© CET
I.I.T. KGP

Partial Differential Equations

Finite Difference Methods

Parabolic | Elliptic | Hyperbolic

Method of characteristics for both 1st & 2nd order hyperbolic

Iterative methods for sparse systems

© CET
I.I.T. KGP

Finite Difference Approximations

Assumption: Functions and derivatives are single-valued, finite and continuous of the independent variables, say, x, y, z, t etc.

$$u(x+h) = u(x) + h\,u'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \cdots + R \quad \text{——} (1)$$

dependent variable | independent variable

© CET
I.I.T. KGP

$$u(x-h) = u(x) - h\,u'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x) + \cdots + R \quad \text{—— (2)}$$

Add (1) & (2) $\Longrightarrow$

$$u(x+h) + u(x-h) = 2u(x) + h^2 \boxed{u''(x)} + O(h^4)$$

$$u''(x) \simeq \frac{u(x+h) + u(x-h) - 2u(x)}{h^2} + O(h^2)$$

$$\simeq \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2). \quad \text{—— (3)}$$

© CET
I.I.T. KGP

Subtract ② from ①

$$u(x+h) - u(x-h) = 2h\,\boxed{u'(x)} + \frac{h^3}{3}\,u'''(x) + O(h^5)$$

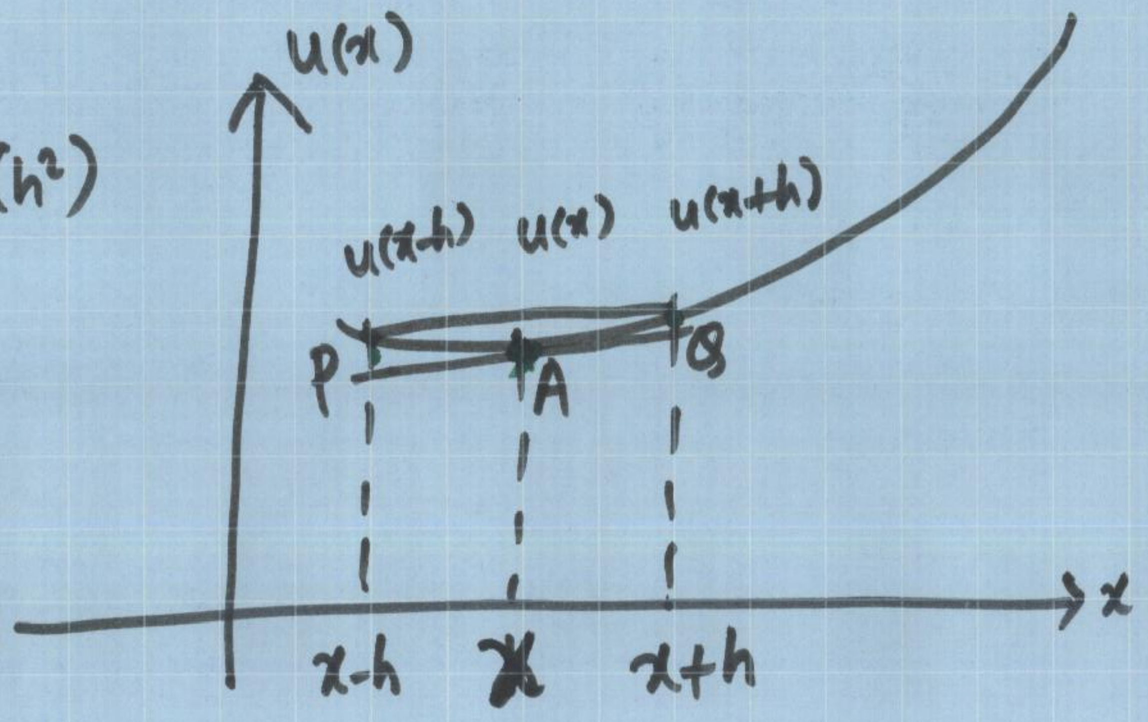
$$u'(x) \simeq \frac{u(x+h) - u(x-h)}{2h} + O(h^2) \quad \text{———} \ (*)$$

(*) approximates the slope of the tangent

© CET
I.I.T. KGP

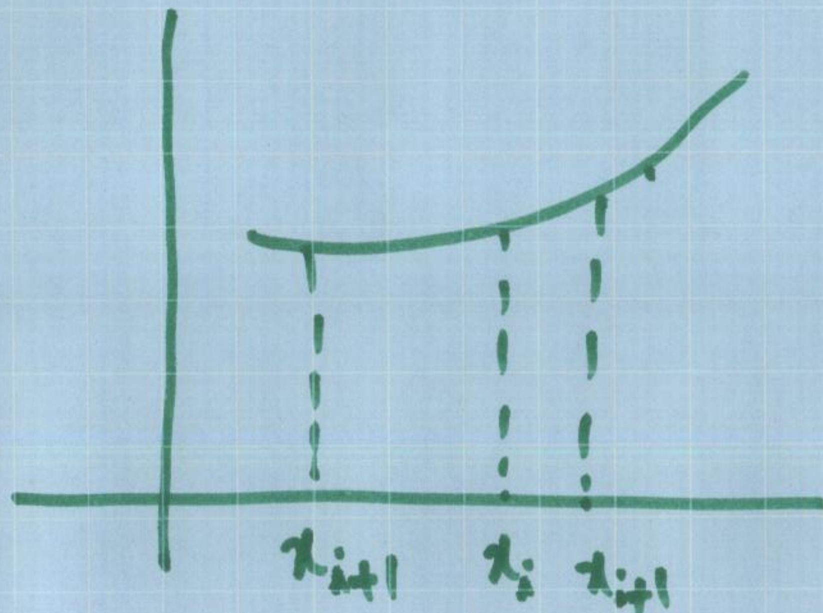
$$u'(x) \simeq \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

