both u and $\frac{\partial u}{\partial n}$ at a same point simultaneously is not possible.

Laplace equation -2D

$$\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

central difference approximations for both $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$

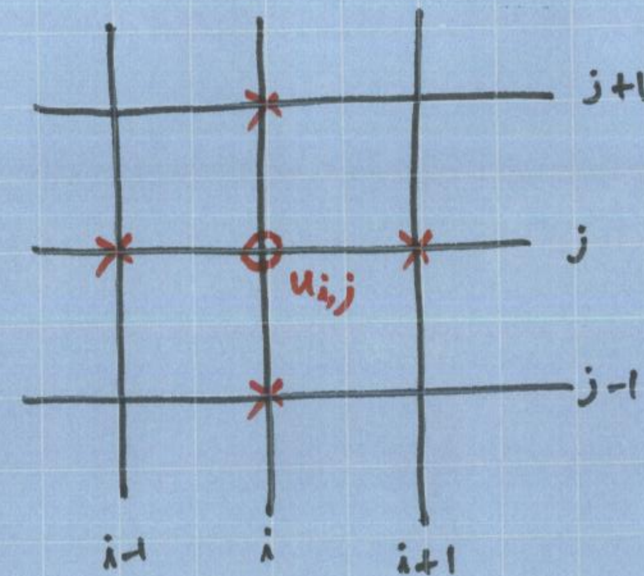
$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\delta y)^2} = 0 \quad \text{--- (2)}$$

if $\delta x = \delta y = h$, then (2) reduces to

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \quad \text{--- (3)}$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

Standard 5 point formula

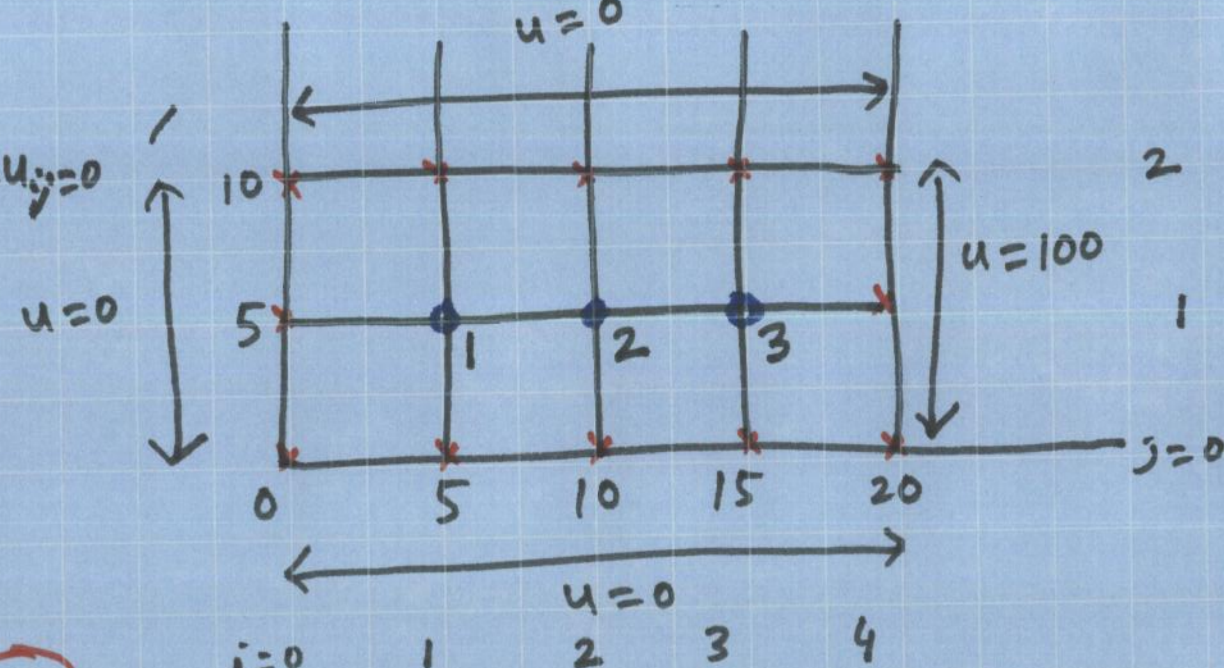


Example: solve $\nabla^2 u = 0$, $u(x,0)=0$; $u(0,y)=0$

$u(x,10)=0$; $u(20,y)=100$

$h=5$

$$u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}=0$$



Point 1: $i=1, j=1$

Point 2: $i=2, j=1$

Point 3: $i=3, j=1$

p_1: $u_{2,1}+u_{0,1}+u_{1,2}+u_{1,0}-4u_{1,1}=0$

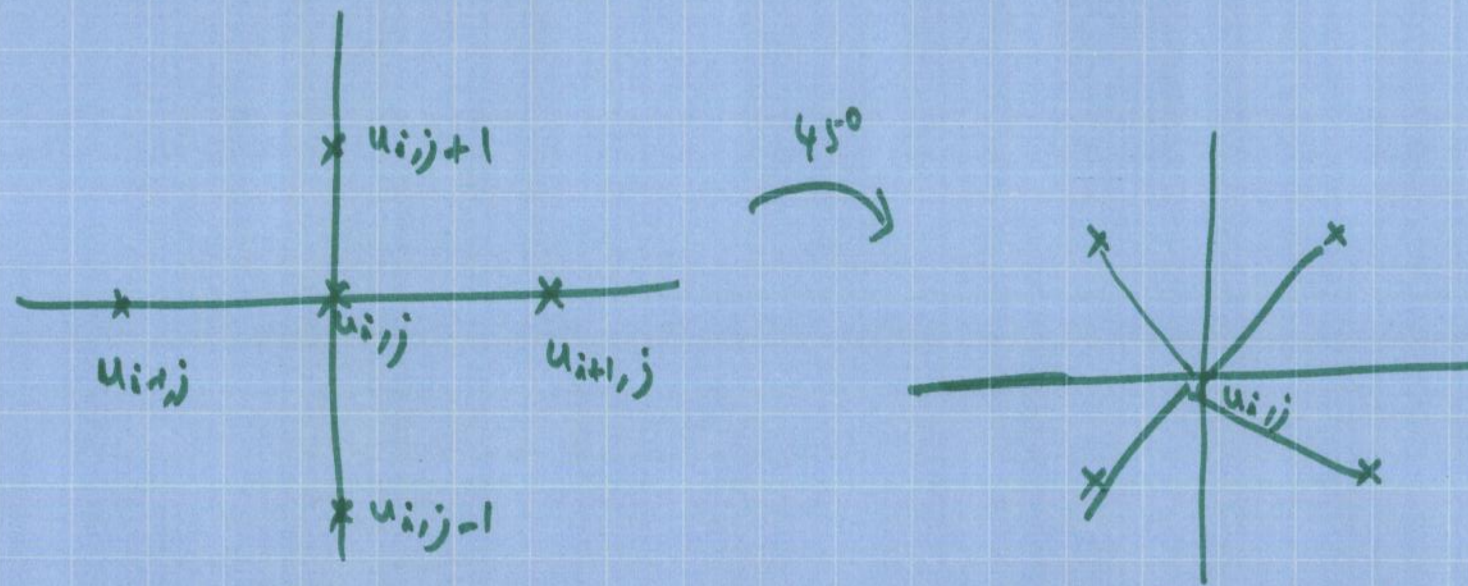
p_2: $u_{3,1}+u_{1,1}+u_{2,2}+u_{2,0}-4u_{2,1}=0$

p_3: $u_{4,1}+u_{2,1}+u_{3,2}+u_{3,0}-4u_{3,1}=0$

use b.c.s, say,

$u(x,0)=0 \Rightarrow u_{i,0}=0$

$u(0,y)=0 \Rightarrow u_{0,j}=0$



Standard 5 point

diagonal 5 point

Finite Difference Approximations to Elliptic PDEs - II

Iterative Methods

$$Ax = B \Rightarrow \underset{\substack{|\\ \text{non-singular and of the same dimension as } A}}{P}X + (A-P)X = B$$

then

$$\boxed{PX_{k+1} = B - (A-P)X_K \;, \quad X_0 : \text{given}}$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j} \quad \text{—} (*)$$

