Spectral theorem and its applications

M. Rajesh Kannan

Department of Mathematics,
Indian Institute of Technology Kharagpur,
email: rajeshkannan1.m@gmail.com, rajeshkannan@maths.iitkgp.ernet.in

24-08-2018 , 28-08-2018

References

A. Ramachandra Rao and P Bhimasankaram , Linear Algebra, Second edition, Hindustan book agency.

Outline

- Eigenvalues and eigenvectors,

Outline

- Eigenvalues and eigenvectors,
- Spectral representation of semi-simple matrices,

Outline

- Eigenvalues and eigenvectors,
- Spectral representation of semi-simple matrices,
- Spectral theorem symmetric matrices, Hermitian matrices and normal matrices,

Outline

- Eigenvalues and eigenvectors,
- Spectral representation of semi-simple matrices,
- Spectral theorem symmetric matrices, Hermitian matrices and normal matrices,
- Schur's lemma,

Outline

- Eigenvalues and eigenvectors,
- Spectral representation of semi-simple matrices,
- Spectral theorem symmetric matrices, Hermitian matrices and normal matrices,
- Schur's lemma,
- Singular value decomposition.

Eigenvalues and eigenvectors

Definition

A complex number λ is said to be an eigenvalue of an $n \times n$ complex matrix A, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.

Eigenvalues and eigenvectors

Definition

A complex number λ is said to be an eigenvalue of an $n \times n$ complex matrix A, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. The vector x is said to be an eigenvector associated with the eigenvalue λ.

Eigenvalues and eigenvectors

Definition

A complex number λ is said to be an eigenvalue of an $n \times n$ complex matrix A, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. The vector x is said to be an eigenvector associated with the eigenvalue λ.

Theorem

A complex number λ is an eigenvalue of a complex matrix A if and only if λ is a root of the characteristic polynomial $\det(A - \lambda I) = 0$.

Eigenvalues and eigenvectors

Definition

A complex number λ is said to be an eigenvalue of an $n \times n$ complex matrix A, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. The vector x is said to be an eigenvector associated with the eigenvalue λ.

Theorem

A complex number λ is an eigenvalue of a complex matrix A if and only if λ is a root of the characteristic polynomial $\det(A - \lambda I) = 0$.

Definition

Two $n \times n$ matrices A and B are said to be similar, if there exists an invertible matrix C such that $B = C^{-1}AC$.

Algebraic and geometric multiplicity

Definition

For an eigenvalue λ of A, the subspace of all eigenvectors of A corresponding to the eigenvalue λ together with the zero vector is called the eigenspace of A corresponding to the λ.

Algebraic and geometric multiplicity

Definition

For an eigenvalue λ of A, the subspace of all eigenvectors of A corresponding to the eigenvalue λ together with the zero vector is called the eigenspace of A corresponding to the λ.

Algebraic multiplicity of an eigenvalue λ of a matrix A is defined as the multiplicity of λ considered as a root of the characteristic polynomial. An eigenvalue λ is said to be simple, if its algebraic multiplicity is 1.

Algebraic and geometric multiplicity

Definition

For an eigenvalue λ of A, the subspace of all eigenvectors of A corresponding to the eigenvalue λ together with the zero vector is called the eigenspace of A corresponding to the λ.

Algebraic multiplicity of an eigenvalue λ of a matrix A is defined as the multiplicity of λ considered as a root of the characteristic polynomial. An eigenvalue λ is said to be simple, if its algebraic multiplicity is 1.

Geometric multiplicity of an eigenvalue λ of a matrix A is defined as the dimension of the eigenspace associated with λ. An eigenvalue λ is said to be regular, if its algebraic multiplicity is equal the geometric multiplicity.

A.M. $\geq$ G.M.

Theorem

For any eigenvalue λ of A, the algebraic multiplicity of λ with respect to A is greater than or equal to the geometric multiplicity of λ, as an eigenvalue of A.

A.M. $\geq$ G.M.

Theorem

For any eigenvalue λ of A, the algebraic multiplicity of λ with respect to A is greater than or equal to the geometric multiplicity of λ, as an eigenvalue of A.

Proof

- Let $\{x_1, \ldots, x_k\}$ be a basis for the eigenspace of λ, and let $\{x_1, x_2, \ldots, x_n\}$ be an extension to a basis basis of $\mathbb{C}^n$.