approximates the slope of the tangent at A by slope of the chord PQ

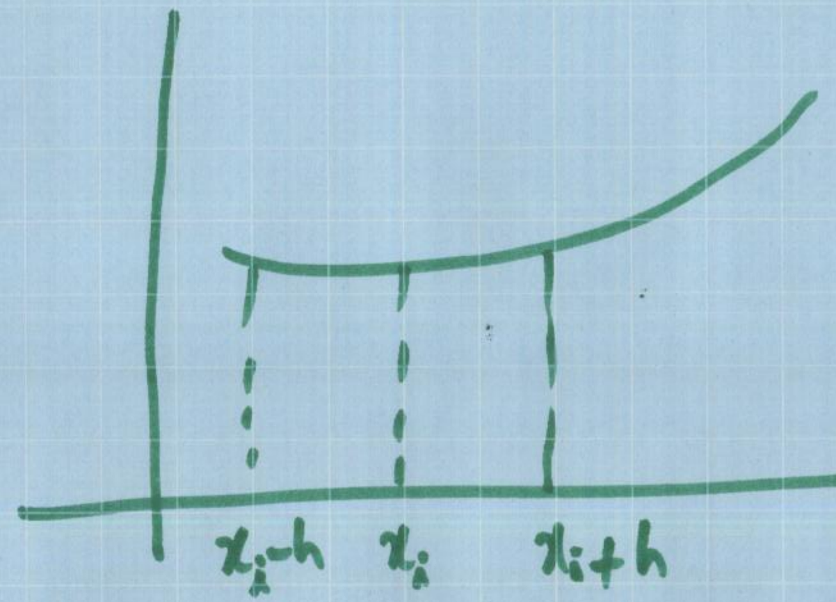


"central-difference" approximation

© CET
I.I.T. KGP



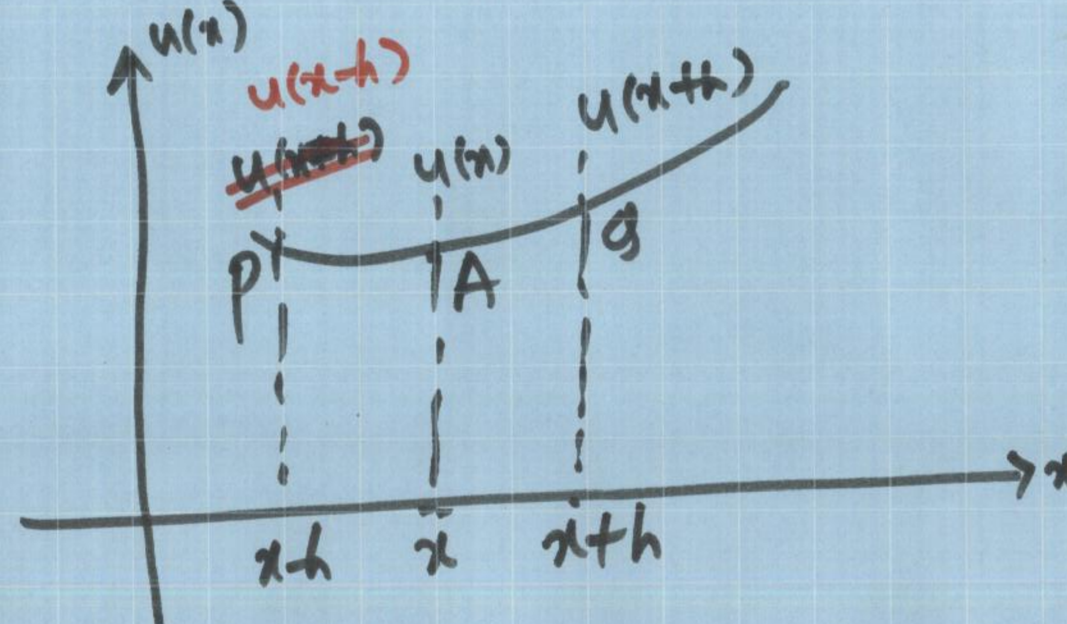
unequal spacing



equal spacing

$$u'(x_i) \simeq \frac{u(x_{i+1}) - u(x_{i-1})}{(x_{i+1} - x_{i-1})} + O(\Delta x_i^2)$$

© CET
I.I.T. KGP



One can approximate the slope of the tangent at A by slope of the chord AQ

$$u'(x) \simeq \frac{u(x+h) - u(x)}{h} \quad \text{-- forward}$$

...... by chord PA, then

$$u'(x) \simeq \frac{u(x) - u(x-h)}{h} \quad \text{-- backward}$$

© CET
I.I.T. KGP

Example. First order PDE

$$\frac{\partial}{\partial t} u(x,t) + a \frac{\partial}{\partial x} u(x,t) = 0 \quad \text{--- } (2.1)$$

(2.2) ... $u(x,0) = u_0(x)$ initial condition (at $t = t_0 = 0$)

(2.3) ... $u(0,t) = u_1(t)$ boundary condition (at $x = x_0 = 0$)

$x: \quad x_{i+1} = x_i + h, \quad i = 0, 1, \ldots$

$t: \quad t_{j+1} = t_j + k, \quad j = 0, 1, \ldots$

$$u(x,t)\Big|_{(x_i, t_j)} = u(x_i, t_j) = u_{i,j}$$

© CET
I.I.T. KGP

$$\frac{\partial}{\partial t} u(x,t) \simeq \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$ forward time

$$\frac{\partial}{\partial x} u(x,t) \simeq \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}$$ central space

$\Delta x = h$

$\Delta t = K$

put in (2.1),

$$a\, \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{u_{i,j+1} - u_{i,j}}{\Delta t} = 0$$

© CET
I.I.T. KGP

$$u_{i,j+1} - u_{i,j} + \frac{a\,\Delta t}{2\,\Delta x}\left(u_{i+1,j} - u_{i-1,j}\right) = 0$$

initial condition $\quad u(x,0) = u_0(x) = f(x)$

i.e $\quad u(x_i,0) = u_0(x_i)$

$$\boxed{u_{i,0} = u_0(x_i) = f_i \quad , \forall i}$$

boundary condition $\quad u(0,t) = u_1(t) = g(t)$

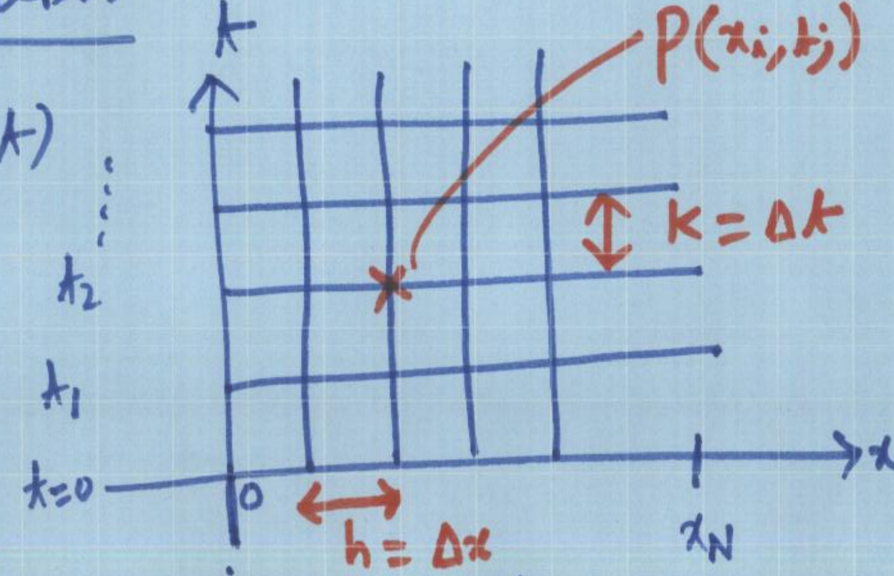
i.e $\quad u(0,t_j) = g(t_j)$

$$\boxed{u_{0,j} = g_j \qquad \forall j}$$

© CET
I.I.T. KGP

Functions of several variables

assume $u = u(x, t)$



$x_0 = 0, \quad x_1 = h \ \cdots$

$$\frac{\partial^2 u(x_i, t_j)}{\partial x^2} \simeq \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

$$\frac{\partial^2 u(x_i, t_j)}{\partial t^2} \simeq \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + O(k^2)$$

© CET
I.I.T. KGP

Parabolic Equations

Heat conduction $\quad \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$

thermal conductivity

non-dimensionalization : $\quad x' = x/L$
$\quad u' = u/u_0$

characteristic quantities

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial x} = \frac{\partial u}{\partial x'} \frac{1}{L}$$

© CET
I.I.T. KGP

$$\frac{1}{K L^{-2}} \frac{\partial u'}{\partial t} = \frac{\partial^2 u'}{\partial x'^2} \implies t' = \frac{Kt}{L^2}$$

$$\implies \frac{\partial u'}{\partial t'} = \frac{\partial^2 u'}{\partial x'^2} \implies \boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$$

© CET
I.I.T. KGP

Explicit Method

Consider $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ — (*)