Then $(*)$ reduces to $AV = F$, $V = [v_1 \cdots v_{n-1}]^T = [u_1 \cdots u_{n-1}]^T$

$$A = \begin{bmatrix} T & I & 0 & 0 & \cdots & 0 \\ I & T & I & 0 & \cdots & 0 \\ & & & & & \\ & & & 0 & I & T \end{bmatrix}, \quad T = \begin{bmatrix} -4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 0 & \cdots & 0 \\ & & & & & \\ & & & 0 & 1 & -4 \end{bmatrix}$$

Note: the system is sparse

$i=1, j=2$, $u_{i,j} = u_{1,2}$

usual

$$u_{i,j} = v_{i+(n-1)(j-1)}$$

$i=1, j=1$

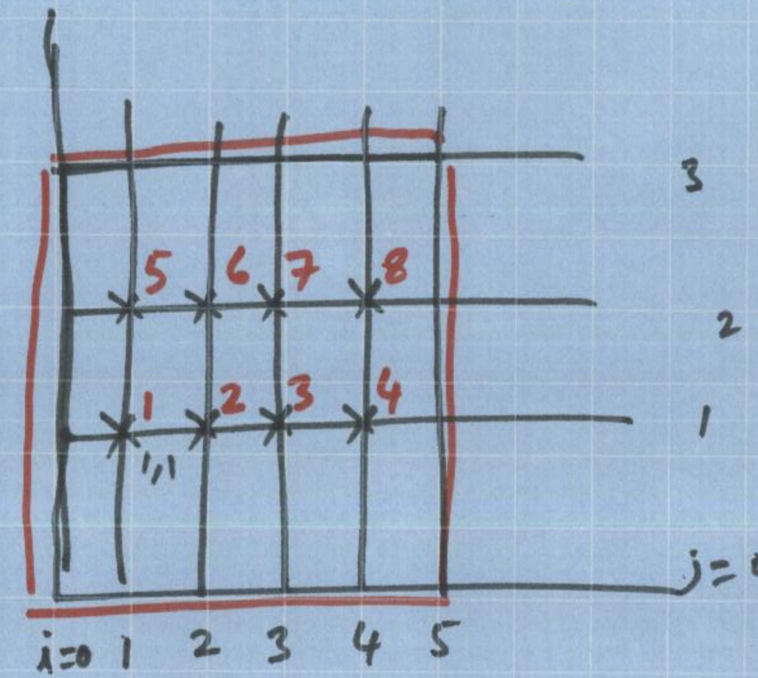
$$u_{1,1} = v_{1+4\times 0} = v_1 = u_1$$

$i=2, j=1$

$$u_{2,1} = v_{2+4\times 0} = v_2 = u_2$$

$i=1, j=2$

$$u_{1,2} = v_{1+4\times 1} = v_5 = u_5$$



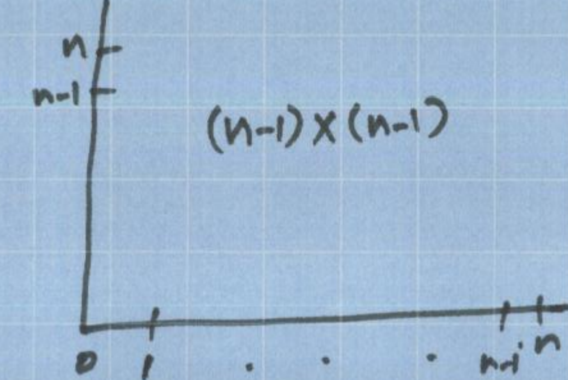
Computations involved

$$u_{i+1,j} + \cancel{u_{i,j-1}} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{i,j}$$

$u_{i-1,j-1}$

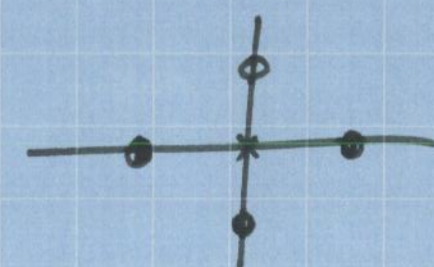
$$, \quad i,j \in (1 \ldots n-1) \times (1 \ldots n-1)$$

one has to solve a set of $(n-1)^2$ linear equations



Note: (1) any node has 4 neighbours contributing, hence the tridiagonal structure is lost

(2) need a method of arranging/counting



Poisson's equation

solve $\nabla^2 \phi = f(\bar{x})$, $\quad \bar{x} \in \Omega \subset \mathbb{R}^n$

2D: $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = f(x,y) \Rightarrow \phi = \phi(x,y)$

example $\quad \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = -20$

5 point formula $\Rightarrow \dfrac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(1/2)^2} + \dfrac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(1/2)^2} = -20$

$h = 1/2$

$$\Rightarrow \underline{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -5}$$

Iterative Methods

Remark: A sparse matrix is useful only when the non-zero entries alone are required to store.

example

if $A = \begin{bmatrix} 0.0 & 1.0 & 2.1 & 0.0 & 0.0 \\ 2.0 & 0.0 & 0.5 & 3.1 & 0.0 \\ 0.0 & -1.0 & -0.6 & 0.0 & 1.3 \\ 0.0 & 0.0 & 0.0 & 3.5 & 0.0 \\ 0.4 & 0.0 & 2.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix}$

$\bar{r} = [0, 2, 5, 8, 9, 11, 12]$

non-zero elements in each row as $(r_{k+1} - r_k)$ (number)

vectors $\bar{a}, \bar{b}, \bar{r}$

non-zero elements $\bar{a} = [1.0, 2.1, 2.0, 0.5, 3.1, -1.0, -0.6, 1.3, 3.5, 0.4, 2.0, 1.0]$

column index $\bar{b} = [2, 3, 1, 3, 4, 2, 3, 5, 4, 1, 2, 3]$

Example $\nabla^2 u = 0$, $u(x,0) = 0$; $u(0,y) = 5$ $h = 5$

$u(x,15) = 0$; $u(15,y) = 50$

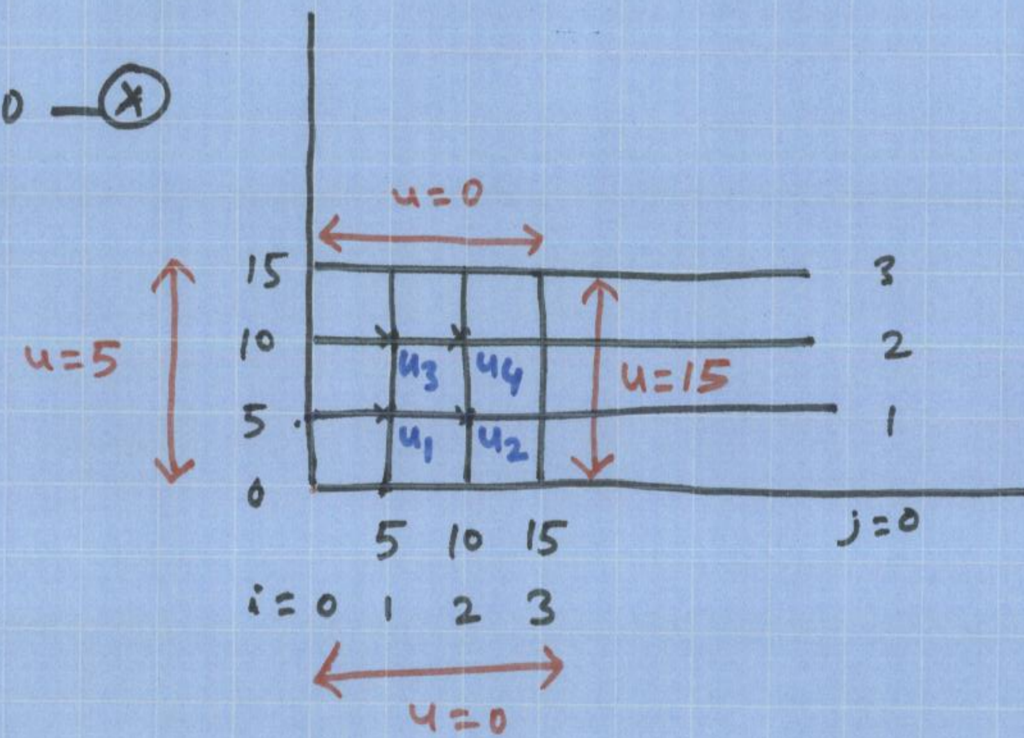
$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \quad \text{— (*)}$$