A.M. $\geq$ G.M.

Theorem

For any eigenvalue λ of A, the algebraic multiplicity of λ with respect to A is greater than or equal to the geometric multiplicity of λ, as an eigenvalue of A.

Proof

- Let $\{x_1, \ldots, x_k\}$ be a basis for the eigenspace of λ, and let $\{x_1, x_2, \ldots, x_n\}$ be an extension to a basis basis of $\mathbb{C}^n$.
- Set $P = [x_1 \; x_2 \; \ldots \; x_n]$.

A.M. $\geq$ G.M.

Theorem

For any eigenvalue λ of A, the algebraic multiplicity of λ with respect to A is greater than or equal to the geometric multiplicity of λ, as an eigenvalue of A.

Proof

- Let $\{x_1, \ldots, x_k\}$ be a basis for the eigenspace of λ, and let $\{x_1, x_2, \ldots, x_n\}$ be an extension to a basis basis of $\mathbb{C}^n$.
- Set $P = [x_1 \ x_2 \ \ldots \ x_n]$.
- Then P is nonsingular, and

A.M. $\geq$ G.M.

Theorem

For any eigenvalue λ of A, the algebraic multiplicity of λ with respect to A is greater than or equal to the geometric multiplicity of λ, as an eigenvalue of A.

Proof

- Let $\{x_1, \ldots, x_k\}$ be a basis for the eigenspace of λ, and let $\{x_1, x_2, \ldots, x_n\}$ be an extension to a basis basis of $\mathbb{C}^n$.
- Set $P = [x_1 \ x_2 \ \ldots \ x_n]$.
- Then P is nonsingular, and

$$\begin{aligned} P^{-1}AP &= P^{-1}[Ax_1 \ Ax_2 \ \ldots \ Ax_k \ \ldots \ Ax_n] \\ &= P^{-1}[\lambda x_1 \ \lambda x_2 \ \ldots \ \lambda x_k \ \ldots \ Ax_n]. \end{aligned}$$

Proof cont...

- Thus,

$$P^{-1}AP = \begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix},$$

for some matrices B and C.

Proof cont...

- Thus,

$$P^{-1}AP = \begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix},$$

  for some matrices B and C.
- Hence, $\chi_A(\alpha) = \chi_{P^{-1}AP}(\alpha) = (\lambda - \alpha)^k \chi_C(\alpha)$.

Proof cont...

- Thus,

$$P^{-1}AP = \begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix},$$

  for some matrices B and C.
- Hence, $\chi_A(\alpha) = \chi_{P^{-1}AP}(\alpha) = (\lambda - \alpha)^k \chi_C(\alpha)$.
- Thus, the algebraic multiplicity of λ is greater than or equal geometric multiplicity of λ.

Properties

- If λ is an eigenvalue of A with eigenvector x, then λ^k is an eigenvalue of A^k with eigenvector x.

Properties

- If λ is an eigenvalue of A with eigenvector x, then λ^k is an eigenvalue of A^k with eigenvector x.
- If $f(\alpha)$ is a polynomial and λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.

Properties

- If λ is an eigenvalue of A with eigenvector x, then λ^k is an eigenvalue of A^k with eigenvector x.
- If $f(\alpha)$ is a polynomial and λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.
- If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A, and $x_1, x_2, \ldots, x_k$ are the corresponding eigenvectors. Then the $x_1, x_2, \ldots, x_k$ are linearly independent.

Properties

- If λ is an eigenvalue of A with eigenvector x, then λ^k is an eigenvalue of A^k with eigenvector x.
- If $f(\alpha)$ is a polynomial and λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.
- If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A, and $x_1, x_2, \ldots, x_k$ are the corresponding eigenvectors. Then the $x_1, x_2, \ldots, x_k$ are linearly independent.
- If λ is a nonzero eigenvalue of a square matrix AB(A and B need not be square), then λ is an eigenvalue of the matrix BA with the same algebraic and geometric multiplicities.