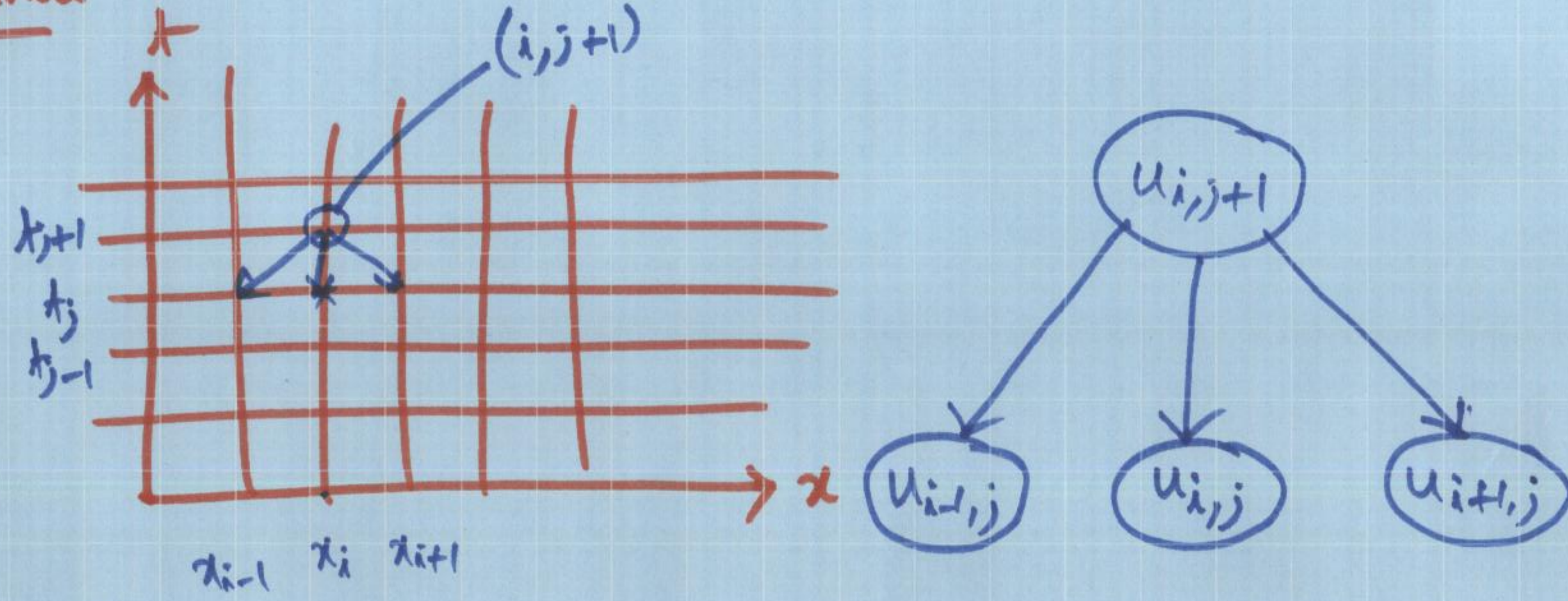
Approximate (*) by forward time and central space

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$+ O(k)$ $\qquad$ $+ O(h^2)$

© CET
I.I.T. KGP

Schmidt Method



$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i+1,j}$$

Explicit nature

© CET
I.I.T. KGP

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(k + h^2)$$

$$u_{i,j+1} = \lambda\, u_{i-1,j} + (1 - 2\lambda)\, u_{i,j} + \lambda\, u_{i+1,j} \quad \text{--- (E)}$$

$\lambda = \dfrac{k}{h^2}$ grid parameter

- Two-level method
- Explicit

one-level time

© CET
I.I.T. KGP

Example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 \le x \le 1$$

$u(x,0) = \sin \pi x$ —— i.c

$u(0,t) = 0; \quad u(1,t) = 0$ —— b.c

i.c $\Rightarrow$ $u(x_i, 0) = u_{i,0} = \sin \pi x_i$

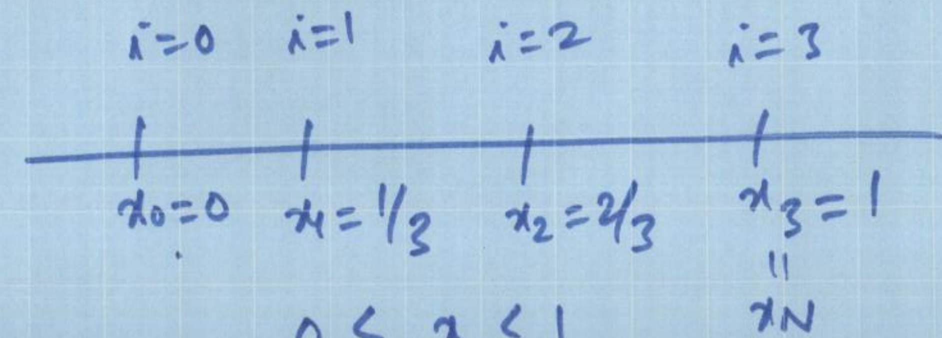
b.c $\Rightarrow$ $u(x_0, t_j) = 0 \; ; \; u(x_N, t_j) = 0$

$x_0 = 0$

$x_N = 1$

let $h = \frac{1}{3}$, $k = 1/36$; $\lambda = \frac{1}{4} = \frac{k}{h^2}$

© CET
I.I.T. KGP



$0 \le x \le 1$

$\underline{j=0}$

$$u_{i,0} = \sin \pi x_i;$$

$\underline{i=0}$ $\quad u_{0,0} = \sin \pi x_0 = 0$

$\underline{i=1}$ $\quad u_{1,0} = \sin \pi x_1 = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\underline{i=2}$ $\quad u_{2,0} = \sin \pi x_2 = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$u_{3,0} = 0$

© CET
I.I.T. KGP

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i+1,j}$$

$\underline{j=0}$

$\underline{i=1}$

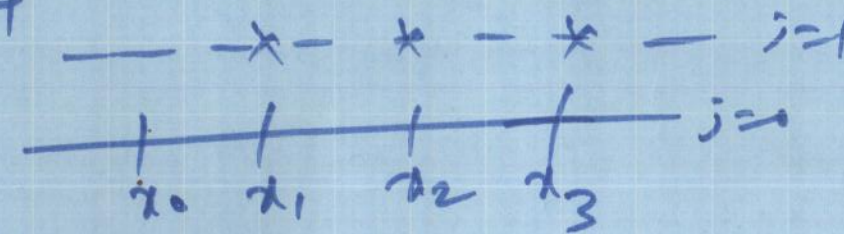
$$u_{1,1} = \lambda u_{0,0} + (1-2\lambda) u_{1,0} + \lambda u_{2,0}$$

$$= \frac{1}{4} u_{0,0} + \frac{1}{2} u_{1,0} + \frac{1}{4} u_{2,0}$$

$$= \frac{1}{4}\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

$i=2$

$$u_{2,1} = \frac{1}{4} u_{1,0} + \frac{1}{2} u_{2,0} + \frac{1}{4} u_{3,0} = \frac{3\sqrt{3}}{8}$$



© CET
I.I.T. KGP

Implicit Methods for parabolic PDEs

Consider $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

backward central

$$\frac{u_{i,j} - u_{i,j-1}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \boxed{O(k+h^2)}$$

© CET
I.I.T. KGP

writing the above approximation at level $(j+1)$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

$$\Rightarrow \quad -\lambda u_{i-1,j+1} + (1+2\lambda) u_{i,j+1} - \lambda u_{i+1,j+1} = u_{i,j}$$