u_1: $i=1, j=1$

(*) at $i=1, j=1$

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = 0$$

$$u_2 + 5 + u_3 + 0 - 4u_1 = 0$$



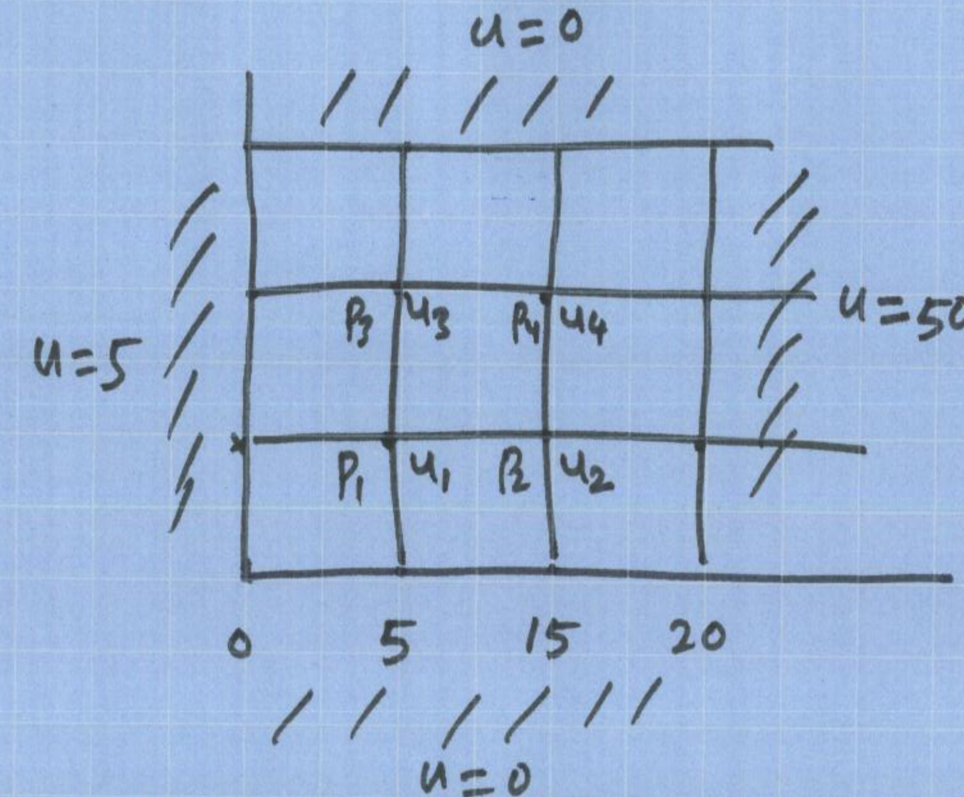
point 1:

$$u_2 + 5 + u_3 + 0 - 4u_1 = 0$$

point 2: $50 + u_1 + u_4 + 0 - 4u_2 = 0$

point 3: $u_4 + 5 + 0 + u_1 - 4u_3 = 0$

point 4: $50 + u_3 + 0 + u_2 - 4u_4 = 0$



$$\Rightarrow \quad \begin{aligned} -4u_1 + u_2 + u_3 &= -5 \\ u_1 - 4u_2 + u_4 &= -50 \\ u_1 - 4u_3 + u_4 &= -5 \\ u_2 + u_3 - 4u_4 &= -50 \end{aligned}$$

initial guess $(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_4^{(0)}) = (0,0,0,0)$

$$u_1 = \frac{u_2 + u_3}{4} + \frac{5}{4}$$

$$u_2 = \frac{u_1 + u_4}{4} + \frac{25}{2}$$

$$u_3 = \frac{u_1 + u_4}{4} + \frac{5}{4}$$

$$u_4 = \frac{u_2 + u_3}{4} + \frac{25}{2}$$

G-Jacobi

$u_1^{(1)} = 5/4 = 1.25$; $u_2^{(1)} = 12.5$

$u_3^{(1)} = 5/4 = 1.25$; $u_4^{(1)} = 12.5$

$$u_1^{(2)} = \frac{u_2^{(1)} + u_3^{(1)}}{4} + \frac{5}{4} = \frac{12.5 + 1.25 + 5}{4} \simeq 4.68$$

$$u_2^{(2)} = \frac{u_1^{(1)} + u_4^{(1)}}{4} + \frac{25}{2} = \frac{1.25 + 12.5 + 50}{4} = 15.93$$

$$u_3^{(2)} = \frac{u_1^{(1)} + u_4^{(1)} + 5}{4} = \frac{1.25 + 12.5 + 5}{4} \simeq 4.68$$

$$u_4^{(2)} = \qquad = 15.93$$

$$u_1 = \frac{u_2 + u_3}{4} + \frac{5}{4}$$

$$u_2 = \frac{u_1 + u_4}{4} + \frac{25}{2}$$

$$u_3 = \frac{u_1 + u_4}{4} + \frac{5}{4}$$

$$u_4 = \frac{u_2 + u_3}{4} + \frac{25}{2}$$

G - Seidel

$(u_1^{(0)}, u_2^{(1)}, u_3^{(1)}, u_4^{(0)}) = (0, 0, 0, 0)$

$u_1 = \frac{u_2 + u_3}{4} + \frac{5}{4}$

$u_2 = \frac{u_1 + u_4}{4} + \frac{50}{4}$ $\quad \frac{51.25}{4} = 12.81$

$u_3 = \frac{u_1 + u_4}{4} + \frac{5}{4}$

$u_4 = \frac{u_2 + u_3}{4} + \frac{50}{4}$

$u_1^{(1)} = 5/4 = 1.25$

$u_2^{(1)} = \frac{u_1^{(1)} + u_4^{(0)}}{4} + \frac{50}{4} = \frac{1.25 + 50}{4} \simeq 12.81$

$u_3^{(1)} = \frac{u_1^{(1)} + u_4^{(0)}}{4} + \frac{5}{4} = \frac{1.25 + 5}{4} = \frac{6.25}{4} \simeq 1.56$

$u_4^{(1)} = \frac{u_2^{(1)} + u_3^{(1)} + 50}{4} = \frac{12.81 + 1.56 + 50}{4}$

$\simeq \frac{64.31}{4} \simeq 16.07$

$u_1^{(2)} = \frac{u_2^{(1)} + u_3^{(1)} + 5}{4} = \frac{12.81 + 1.56 + 5}{4} \simeq \frac{19.37}{4} \simeq 4.89$

$$u_7 = u_1 - \frac{1}{2} u_4$$

$$u_{i,3} - u_{i,1} = -\frac{1}{2} u_{i,2}$$

$$u_8 = u_2 - \frac{1}{2} u_5$$

$$u_9 = u_3 - \frac{1}{2} u_6$$

P_1: $u_2 + 2u_4 - 6u_1 = 8$; P_2: $u_3 + u_1 + 2u_5 - 6u_2 = 8$

P_3: $u_2 + 2u_6 - 6u_3 = 8$;