Properties

- If λ is an eigenvalue of A with eigenvector x, then λ^k is an eigenvalue of A^k with eigenvector x.
- If $f(\alpha)$ is a polynomial and λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.
- If $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues of A, and $x_1, x_2, \dots, x_k$ are the corresponding eigenvectors. Then the $x_1, x_2, \dots, x_k$ are linearly independent.
- If λ is a nonzero eigenvalue of a square matrix AB(A and B need not be square), then λ is an eigenvalue of the matrix BA with the same algebraic and geometric multiplicities. If $x_1, x_2, \dots, x_r$ are linearly independent eigenvectors of AB corresponding to to λ, then $Bx_1, \dots, Bx_r$ are linearly independent eigenvectors of BA corresponding to λ.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction. If $n = 1$, then we are done.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction. If $n = 1$, then we are done.
- Assume the result is true for $(n-1) \times (n-1)$ matrices.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction. If $n = 1$, then we are done.
- Assume the result is true for $(n-1) \times (n-1)$ matrices.
- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction. If $n = 1$, then we are done.
- Assume the result is true for $(n-1) \times (n-1)$ matrices.
- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.
- Let P be a nonsingular matrix with x as the first column.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction. If $n = 1$, then we are done.
- Assume the result is true for $(n-1) \times (n-1)$ matrices.
- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.
- Let P be a nonsingular matrix with x as the first column.
- Then, $P^{-1}AP = \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix}$, for some $1 \times n-1$ vector y^T and $(n-1) \times (n-1)$ matrix C.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction. If $n = 1$, then we are done.
- Assume the result is true for $(n-1) \times (n-1)$ matrices.
- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.
- Let P be a nonsingular matrix with x as the first column.
- Then, $P^{-1}AP = \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix}$, for some $1 \times n-1$ vector y^T and $(n-1) \times (n-1)$ matrix C.
- By induction, there exists a non-singular matrix W such that $T = W^{-1}CW$ is upper triangular.

Weaker version of Schur's lemma

Theorem

Every matrix A is similar to an upper triangular matrix over $\mathbb{C}$.

Proof

- Proof by induction. If $n = 1$, then we are done.
- Assume the result is true for $(n-1) \times (n-1)$ matrices.
- Let A be an $n \times n$ matrix, and λ be an eigenvalue of A with eigenvector x.
- Let P be a nonsingular matrix with x as the first column.
- Then, $P^{-1}AP = \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix}$, for some $1 \times n-1$ vector y^T and $(n-1) \times (n-1)$ matrix C.
- By induction, there exists a non-singular matrix W such that $T = W^{-1}CW$ is upper triangular.
- Set $Q = \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix}$.

Proof cont...

- $(PQ)^{-1}A(PQ) = \begin{pmatrix} 1 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & y^T W \\ 0 & T \end{pmatrix}$,

Proof cont...

- $(PQ)^{-1}A(PQ) = \begin{pmatrix} 1 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & y^T W \\ 0 & T \end{pmatrix}$, which is upper triangular .

Corollary

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the eigenvalues of the matrix* A *and let* $f(\alpha)$ *be a polynomial. Then* $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_k)$ *are the eigenvalues of* $f(A)$*.*

Corollary

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the eigenvalues of the matrix* A *and let* $f(\alpha)$ *be a polynomial. Then* $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_k)$ *are the eigenvalues of* $f(A)$*.*

Proof

Corollary

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the eigenvalues of the matrix* A *and let* $f(\alpha)$ *be a polynomial. Then* $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_k)$ *are the eigenvalues of* $f(A)$*.*

Proof Exercise!

Semi-simple matrices

Definition

A matrix is said to be semi-simple or diagonalizable if it is similar to a diagonal matrix.

Semi-simple matrices

Definition

A matrix is said to be semi-simple or diagonalizable if it is similar to a diagonal matrix.

Remark

If A is semi-simple and is similar to the diagonal matrix diagonal entries are $d_1, d_2, \ldots, d_n$, then the eigenvalues of the matrix A are $d_1, d_2, \ldots, d_n$.

Semi-simple matrices

Definition

A matrix is said to be semi-simple or diagonalizable if it is similar to a diagonal matrix.

Remark

If A is semi-simple and is similar to the diagonal matrix diagonal entries are $d_1, d_2, \ldots, d_n$, then the eigenvalues of the matrix A are $d_1, d_2, \ldots, d_n$.

Observation

If $P^{-1}AP = D$ is a diagonal matrix, then $AP = DP$.