- 2 levels; but more terms of level above

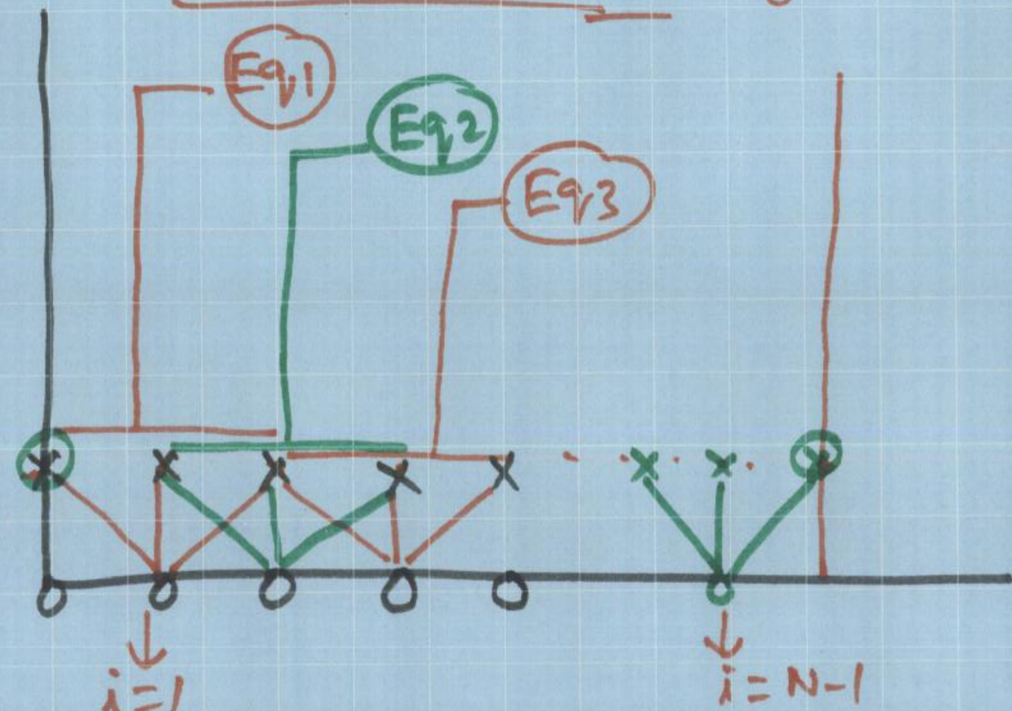
© CET
I.I.T. KGP

$$-\lambda u_{i-1,j+1} + (1+2\lambda)\, u_{i,j+1} - \lambda u_{i+1,j+1} = u_{i,j}$$

$$\begin{tikzcd}
t=\Delta t \\ j=1 & \boxed{u_{i-1,j+1}} \arrow[dr] & \boxed{u_{i,j+1}} \arrow[d] & \boxed{u_{i+1,j+1}} \arrow[dl] \\
t=0;\ j=0 & & \boxed{u_{i,j}} &
\end{tikzcd}$$

© CET
I.I.T. KGP

implicit → system of equations



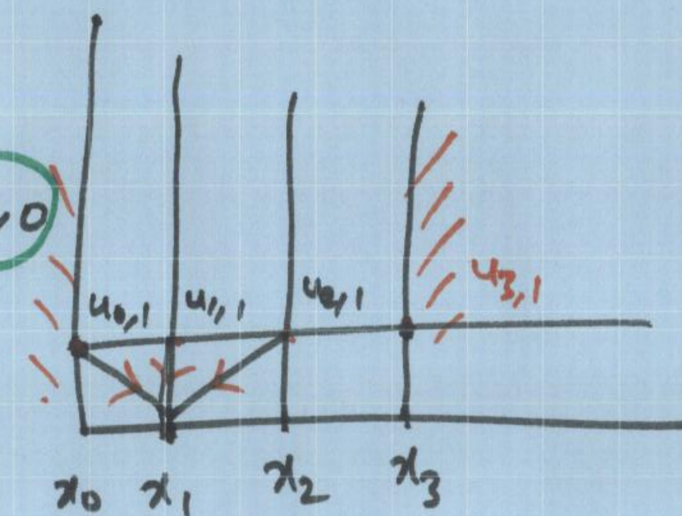
© CET
I.I.T. KGP

$\underline{j=0}$ $\quad -\lambda u_{i-1,1} + (1+2\lambda) u_{i,1} - \lambda u_{i+1,1} = u_{i,0}$

$\underline{i=1}$ $\quad -\lambda u_{0,1} + (1+2\lambda) \boxed{u_{1,1}} - \lambda \boxed{u_{2,1}} = u_{1,0}$

unknowns

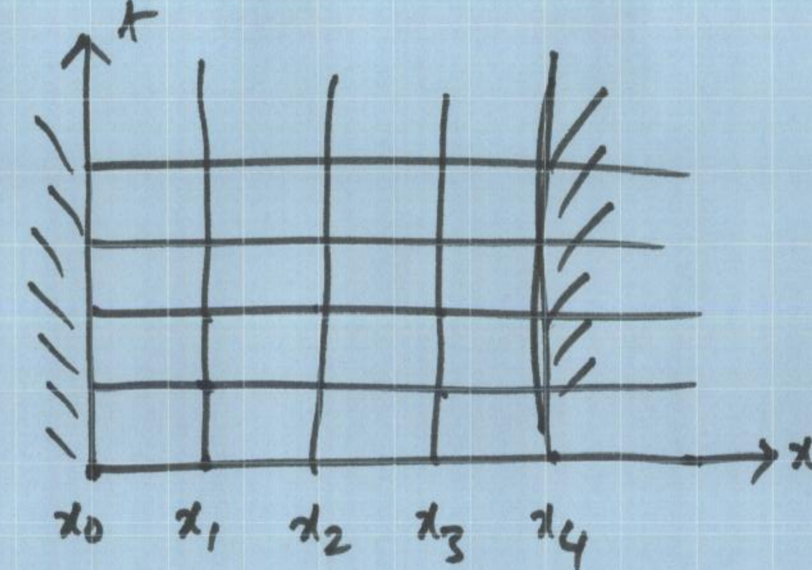
$\underline{i=2}$ $\quad -\lambda \boxed{u_{1,1}} + (1+2\lambda) \boxed{u_{2,1}} - \lambda u_{3,1} = u_{2,0}$



$\underline{i=3}$. λu.

© CET
I.I.T. KGP

$$-\lambda u_{i-1,j+1} + (1+2\lambda) u_{i,j+1} - \lambda u_{i+1,j+1} = u_{i,j}$$



$\underline{j=0}$

$$-\lambda u_{i-1,1} + (1+2\lambda) u_{i,1} - \lambda u_{i+1,1}$$
$$= u_{i,0}$$

© CET
I.I.T. KGP

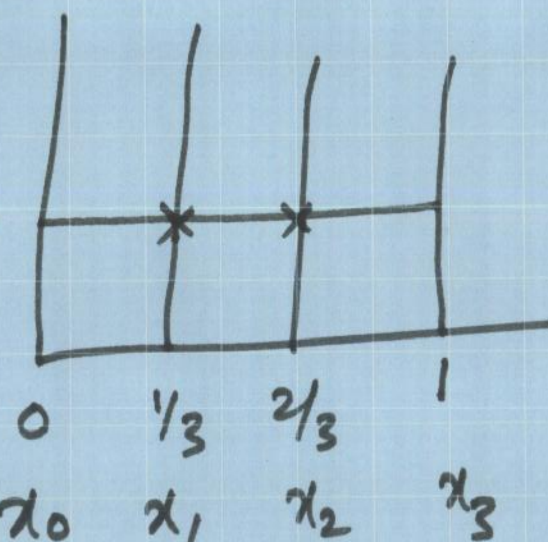
Example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$u(x,0) = \sin \pi x$, $0 \le x \le 1$, $\lambda = 1/4$