P_4: $u_5 + 4u_1 - 5u_4 = 8$; P_5: $u_6 + u_4 + 2u_2 + 2\left(u_2 - \frac{1}{2} u_5\right) - 6u_5 = 8$

$$u_6 + u_4 + 4u_2 - 7u_5 = 8$$

P_6: $u_5 + 2u_3 + 2\left(u_3 - \frac{1}{2} u_6\right) - 6u_6 = 8$

$$u_5 + 4u_3 - 7u_6 = 8$$

Accelerating : Successive overrelaxation (S.O.R)

$$u_{i,j}^{(k+1)} = \frac{u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)}}{4} + u_{i,j}^{(k)} - u_{i,j}^{(k)}$$

$$\Rightarrow u_{i,j}^{(k+1)} = u_{i,j}^{(k)} + \left[\underbrace{\frac{u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} - 4u_{i,j}}{4}}_{\text{Residual}}\right]\omega$$

relaxation parameter

$1.0 < \omega < 0.2$

Accuracy and Stability

$$T_{ij} = \frac{1}{h^2}\left\{ u(x_{i-1}, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}) - 4u(x_i, y_j) \right\} - f(x_i, y_j)$$

$$= \frac{1}{12} h^2 \left(u_{xxxx} + u_{yyyy} \right) + O(h^4)$$

$E_{ij} = u_{ij} - u(x_i, y_j)$ then

$$A^h E^h = -T^h$$

$$T = \begin{bmatrix} -4 & 1 & 0 & \cdot & \cdots & 0 \\ 1 & -4 & 1 & 0 & \cdots & \\ & & & & & \end{bmatrix}$$

$$A^h = \begin{bmatrix} T & I & 0 & 0 & \cdot & \cdots \\ I & T & I & 0 & \cdot & \cdots \\ & & & & & \\ & & & 0 & I & T \end{bmatrix}$$

For the method to be stable $\|(A^h)^{-1}\|$ is uniformly bounded as $h \to 0$

Corresponding to the matrix A^h,

the (p,k) eigenvector $u^{p,k}$ has m^2 elements

$$u_{i,j}^{p,k} = \sin(p\pi i h)\sin(k\pi j h)$$

and the corresponding eigenvalue is

$$\lambda_{p,k} = \frac{2}{h^2}\Big((\cos(p\pi h)-1) + (\cos(k\pi h)-1)\Big)$$

$$\lambda_{1,1} = -2\pi^2 + O(h^2) \quad \text{close to } 0$$

Spectral radius of $(A^h)^{-1}$: $\frac{1}{\lambda_{1,1}} \simeq -\frac{1}{2\pi^2}$ $\therefore$ stable

Finite Difference Approximations to Elliptic PDEs – III

Derivative Boundary condition

solve $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial y^2} = 32$ subject to

$u = 0$ on $x = 0$ and $x = 2$

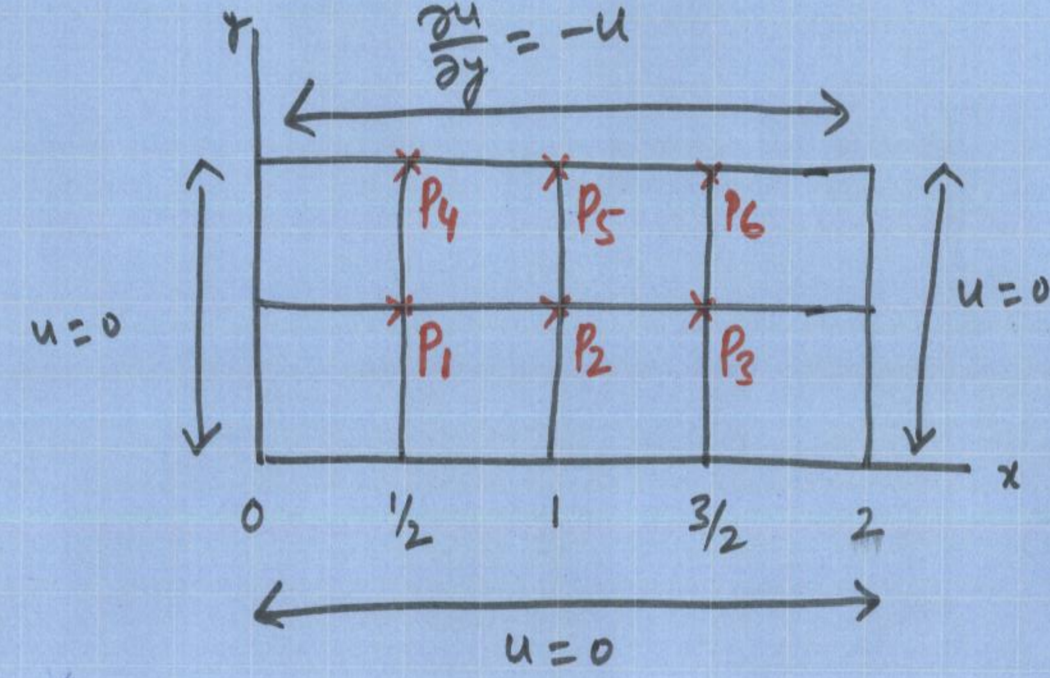
$u = 0$ on $y = 0$

$\dfrac{\partial u}{\partial y} = -u$ on $y = 1$

$h = 1/2$

d2

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} = 32 \quad , \quad h = 1/2$$



$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$+ 2\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right) = 32$$

$$\Rightarrow \quad u_{i+1,j} + 2u_{i,j+1} + u_{i-1,j} + 2u_{i,j-1} - 6u_{i,j} = 8$$

d3

$$u_{i+1,j} + 2u_{i,j+1} + u_{i-1,j} + 2u_{i,j-1} - 6u_{i,j} = 8$$

P_1: $u_2 + 0 + 2u_4 + 0 - 6u_1 = 8$

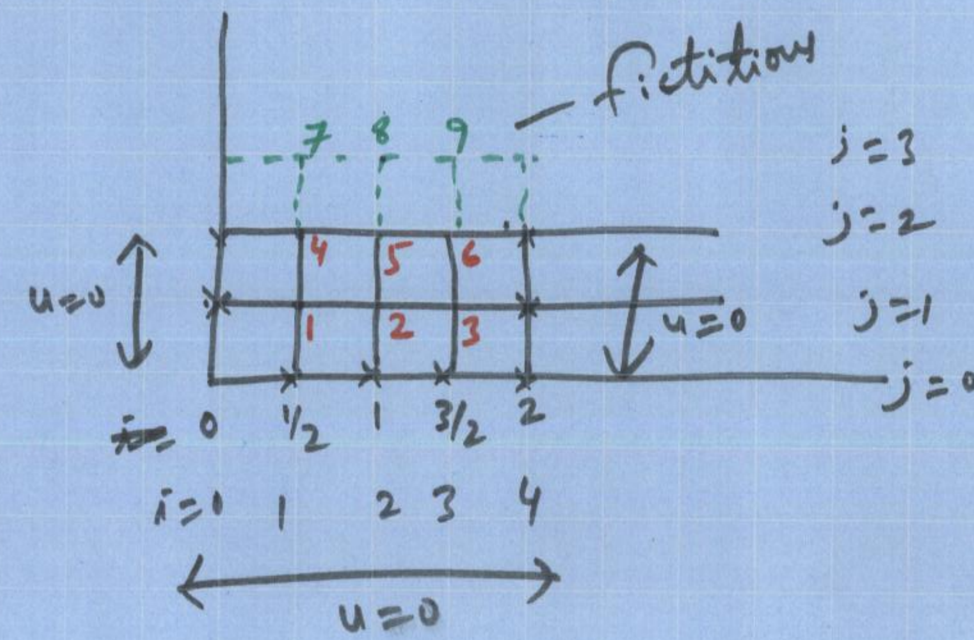
P_2: $u_3 + u_1 + 2(u_5 + 0) - 6u_2 = 8$

P_3: $0 + u_2 + 2u_6 + 0 - 6u_3 = 8$

P_4: $u_5 + 0 + 2(\boxed{u_7} + u_1) - 6u_4 = 8$

P_5: $u_6 + u_4 + 2(\boxed{u_8} + u_2) - 6u_5 = 8$

P_6: $0 + u_5 + 2(\boxed{u_9} + u_3) - 6u_6 = 8$



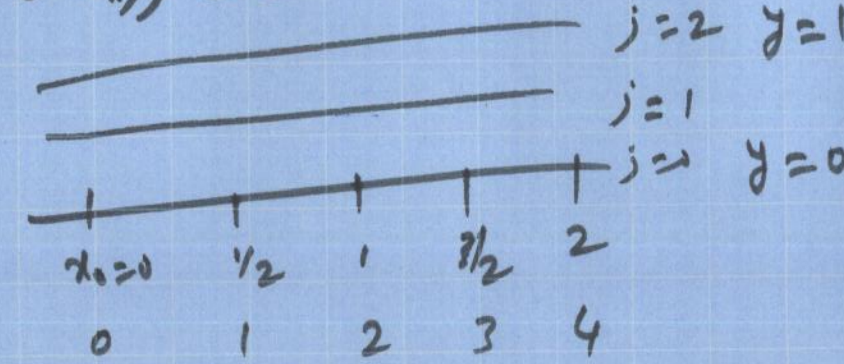
$$\frac{\partial u}{\partial y} = -u \quad \text{at } y=1$$

$$u_{i,3} - u_{i,1} = \frac{-1}{2} u_{i,2} \quad (*)$$

eliminate u_7, u_8, u_9 using $(*)$

$$\begin{bmatrix} -6 & 1 & 0 & 2 & 0 & 0 \\ 1 & -6 & 1 & 0 & 2 & 0 \\ 0 & 1 & -6 & 0 & 0 & 2 \\ 4 & 0 & 0 & -7 & 1 & 0 \\ 0 & 4 & 0 & 1 & -7 & 1 \\ 0 & 0 & 4 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = 8 \begin{pmatrix} 1 \\ 1 \\ \cdot \\ 1 \\ 1 \end{pmatrix}$$

$$u_{i+1,j} + 2u_{i,j+1} + u_{i-1,j} + 2u_{i,j-1} - 6u_{i,j} = 8$$



$$u(0,y) = 0 \Rightarrow u_{0,j} = 0$$

$$u(2,y) = 0 \Rightarrow u_{4,j} = 0$$

$$u(x,0) = 0 \Rightarrow u_{i,0} = 0$$

$$\frac{\partial u}{\partial y}(x,1) = -u(x,1) \Rightarrow \boxed{u_{i,3} - u_{i,1} = -\frac{1}{2}u_{i,2}}$$

$$\frac{u_{i,j+1} - u_{i,j-1}}{2h} = -u_{i,j} \Rightarrow$$

Laplace equation - "Axisymmetric" - cylindrical coordinates

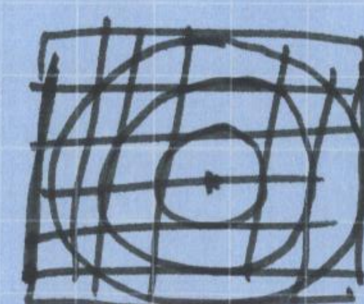
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 , \quad \begin{array}{l} 0 \le r \le R \\ 0 \le z \le c \end{array}$$

$u(r,0) = f(r)$; $u(r,c) = h(r)$

$u(R,z) = g(z)$; $\frac{\partial u}{\partial r}(0,z) = 0$

discretizing the domain: $r_l = lh$, $l = 1, 2, \cdots L$

$z_m = mk$, $m = 1 \cdots M$



C2

$$\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right)\Bigg|_{(r_l, z_m)} \simeq a_0 u_{l,m} + a_1 u_{l+1,m} + a_3 u_{l,m+1} + a_4 u_{l,m-1} + a_5 u_{l-1,m} + \text{error}$$

$$\Rightarrow \quad a_0 + a_1 + 2a_3 + a_5 = 0$$

$$\frac{1}{lh} - (a_1 - a_5)h = 0$$

$$1 - \frac{h^2}{2}(a_1 + a_5) = 0$$

$$1 - k^2 a_3 = 0, \quad a_3 = a_4$$

$r_l = lh$; $z_m = mk = msh$, $k = sh$

$$-2\left(1+\frac{1}{s^2}\right)u_{l,m} + \left(1+\frac{1}{2l}\right)u_{l+1,m} + \left(1-\frac{1}{2l}\right)u_{l-1,m}$$

$$+\frac{1}{s^2}\left(u_{l,m-1}+u_{l,m+1}\right) = h^2 f(r_l, z_m, u_{l,m}) \quad \text{--- (1)}$$

as $r \to 0$, the differential equation becomes

$$2\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} = f(0, z, u)$$

$$-2\left(2+\frac{1}{s^2}\right)u_{l,m} + 2\left(u_{l+1,m}+u_{l-1,m}\right) + \frac{1}{s^2}\left(u_{l,m-1}+u_{l,m+1}\right) = h^2 f_{l,m}$$

— (2) for $l = 0$.

$\frac{\partial u}{\partial r} = 0$ at $r = 0 \Rightarrow u_{-1,m} = u_{1,m}$ — (3)

$\nabla^2 u = 0, \quad 0 \le r < 1, \quad -1 < z < 1$

$u = 0$ on the boundary, $h = k = 1/2$

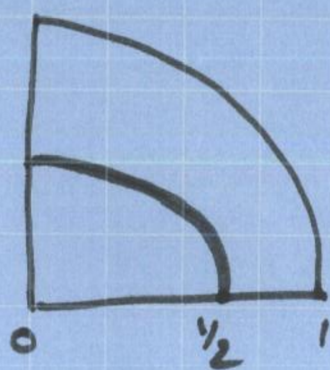
$r_l = l/2, \quad z_m = m/2, \quad l = 0, \pm 1, \pm 2, \dots$

$m = 0, \pm 1, \pm 2, \dots$

b.c.s: $u_{l,2} = 0, \quad l = 0, 1, 2 \dots$

$u_{2,m} = 0, \quad m = 0, 1, 2 \dots$

points involved: $(0,0), (1/2, 0), (0, 1/2), (1/2, 1/2)$



$(0,0): \quad -6 u_{0,0} + 2(u_{1,0} + u_{0,1}) = -\frac{1}{4}$

$(1/2, 0): \quad -4 u_{1,0} + \frac{3}{2} u_{2,0} + \frac{1}{2} u_{0,0} = -\frac{1}{4}$

Finite differences - polar coordinates

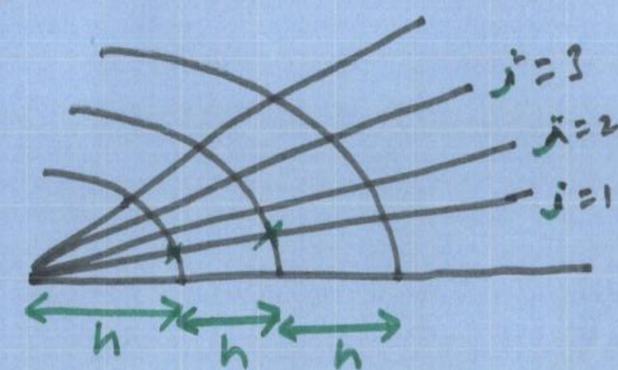
$\nabla^2 u = 0 \Rightarrow$ "(r, θ) polar coordinates"

$$\Rightarrow \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$r\theta$-plane

$r = ih \quad , \quad i = 1, 2, \cdots$

$\theta = j\,\delta\theta = jk \quad , \quad j = 0, 1, \cdots$



C5

example

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{1}{ih}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h}\right)$$

$$+ \frac{1}{i^2h^2}\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right) = 0$$

$$u_{i-1,j}\left(1 - \frac{1}{2i}\right) + u_{i+1,j}\left(1 + \frac{1}{2i}\right) - 2\left(1 + \frac{1}{i^2k^2}\right)u_{i,j}$$