$u(0,t) = u(1,t)$,

$$-\lambda u_{i-1,j+1} + (1+2\lambda) u_{i,j+1} - \lambda u_{i+1,j+1} = u_{i,j}$$



$j=0$

$\cancel{-\lambda u_{i-1,j+1}}$

$$-\lambda u_{i-1,1} + (1+2\lambda) u_{i,1} - \lambda u_{i+1,1} = u_{i,0}$$

© CET
I.I.T. KGP

$\underline{i=1}$ $\quad -\lambda u_{0,1} + (1+2\lambda) u_{1,1} - \lambda u_{2,1} = u_{1,0}$

$\underline{i=2}$ $\quad -\lambda u_{1,1} + (1+2\lambda) u_{2,1} - \lambda u_{3,1} = u_{2,0}$

$$\Rightarrow \quad -\frac{1}{4} u_{0,1} + \frac{3}{2} \boxed{u_{1,1}} - \frac{1}{4} \boxed{u_{2,1}} = u_{1,0}$$

$$-\frac{1}{4} \boxed{u_{1,1}} + \frac{3}{2} \boxed{u_{2,1}} - \frac{1}{4} u_{3,1} = u_{2,0}$$

$\Rightarrow$ solve for $u_{1,1}$ and $u_{2,1}$

© CET
I.I.T. KGP

Local Truncation Error :

Consider $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ approximated by $\boxed{F_{i,j}(u) = 0}$

$$Lu = 0 \simeq L_{i,j}u = 0$$

Let $\bar{u}$ be the exact solution,

$$L_{i,j}u \simeq 0 \Rightarrow Lu - L_{i,j}u \simeq 0 = \boxed{T_{i,j}}$$

$$L_{i,j}u = \frac{u_{i,j+1} - u_{i,j}}{k} - \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \simeq 0$$

$$+O(k) \qquad\qquad +O(h^2)$$

© CET
I.I.T. KGP

$$T_{i,j} = \frac{\bar{u}_{i,j+1} - \bar{u}_{i,j}}{k} - \frac{\bar{u}_{i-1,j} - 2\bar{u}_{i,j} + \bar{u}_{i+1,j}}{h^2}$$

Taylor's expansion

$$\begin{aligned}\bar{u}_{i+1,j} &= \bar{u}(x_{i+1}, t_j) \\ &= \bar{u}(x_i, t_j) + h\frac{\partial \bar{u}}{\partial x_i}\Big|_{(x_i,t_j)} + \frac{h^2}{2!}\frac{\partial^2 \bar{u}}{\partial x_i^2}\Big|_{(x_i,t_j)} + \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x^3} + \cdots\end{aligned}$$

$$\bar{u}_{i-1,j} = \bar{u}(x_i, t_j) - h\frac{\partial \bar{u}}{\partial x_i}\Big|_{(x_i,t_j)} + \frac{h^2}{2!}\frac{\partial^2 \bar{u}}{\partial x^2}\Big|_{(x_i,t_j)} \cancel{+} - \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x^3} + \cdots$$

$$\bar{u}_{i,j+1} = \bar{u}(x_i, t_j) + k\frac{\partial \bar{u}}{\partial t}\Big|_{(x_i,t_j)} + \frac{k^2}{2!}\frac{\partial^2 \bar{u}}{\partial t^2}\Big|_{(x_i,t_j)} + \frac{k^3}{6}\frac{\partial^3 \bar{u}}{\partial t^3} + \cdots$$

© CET
I.I.T. KGP

$$T_{i,j} = \frac{\bar{u}_{i,j+1} - \bar{u}_{i,j}}{k} - \frac{\bar{u}_{i-1,j} - 2\bar{u}_{i,j} + \bar{u}_{i+1,j}}{h^2}$$

$$= \frac{1}{k}\left[\bar{u} + k\frac{\partial \bar{u}}{\partial t} + \frac{k^2}{2}\frac{\partial^2 \bar{u}}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 \bar{u}}{\partial t^3} + \cdots - \bar{u} \right]\Bigg|_{(x_i, t_j)}$$

$$- \frac{1}{h^2}\left[\bar{u} - h\frac{\partial \bar{u}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 \bar{u}}{\partial x^4} - \cdots \right.$$

$$+ \frac{h^6}{\vdots}$$

$$- 2\bar{u}$$

$$\left. + \bar{u} + h\frac{\partial \bar{u}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 \bar{u}}{\partial x^4} + \cdots \right]\Bigg|_{(x_i, t_j)}$$

$$+ \frac{h^6}{\vdots}$$

© CET
I.I.T. KGP

$$T_{i,j} = \underbrace{\boxed{\left(\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}}{\partial x^2}\right)}}_{=0} + \frac{k}{2}\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{h^2}{12}\frac{\partial^4 \bar{u}}{\partial x^4}$$

$$+ \frac{k^2}{6}\frac{\partial^3 \bar{u}}{\partial t^3} - \frac{1}{360} h^4 \frac{\partial^6 \bar{u}}{\partial x^6}$$

$\because$ $\bar{u}$ is the exact solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, we have $\boxed{} = 0$

the leading non-zero term (principal part) of the local truncation error is $\left(\frac{1}{2} k \frac{\partial^2 \bar{u}}{\partial t^2} - \frac{1}{12} h^2 \frac{\partial^4 \bar{u}}{\partial x^4}\right)$

© CET
I.I.T. KGP

$$T_{i,j} = k\frac{1}{2}\frac{\partial^2 \bar{u}}{\partial t^2} - h^2\frac{1}{12}\frac{\partial^4 \bar{u}}{\partial x^4}$$

$$+ k^2\frac{1}{6}\frac{\partial^3 \bar{u}}{\partial t^3} - h^4\frac{1}{360}\frac{\partial^6 \bar{u}}{\partial x^6} + \cdots$$

$$= k\frac{1}{2}\frac{\partial^2 \bar{u}}{\partial t^2} - h^2\frac{1}{12}\frac{\partial^4 \bar{u}}{\partial x^4} + O(k^2 + h^4)$$

$$\frac{\partial \cdot}{\partial t} = \frac{\partial^2 \cdot}{\partial x^2} \Rightarrow \frac{\partial}{\partial t}\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial^4 u}{\partial x^4}$$

$$\therefore T_{i,j} = \underbrace{\left(\frac{k}{2} - \frac{h^2}{12}\right)}_{= 0 \text{ if } \frac{6k}{h^2} = 1}\frac{\partial^4 \bar{u}}{\partial x^4} + O(k^2 + h^4)$$

$$T_{i,j} \simeq \left(k \boxed{\frac{1}{2} \frac{\partial^2 \bar{u}}{\partial t^2}} - h^2 \boxed{\frac{1}{12} \frac{\partial^4 \bar{u}}{\partial x^4}} \right) \Bigg|_{(x_i, t_j)}$$

$$+ k^2 \frac{1}{6} \frac{\partial^3 \bar{u}}{\partial t^3} - h^4 \frac{1}{360} \frac{\partial^6 \bar{u}}{\partial x^6} + \cdots$$

lead term : $c_1 k + c_2 h^2 \in O(k + h^2)$

$\therefore T_{i,j} \in O(k + h^2)$

Can we minimize the error further?