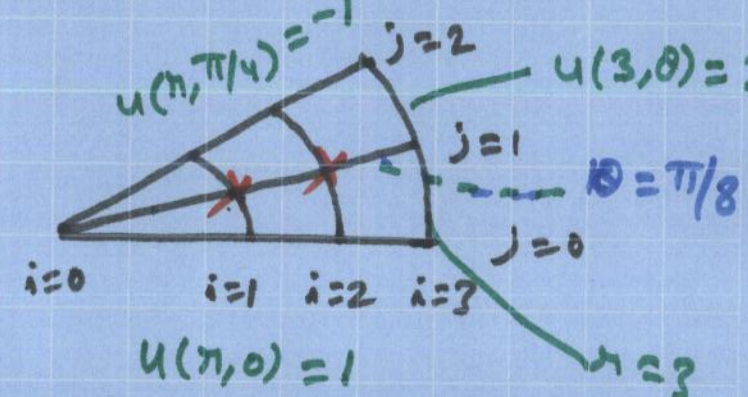
$$+ u_{i,j+1}\frac{1}{i^2h^2} + \frac{1}{i^2h^2}u_{i,j-1} = 0$$

C6

$$\Rightarrow A_i u_{i-1,j} + B_i u_{i+1,j} + C_i u_{i,j} + D_i u_{i,j+1} + E_i u_{i,j-1} = 0$$

where $A_i = 1 - \frac{1}{2i}$; $B_i = 1 + \frac{1}{2i}$

$C_i = -2\left(1 + \frac{1}{i^2 k^2}\right)$; $D_i = E_i = \frac{1}{i^2 h^2}$

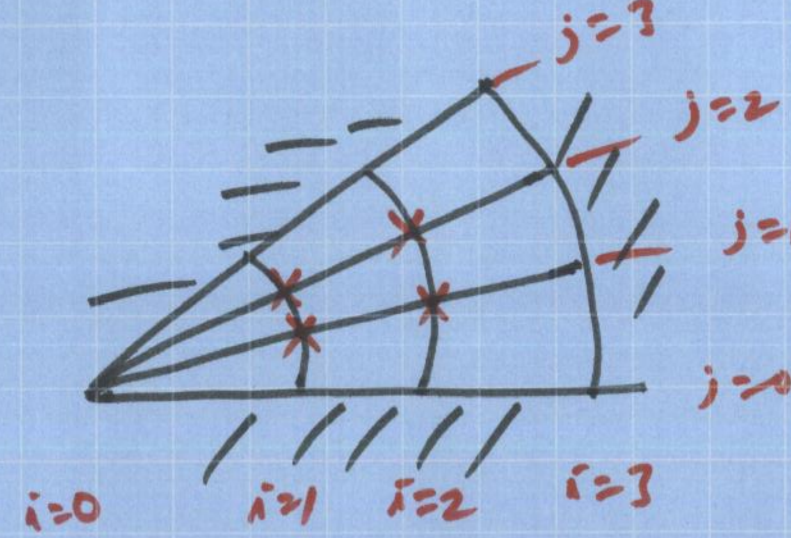


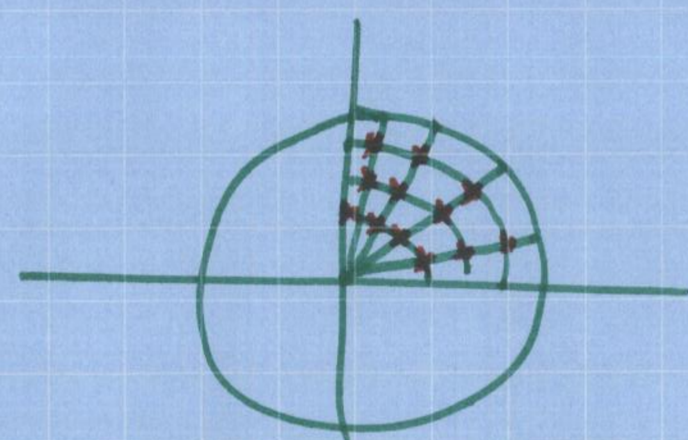
$i=1, j=1$ $\Rightarrow \frac{1}{2} u_{0,1} + \frac{3}{2} u_{2,1} - 2\left(1 + \frac{64}{\pi^2}\right) u_{1,1} + u_{1,2} + u_{1,0} = 0$

$i=2, j=1$ $\Rightarrow \frac{3}{4} u_{1,1} + \frac{5}{4} u_{3,1} - 2\left(1 + \frac{256}{\pi^2}\right) u_{2,1} + u_{2,2} + u_{2,1} = 0$

$r = ih$, $h = 1$
$\theta = jk$, $k = 1/2$

solve for $u_{1,1}$ and $u_{2,1}$


j=3
j=2
j=1
j=0
i=0
i=1
i=2
i=3



C8

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{1}{ih}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h}\right)$$

$$+ \frac{1}{(ih)^2}\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\delta\theta)^2}\right) = 0$$

$$\Rightarrow \left(1 - \frac{1}{2i}\right)u_{i-1,j} + \left(1 + \frac{1}{2i}\right)u_{i+1,j} - 2\left(1 + \frac{1}{i(\delta\theta)^2}\right)u_{i,j}$$

$$+ \frac{1}{i(\delta\theta)^2}u_{i,j-1} + \frac{1}{i(\delta\theta)^2}u_{i,j+1} = 0$$

9

$$B_{ij} = \begin{pmatrix} -2\left(1+\frac{1}{(j\delta\theta)^2}\right) & \frac{1}{(j\delta\theta)^2} & & & \\ \frac{1}{(j\delta\theta)^2} & -2\left(1+\frac{1}{(j\delta\theta)^2}\right) & \frac{1}{(j\delta\theta)^2} & & \\ & & \ddots & & \\ & & \frac{1}{(j\delta\theta)^2} & -2\left(1+\frac{1}{(j\delta\theta)^2}\right) & \frac{1}{(j\delta\theta)^2} \\ & & 0 & \frac{1}{(j\delta\theta)^2} & -2\left(1+\frac{1}{(j\delta\theta)^2}\right) \end{pmatrix}$$

C9

C10

$$A\bar{u} = \bar{b}, \qquad \bar{u} = [u_{1,1}, u_{1,2}, \ldots u_{1,m}; u_{2,1}, u_{2,2} \ldots u_{2,m}, \ldots u_{n,m}]$$

$$A = \begin{bmatrix} B_1 & (1+\frac{1}{2})I & & & & \\ (1-\frac{1}{4})I & B_2 & (1+\frac{1}{4})I & & & \\ & \cancel{(1-\frac{1}{6})I} & \cancel{(1+\frac{1}{6})} & & & \\ & (1-\frac{1}{6})I & B_3 & (1+\frac{1}{6})I & & \\ & & & (1-\frac{1}{2(n-1)})I & B_{n-1} & (1+\frac{1}{2(n-1)})I \\ & & & 0 & (1-\frac{1}{2n})I & B_n \end{bmatrix}$$

Non-uniform grid - Laplace / Poisson equation

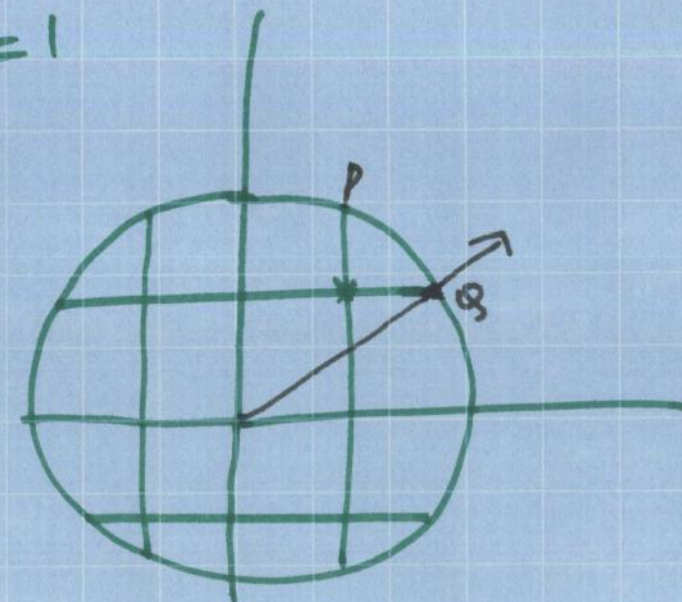
example

solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $\quad 0 \le x^2 + y^2 \le 1$

$x \geq 0, \; y \geq 0$

$u = 2$ on $x = 0, \; y = 0$

$\dfrac{\partial u}{\partial n} = (x^2 - y^2)$ on $x^2 + y^2 = 1$



Consider $\left. A u_{xx} + C u_{yy} + D u_x + E u_y + F u \right|_{(x_i, y_j)} = 0$

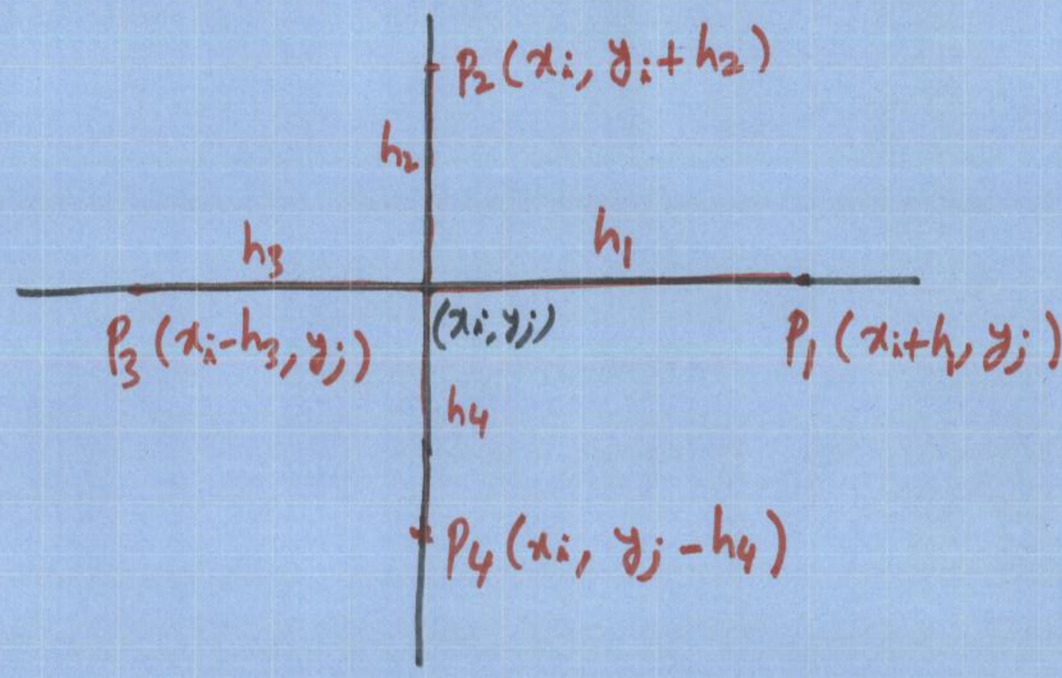
$\simeq \alpha_0 u(x_i, y_j)$

$+ \alpha_1 u(x_i + h_1, y_j)$

$+ \alpha_2 u(x_i, y_i + h_2)$

$+ \alpha_3 u(x_i - h_3, y_j)$

$+ \alpha_4 u(x_i, y_j - h_4)$



expand in Taylor-series

coeff of $u(x_i, y_j)$: $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = F$

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$$\Rightarrow \left.\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 &= F_{i,j} \\
h_1\alpha_1 - h_3\alpha_3 &= D_{i,j} \\
h_2\alpha_2 - h_4\alpha_4 &= E_{i,j} \\
h_1^2\alpha_1 + h_3^2\alpha_3 &= 2A_{i,j} \\
h_2^2\alpha_2 + h_4^2\alpha_4 &= 2C_{i,j}
\end{aligned}\right\} \circledast$$

solution of $\circledast$

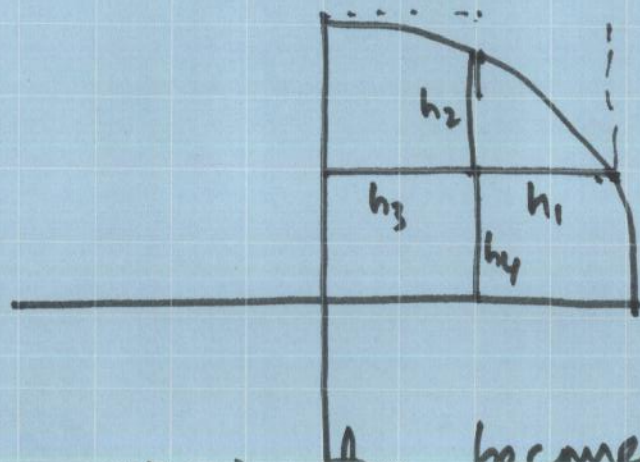
Solution of *

$$\alpha_0 = F_{i,j} - \left[\frac{1}{h_1 h_3}\left(2A_{i,j} + (h_3 - h_1) D_{i,j}\right) + \frac{1}{h_2 h_4}\left(2C_{i,j} + (h_4 - h_2) F_{i,j}\right) \right]$$

$$\alpha_1 = \frac{2A_{i,j} + h_3 D_{i,j}}{h_1(h_1 + h_3)} \; ; \quad \alpha_2 = \frac{2C_{i,j} + h_4 E_{i,j}}{h_2(h_2 + h_4)}$$

$$\alpha_3 = \frac{2A_{i,j} - h_1 D_{i,j}}{h_3(h_1 + h_3)} \; ; \quad \alpha_4 = \frac{2C_{i,j} - h_2 E_{i,j}}{h_4(h_2 + h_4)}$$

if $h_1 = h_3 = h$; $h_2 = h_4 = k$, $\quad O(k^2 + h^2)$



$h_1 = h_2$

$h_3 = h_4$

$u_{xx} + u_{yy} = 0$

$A = 1, \quad C = 1$

$D = 0, \quad E = 0, \quad F = 0$

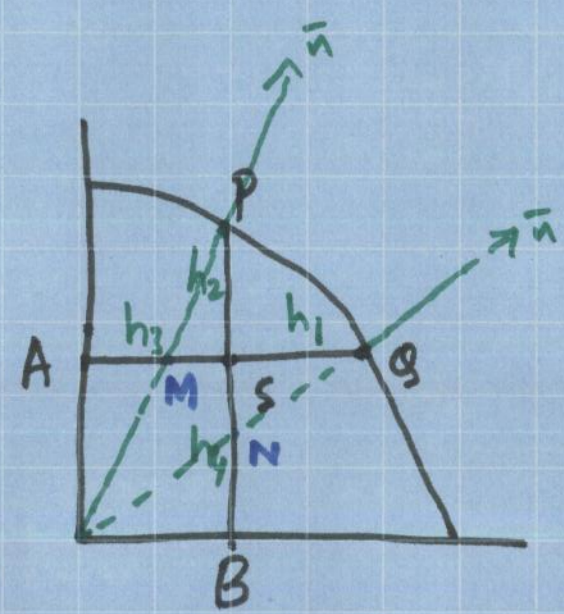
difference approximation becomes

$$" -2\left(\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right) u_{i,j} + \frac{2}{(h_1+h_3)h_1} u_{i+1,j} + \frac{2}{h_2(h_2+h_4)} u_{i,j+2}$$

$$+ \frac{2}{h_3(h_1+h_3)} u_{i-3,j} + \frac{2}{h_4(h_2+h_4)} u_{i,j-4} = 0. "$$

$u_{i,j+2} = u(x_i, y_j + h_2)$

$u_{i,j-4} = u(x_i, y_j - h_4)$



<u>Recall</u> $h_1 = h_2$; $h_3 = h_4$

$$-2\left(\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right) u_S + \frac{2}{(h_1+h_3)h_1} u_Q + \frac{2}{h_2(h_2+h_4)} u_P$$