Yes, in this case!

© CET I.I.T. KGP

$T_{i,j} \simeq O(k^2 + h^4)$ for $k = h^2/6$

© CET
I.I.T. KGP

Other Numerical Methods for parabolic PDEs

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

forward time

at (x_i, t_j) by average of values at j & $(j+1)$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}\left\{ \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \right\}$$

© CET
I.I.T. KGP

with $\lambda = k/h^2$

$$\Rightarrow \quad -\lambda u_{i-1,j+1} + 2(1+\lambda) u_{i,j+1} - \lambda u_{i+1,j+1}$$

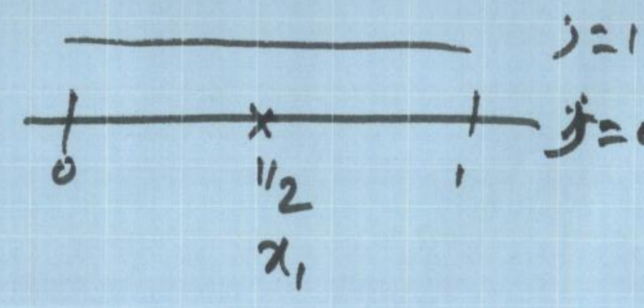
$$= \lambda u_{i-1,j} + 2(1-\lambda) u_{i,j} + \lambda u_{i+1,j}$$

"Crank - Nicolson Implicit scheme"

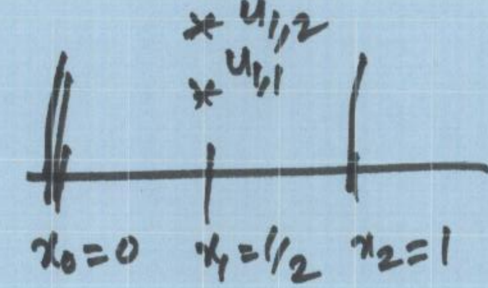
Example $\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(1,t) = 0, \ \forall t \geq 0$

$u(x,0) = x - x^2, \quad 0 \leq x \leq 1$

$h = 1/2, \ k = 1/4 \Rightarrow \lambda = 1$



© CET
I.I.T. KGP



$$u(0,t)=0 \implies u_{0,j}=0$$

$$u(1,t)=0 \implies u_{2,j}=0$$

$$u(x,0)=x-x^2 \implies u_{i,0}=x_i-x_i^2$$

$$u_{0,0}=0, \quad u_{1,0}=x_1-x_1^2=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}, \quad \lambda=1$$

$$u_{2,0}=0$$

$$-\lambda u_{i-1,j+1}+2(1+\lambda)u_{i,j+1}-\lambda u_{i+1,j+1}$$
$$=\lambda u_{i-1,j}+2(1-\lambda)u_{i,j}+\lambda u_{i+1,j}$$

$\underline{j=0}$, $i=1$ $\implies -\underbrace{u_{0,1}}_{=0}+4u_{1,1}-\underbrace{u_{2,1}}_{=0}=\underbrace{u_{0,0}}_{=0}+\underbrace{u_{2,0}}_{=0} \implies u_{1,1}=0$

$\underline{j=1}$ $\implies -\underbrace{u_{0,2}}_{=0}+4u_{1,2}-\underbrace{u_{2,2}}_{=0}=\underbrace{u_{0,1}}_{=0}+\underbrace{u_{2,1}}_{=0} \implies u_{1,2}=0$

© CET
I.I.T. KGP

Consistency, Stability and Convergence

Convergence

A one-step finite difference scheme approximating a PDE is a convergent scheme if the solution of the finite difference scheme $u_{i,j}$ converges to $\bar{u}(x,t)$, any solution of the PDE as $\Delta x, \Delta t \to 0$.

© CET
I.I.T. KGP

Stability: The error caused by a small perturbation in the numerical method remains bounded.

- Could happen unconditionally in the entire domain
- Conditionally within a range

Consistency: Given a PDE $Lu = f$ approximated by $L_{i,j} u = f$.

The finite difference scheme $L_{i,j} u = f$ is consistent with the PDE if $(L\psi - L_{i,j}\psi) \longrightarrow 0$ as $\Delta x, \Delta t \longrightarrow 0$ for ψ smooth enough.

© CET
I.I.T. KGP

Lax- Richtmyer Equivalence Theorem

(Fundamental Theorem of Numerical Analysis)

consistency + stability $\Longleftrightarrow$ convergence

If a linear finite difference scheme is consistent with a well defined linear IVP then stability guarantees convergence as mesh length $\to 0$.

- "Linear"

© CET
I.I.T. KGP

Matrix Stability Analysis

stability $\rightarrow$ boundedness of the solution

let $\bar{u}_{i+1} = A\,\bar{u}_i + \bar{b}_i$ be an approximation

idea: for errors to be bounded, one expects a condition on A !

© CET
I.I.T. KGP

assume that the values at $(j+1)$ and j are related by

$$\boxed{\begin{aligned} & b_{i-1}\, u_{i-1,j+1} + b_i\, u_{i,j+1} + b_{i+1,j+1} \\ & \qquad = c_{i-1}\, u_{i-1,j} + c_i\, u_{i,j} + c_{i+1}\, u_{i+1,j} \end{aligned}} \quad (*)$$

if the boundary values at $i = 0$ ~~and~~ and N, $j > 0$ are known

run $(*)$ at $i = 1$ to $N-1$

<u>$i=1$</u>

$$b_0 \overset{\text{known}}{\boxed{u_{0,j+1}}} + b_1\, u_{1,j+1} + b_2\, u_{2,j+1}$$

$$= c_0 \overset{\text{known}}{\boxed{u_{0,j}}} + c_1\, u_{1,j} + c_2\, u_{2,j}$$

© CET
I.I.T. KGP

$\underline{i=2}$

$$b_1 u_{1,j+1} + b_2 u_{2,j+1} + b_3 u_{3,j+1}$$
$$= c_1 u_{1,j} + c_2 u_{2,j} + c_3 u_{3,j}$$

$\vdots$

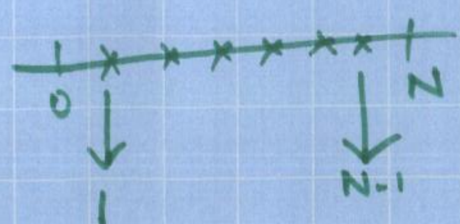
$\underline{i=N-2}$

$$b_{N-3} u_{N-3,j+1} + b_{N-2} u_{N-2,j+1} + b_{N-1} u_{N-1,j+1}$$
$$= c_{N-3} u_{N-3,j} + c_{N-2} u_{N-2,j} + c_{N-1} u_{N-1,j}$$

$\underline{i=N-1}$

known

$$b_{N-2} u_{N-2,j+1} + b_{N-1} u_{N-1,j+1} + b_N \boxed{u_{N,j+1}}$$
$$= c_{N-2} u_{N-2,j} + c_{N-1} u_{N-1,j} + c_N \boxed{u_{N,j}}$$



© CET
I.I.T. KGP

$$\underbrace{\begin{bmatrix} b_1 & b_2 & 0 & & & & \\ b_1 & b_2 & b_3 & 0 & 0 & \cdots & \\ \vdots & & & & & & \\ \vdots & & & & \cdots & & \\ & & & & b_{N-3} & b_{N-2} & b_{N-1} \\ \vdots & & & & 0 & b_{N-2} & b_{N-1} \end{bmatrix}}_{B} \overbrace{\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{N-2,j+1} \\ u_{N-1,j+1} \end{bmatrix}}^{\bar{u}_{j+1}} \quad \text{——} \; ** $$