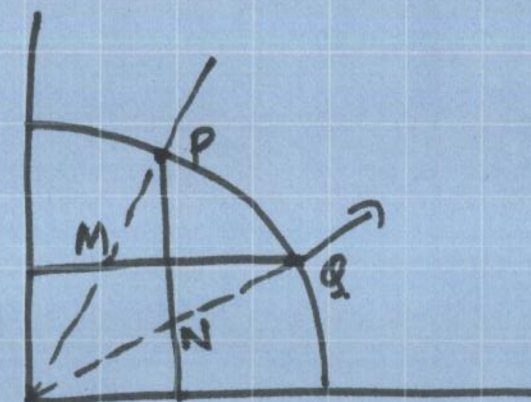
$$+ \frac{2}{h_3(h_1+h_3)} u_A + \frac{2}{h_4(h_2+h_4)} u_B = 0$$

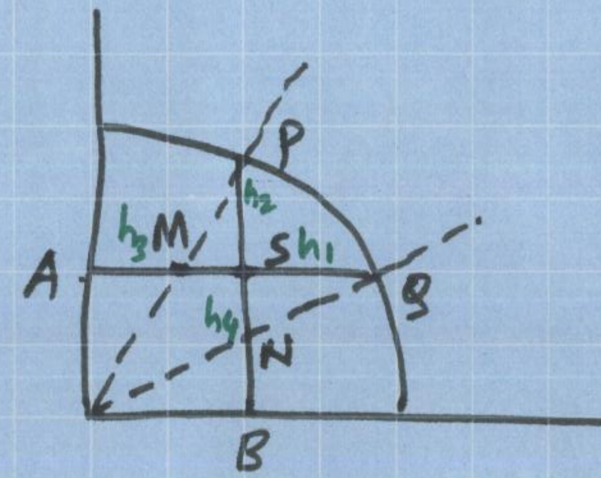
$$-\frac{4}{h_1 h_3} u_S + \frac{2}{(h_1+h_3)h_1} u_Q + \frac{2}{h_1(h_1+h_3)} u_P + \frac{2}{h_3(h_1+h_3)} u_A$$

$$+ \frac{2}{h_3(h_1+h_3)} u_B = 0 \quad \text{—— (*)}$$

$$\left(\frac{\partial u}{\partial n}\right)_Q = \frac{u_Q - u_N}{QN} + O(QN)$$

$$\left(\frac{\partial u}{\partial n}\right)_P = \frac{u_P - u_M}{PM} + O(PM)$$





$$u_N = \frac{SN \cdot u_B + BN \cdot u_S}{h_4}$$

$$u_M = \frac{SM \cdot u_A + AM \cdot U_S}{h_3}$$

using $\left(\frac{\partial u}{\partial n}\right)_P = (x^2-y^2)_P$, $\left(\frac{\partial u}{\partial n}\right)_Q = (x^2-y^2)_Q$

$$\frac{u_Q - u_N}{QN} = (x^2-y^2)_Q \Rightarrow \frac{u_Q - \dfrac{SN \cdot u_B + BN \cdot u_S}{h_4}}{QN} = (x^2-y^2)_Q .$$

$$\frac{u_P - u_M}{PM} = (x^2 - y^2)_P$$

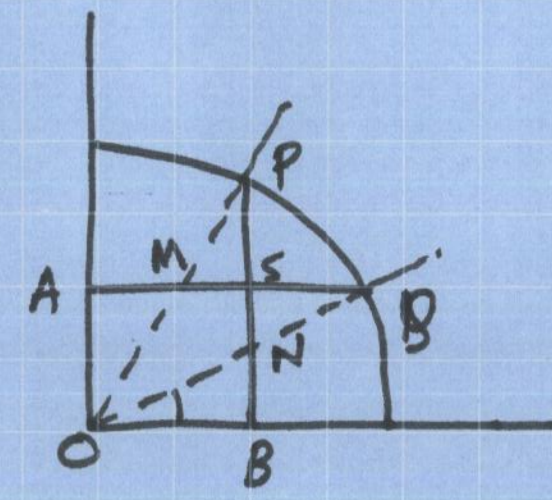
$$\Rightarrow \quad \frac{u_P - \dfrac{SM \cdot u_A + AM \cdot u_S}{h_3}}{PM} = (x^2 - y^2)_P$$

$u = 2$ on $x = 0,\ y = 0$ $\Rightarrow$ $u_A = 2,\ u_B = 2$

$$\therefore \quad \frac{u_Q - \dfrac{(2SN + BN \cdot u_S)}{h_3}}{QN} = (x^2 - y^2)_Q$$

$$\frac{u_P - \dfrac{(2SM + AM \cdot u_S)}{h_3}}{PM} = (x^2 - y^2)_P.$$

$\tan \pi/6 = \frac{NB}{OB} \Rightarrow NB = \frac{1}{2\sqrt{3}}$



$|||^{ly}$ $AM = \frac{1}{2\sqrt{3}}$

also $SQ = PS = \frac{\sqrt{3}-1}{2}$

$\therefore h_1 = \frac{\sqrt{3}-1}{2}, \quad h_3 = \frac{1}{2}$

$Q = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \Rightarrow QN = 1 - \frac{1}{\sqrt{3}}$

$PM = 1 - \frac{1}{\sqrt{3}}$

$$-\frac{4}{\frac{\sqrt{3}-1}{4}}u_s+\frac{2}{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}-1}{2}}u_q+\frac{2}{\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt{3}}{2}}u_p+\frac{8}{\sqrt{3}}\cdot 2+\frac{8}{\sqrt{3}}\cdot 2=0$$

$$\Rightarrow -\frac{16}{\sqrt{3}-1}u_s+\frac{8}{\sqrt{3}(\sqrt{3}-1)}u_q+\frac{8}{\sqrt{3}(\sqrt{3}-1)}u_p+\frac{32}{\sqrt{3}}=0.$$

$$u_p-\frac{2}{\sqrt{3}}(\sqrt{3}-1)-\frac{u_s}{\sqrt{3}}=-\frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right).$$

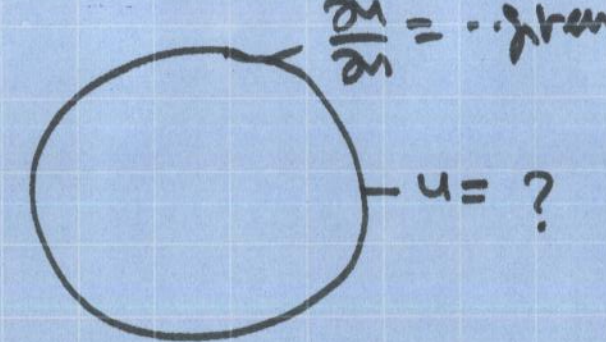
$$u_q-\frac{u_s}{\sqrt{3}}=\frac{5}{2}\,\frac{\sqrt{3}-1}{\sqrt{3}}$$

$$u_p=\frac{u_s}{\sqrt{3}}+\frac{3}{2}\,\frac{\sqrt{3}-1}{\sqrt{3}}$$

$$\frac{-16}{(\sqrt{3}-1)} u_s + \frac{8}{\sqrt{3}(\sqrt{3}-1)} u_\varphi + \frac{8}{\sqrt{3}(\sqrt{3}-1)} u_p = -\frac{32}{\sqrt{3}}$$

$$u_p = \frac{u_s}{\sqrt{3}} + \frac{3}{2}\frac{(\sqrt{3}-1)}{\sqrt{3}}$$

$$u_\varphi = \frac{u_s}{\sqrt{3}} + \frac{5}{2}\frac{(\sqrt{3}-1)}{\sqrt{3}}$$



$$\Rightarrow \quad u_p = \frac{3\sqrt{3}+1}{2\sqrt{3}}, \quad u_q = \frac{1}{2\sqrt{3}}(5\sqrt{3}-1)$$

$$u_s = 2$$