$i = 1 \ldots N-1$

$j = 0 \ldots J$

$$= \underbrace{\begin{bmatrix} c_1 & c_2 & & & \\ c_1 & c_2 & c_3 & & \\ & & \cdot & & \\ & & c_{N-3} & c_{N-2} & c_{N-1} \\ & & 0 & c_{N-2} & c_{N-1} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-2,j} \\ u_{N-1,j} \end{bmatrix}}_{\bar{u}_j} + \underbrace{\begin{bmatrix} c_0 u_{0,j} - b_0 u_{0,j+1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ c_N u_{N,j} - b_N u_{N,j+1} \end{bmatrix}}_{d_j}$$

© CET
I.I.T. KGP

(**) can be put in a form

$$B\,\bar{u}_{j+1} = C\,\bar{u}_j + d_j$$

$$\Rightarrow \bar{u}_{j+1} = B^{-1}C\,\bar{u}_j + B^{-1}d_j$$

$$\Rightarrow \bar{u}_{j+1} = A\bar{u}_j + f_j \quad ; \quad A = B^{-1}C$$

$$f_j = B^{-1}d_j$$

$$\bar{u}_j = A\,\bar{u}_{j-1} + f_{j-1} = A\left(A\,\bar{u}_{j-2} + f_{j-2}\right) + f_{j-1}$$

$$= A^2\,\bar{u}_{j-2} + A f_{j-2} + f_{j-1}$$

$$= \cdots$$

$$= A^j\,\bar{u}_0 + A^{j-1}f_0 + A^{j-2}f_1 + \cdot + A f_{j-2} + f_{j-1}$$

© CET
I.I.T. KGP

$$\bar{u}_j = A^j \bar{u}_0 + A^{j-1} f_0 + A^{j-2} f_1 + \cdots + A f_{j-2} + f_{j-1} \quad \text{— (A)}$$

initial values

let us perturb $\bar{u}_0$ to $\bar{u}_0^*$, then

$$\bar{u}_0^* = A^j \bar{u}_0^* + A^{j-1} f_0 + A^{j-2} f_1 + \cdots + A f_{j-2} + f_{j-1} \quad \text{— (B)}$$

define $\bar{e}_j = \bar{u}_j - \bar{u}_j^*$

(A) & (B) $\Rightarrow$ $\bar{e}_j = A \bar{e}_{j-1} = A^2 \bar{e}_{j-2} = \cdots = A^j \bar{e}_0, \; j = 1 \ldots J$

$$\therefore \; \bar{e}_j = A^j \bar{e}_0$$

with respect to a suitable norm (vector & matrix)

$$\underline{\|\bar{e}_j\| \leq \|A^j\| \, \|\bar{e}_0\|}$$

© CET
I.I.T. KGP

$$\|\bar{e}_j\| \le \|A^j\| \, \|\bar{e}_0\|$$

As per Lax–Richtmyer, difference scheme is stable when $\exists$ a +ve number M, independent of j, h and k $\ni$

$$\|\bar{e}_j\| \le M \|\bar{e}_0\|$$ i.e amplification of initial rounding is limited

look for $\|A^j\| \le M$

OR $\|A^j\| = \|A A^{j-1}\| \le \|A\| \, \|A^{j-1}\| \le \cdots \le \|A\|^j$

method is stable if $\boxed{\|A\| \le 1}$

© CET
I.I.T. KGP

Convergence via differential equation for the error

(Direct method)

Consider $\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial x^2}$, $0 < x < 1$, $t > 0$ —— (*)

i.c $\bar{u}(x, 0) =$ given

b.c $\bar{u}(a, t) =$ given; $\bar{u}(b, t) =$ given $\forall t$

$a = 0$, $b = 1$

finite difference approximating (*) by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

© CET
I.I.T. KGP

at mesh points $u_{i,j} = \bar{u}_{i,j} - \boxed{e_{i,j}}$ error

$$u_{i,j+1} = \bar{u}_{i,j+1} - e_{i,j+1} \quad \text{etc.}$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$\Rightarrow \boxed{u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i+1,j}}$$

$$e_{i,j+1} = \lambda e_{i-1,j} + (1-2\lambda) e_{i,j} + \lambda e_{i+1,j}$$
$$+ \bar{u}_{i,j+1} - \bar{u}_{i,j} + \lambda (2\bar{u}_{i,j} - \bar{u}_{i-1,j} - \bar{u}_{i+1,j})$$

© CET
I.I.T. KGP

$$\bar{u}_{i+1,j} = \bar{u}(x_i+h, t_j) = \bar{u}_{i,j} + h \left.\frac{\partial \bar{u}}{\partial x}\right|_{(x_i,t_j)} + \frac{h^2}{2}\frac{\partial^2 \bar{u}}{\partial x^2}(x_i+\theta_1 h, t_j)$$

$$\bar{u}_{i-1,j} = \bar{u}(x_i-h, t_j) = \bar{u}_{i,j} - h\frac{\partial \bar{u}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \bar{u}}{\partial x^2}(x_i-\theta_2 h, t_j)$$

$$\bar{u}_{i,j+1} = \bar{u}_{i,j} + k\frac{\partial \bar{u}}{\partial t}(x_i, \cancel{t_j}+\theta_3 k) \quad , \quad \begin{array}{l} 0<\theta_1<1 \\ 0<\theta_2<1 \\ 0<\theta_3<1 \end{array}$$

$$e_{i,j+1} = \lambda e_{i-1,j} + (1-2\lambda)e_{i,j} + \lambda e_{i+1,j} \qquad (**)$$

$$+ k\left\{\frac{\partial \bar{u}}{\partial t}(x_i, t_j+\theta_3 k) - \frac{\partial^2 \bar{u}}{\partial x^2}(x_i+\theta_4 h, t_j)\right\}$$

$$-1<\theta_4<1$$

(**) is a difference equation for $e_{i,j}$

© CET
I.I.T. KGP

let $E_j = \max(e_{i,j})$ — j-level

$M = \max\left(\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}}{\partial x^2}\right) \quad \forall i, j$

for $\lambda \le 1/2$, coefficients of $e_{i,j}$ in (**) are +ve or zero.

$$\therefore \quad |e_{i,j+1}| \le \lambda |e_{i-1,j}| + (1-2\lambda)|e_{i,j}| + \lambda |e_{i+1,j}| + kM$$

$$\le \lambda E_j + (1-2\lambda) E_j + \lambda E_j + kM$$

$$= E_j + kM$$

© CET
I.I.T. KGP

$|e_{i,j+1}| \leq E_j + kM$, which is true $\forall i$,

true for $\max |e_{i,j+1}|$

$$\therefore \quad E_{j+1} \leq E_j + kM \leq E_{j-1} + kM + kM \leq E_{j-2} + 3kM$$

$$\leq E_0 + jkM = jkM \quad (\because E_0 = 0)$$

$$= tM \quad (\because jk = t)$$

Further, when $h \to 0$, $k = \lambda h^2 \to 0$, $M \to \left(\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}}{\partial x^2}\right)\Big|_{i,j} \to 0$

$\therefore E_{j+1} \to 0$

as $|\bar{u}_{i,j+1} - u_{i,j}| \leq E_j \Rightarrow$ $\boxed{u \to \bar{u} \text{ as } h \to 0 \text{ when } \lambda \leq 1/2,\ t\text{-finite}}$

$\lambda > 1/2$: blows up

© CET
I.I.T. KGP

Fourier Series Stability Analysis

(Von Neumann's stability)

idea: Assume that the initial error is expanded in terms of a Fourier series and obtain conditions restricting the amplification.

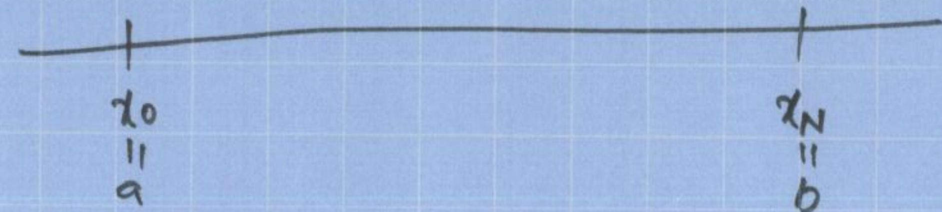
Consider Explicit scheme (Schmidt's method)

$$u_{i,j+1} = (1-2\lambda)u_{i,j} + \lambda(u_{i-1,j} + u_{i+1,j}) \quad \text{— SM}$$

error satisfies $\epsilon_{i,j+1} = (1-2\lambda)\epsilon_{i,j} + \lambda(\epsilon_{i-1,j} + \epsilon_{i+1,j}) \quad \text{— e}$

$$\epsilon_{i,j+1} = \epsilon_{p,q_n} = \sum_q A_q e^{i\beta ph} \underbrace{e^{\alpha qk}}_{= \xi^q}$$

© CET
I.I.T. KGP



$$l = (b-a) = Nh$$

$$x = x_0 + mh$$

$$m = 0, 1, \dots N$$

$$u_{i,j} = u_{p,q} = u(ph, qk)$$

initial values at $t=0$: $u(ph, 0) = u_{p,0}$, $p = 0, 1 \dots N$

initial error $\epsilon_{p,0} = \sum_{n=0}^{N} A_n e^{i\beta x} e^{\alpha t}$

$$= \sum_{n=0}^{N} A_n e^{i\beta ph} e^{\alpha qk} \Big|_{t=0} = \sum_{n=0}^{N} A_n e^{i\beta ph}$$

$$\boxed{\epsilon_{p,q} = \sum_{q} A_q e^{i\beta ph} e^{\alpha qk}}$$

© CET
I.I.T. KGP

$$\epsilon_{p,0} = \sum_{n=0}^{N} A_n e^{i\beta p h}$$

$$\epsilon_{p,q} = \sum_{q} A_q e^{i\beta p h} e^{\alpha q k} = \sum A_q e^{i\beta p h} \xi^q, \quad \xi = e^{\alpha k}$$

note that the scheme is linear, just consider one term

$$\boxed{\epsilon_{p,q} = A \xi^q e^{i\beta p h}} \text{ — } (*), \quad \xi \text{ — amplification factor}$$

<u>Lax theorem</u> : finite difference scheme is stable if $|u_{p,q}|$ is bounded

$$\Rightarrow |\xi| \leq 1$$

substituting $(*)$ form in (2)

© CET
I.I.T. KGP

$$A\,\xi^{q+1} e^{i\beta p h} = (1-2\lambda) A\,\xi^{q} e^{i\beta p h} + \lambda\left(A\,\xi^{q} e^{i\beta(p-1)h} + A\,\xi^{q} e^{i\beta(p+1)h}\right)$$

$$\Rightarrow \quad \xi = (1-2\lambda) + \lambda\left(e^{-i\beta h} + e^{i\beta h}\right)$$

$$= 1 + \lambda\left(e^{i\beta h} - 2 + e^{-i\beta h}\right)$$

$$= 1 + 2\lambda(\cos\beta h - 1)$$

$$= 1 - 4\lambda \sin^2 \beta h/2$$

$$|\xi| \le 1 \Rightarrow -1 \le 1 - 4\lambda\sin^2\beta h/2 \le 1 \Rightarrow \lambda \le \frac{1}{2\,\underbrace{\sin^2 \beta h/2}}$$

$$\Rightarrow \boxed{0 \le \lambda \le 1/2}$$

© CET
I.I.T. KGP

Example Discuss the stability of the finite difference method

$$\frac{1}{k}\left(u_{p,q+1} - u_{p,q}\right) = \frac{a}{3}\frac{1}{h^2}\left(u_{p-1,q} - 2u_{p,q} + u_{p+1,q}\right) + b\,u_{p,q}$$

approximating $\dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + bu$, $\quad a, b-$ constants

Sol. the equation satisfied by the error is

$$e^{i\beta ph}\left(\xi^{q+1} - \xi^{q}\right) = \frac{a}{3}\lambda\left(\xi^q e^{i\beta(p-1)h} - 2\xi^q e^{i\beta ph} + \xi^q e^{i\beta(p+1)h}\right) + kb\,\xi^q e^{i\beta ph}$$

$$\Rightarrow \quad (\xi - 1) = \frac{\lambda a}{3}\left(e^{-i\beta h} - 2 + e^{i\beta h}\right) + kb$$

© CET
I.I.T. KGP

$$\Rightarrow \quad \xi = 1 - 4\frac{\lambda a}{3}\sin^2 \beta h/2 + kb$$

$|\xi| \leq \left|1 - 4\frac{\lambda a}{3}\sin^2\beta h/2\right| + kb$ if we need $|\xi| \leq 1$. then

$$\Rightarrow \quad \boxed{\lambda \leq \frac{3}{2}a}$$

© CET
I.I.T. KGP

Sol: $u_{p,q+1} - u_{p,q} = \lambda \left(u_{p-1,q} - 2u_{p,q} + u_{p+1,q} \right)$

$$\epsilon_{p,q+1} - \epsilon_{p,q} = \lambda \left(\epsilon_{p-1,q} - 2\epsilon_{p,q} + \epsilon_{p+1,q} \right)$$

$$e^{i\beta p h} \xi^{q+1} - e^{i\beta p h} \xi^{q} = \lambda \left(e^{i\beta (p-1) h} \xi^{q} - 2 e^{i\beta p h} \xi^{q} + e^{i\beta (p+1) h} \xi^{q} \right)$$

$$\Rightarrow \quad \xi - 1 = \lambda \left(e^{-i\beta h} - 2 + e^{i\beta h} \right)$$

$$\Rightarrow \quad \xi = 1 - 2\lambda (1 - \cos\beta h)$$

$|\xi| \le 1 \Rightarrow \lambda \le \dfrac{1}{(1 - \cos\beta h)}$ need a tighter ~~value~~ bound

$$\Rightarrow \boxed{\lambda \le 1/2}$$

© CET
I.I.T. KGP

Matrix - Stability - Crank - Nicolson scheme

$$-\lambda u_{i-1,j+1} + 2(1+\lambda) u_{i,j+1} - \lambda u_{i+1,j+1}$$
$$= \lambda u_{i-1,j} + 2(1-\lambda) u_{i,j} + \lambda u_{i+1,j} \quad , i = 1 \cdots N-1$$

$$\begin{bmatrix} 2(1+\lambda) & -\lambda & 0 & & \\ -\lambda & 2(1+\lambda) & -\lambda & & \\ & & & & \\ & & -\lambda & 2(1+\lambda) & -\lambda \\ & & 0 & -\lambda & 2(1+\lambda) \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ \vdots \\ \vdots \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda) & \lambda & 0 & \\ \lambda & 2(1-\lambda) & \lambda & \\ & & & \\ & & \cdots & \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-1,j} \end{bmatrix}$$

$$+ b_i$$