Semi-simple matrices

Definition

A matrix is said to be semi-simple or diagonalizable if it is similar to a diagonal matrix.

Remark

If A is semi-simple and is similar to the diagonal matrix diagonal entries are $d_1, d_2, \ldots, d_n$, then the eigenvalues of the matrix A are $d_1, d_2, \ldots, d_n$.

Observation

If $P^{-1}AP = D$ is a diagonal matrix, then $AP = DP$. We can see that, d_i is an eigenvalue of A with i^{th} column of P as the corresponding eigenvector.

Semi-simple matrices

Definition

A matrix is said to be semi-simple or diagonalizable if it is similar to a diagonal matrix.

Remark

If A is semi-simple and is similar to the diagonal matrix diagonal entries are $d_1, d_2, \ldots, d_n$, then the eigenvalues of the matrix A are $d_1, d_2, \ldots, d_n$.

Observation

If $P^{-1}AP = D$ is a diagonal matrix, then $AP = DP$. We can see that, d_i is an eigenvalue of A with i^{th} column of P as the corresponding eigenvector. Conversely, if A has n linear independent eigenvectors, and P is the matrix formed with these vectors as eigenvectors, then $P^{-1}AP$ is diagonal.

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*
2. *algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity of it,*

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*
2. *algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity of it,*
3. *A has n linearly independent eigenvectors*

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*
2. *algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity of it,*
3. *A has n linearly independent eigenvectors*
4. *the minimal polynomial of A is a product of distinct linear factors.*

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*
2. *algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity of it,*
3. *A has n linearly independent eigenvectors*
4. *the minimal polynomial of A is a product of distinct linear factors.((section 8.5) for minimal polynomial)*

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*
2. *algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity of it,*
3. *A has n linearly independent eigenvectors*
4. *the minimal polynomial of A is a product of distinct linear factors.((section 8.5) for minimal polynomial)*

Proof

- (1) implies (2) is clear.

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*
2. *algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity of it,*
3. *A has n linearly independent eigenvectors*
4. *the minimal polynomial of A is a product of distinct linear factors.((section 8.5) for minimal polynomial)*

Proof

- (1) implies (2) is clear.
- (2) implies (3) is clear.

Characterization of semisimple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple,*
2. *algebraic multiplicity of every eigenvalue is equal to the geometric multiplicity of it,*
3. *A has n linearly independent eigenvectors*
4. *the minimal polynomial of A is a product of distinct linear factors.((section 8.5) for minimal polynomial)*

Proof

- (1) implies (2) is clear.
- (2) implies (3) is clear.
- (2) implies (1) follows from the observation.

Applications:

1. If an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable.

Applications:

1. If an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable.
2. Any idempotent matrix is diagonalizable($P^2 = P$).

Applications:

1. If an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable.
2. Any idempotent matrix is diagonalizable($P^2 = P$).
3. Any nonzero nilpotent matrix is not diagonalizable($A^k = 0$, for some integer k).

Spectral representation for semi-simple matrices

Theorem

The following statements about an $n \times n$ matrix A are equivalent:

1. *A is semi-simple and has rank r,*
2. *there exists a non-singular matrix P of order n, and a diagonal nonsingular matrix Δ of order r such that* $A = P \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$,
3. *There exists nonzero scalars $\gamma_1, \gamma_2, \dots, \gamma_r$ and vectors $u_1, \dots, u_r$ and $v_1, \dots, v_r$ in $\mathbb{C}^n$ such that ${v_i}^T u_j = \delta_{ij}$ for all i, j and*

$$A = \sum_{i=1}^{r} \gamma_i u_i {v_i}^T$$

Proof

- (1) implies (2). Permutation of D.

Proof

- (1) implies (2). Permutation of D.
- (2) implies (1). Trivial.

Proof

- (1) implies (2). Permutation of D.
- (2) implies (1). Trivial.
- (2) implies (3). Set $\delta_i = i^{th}$ diagonal entry of Δ, $u_i = i^{th}$ column of P, and ${v_i}^T = i^{th}$ row of P^{-1} .

Proof

- (1) implies (2). Permutation of D.
- (2) implies (1). Trivial.
- (2) implies (3). Set $\delta_i = i^{th}$ diagonal entry of Δ, $u_i = i^{th}$ column of P, and $v_i^T = i^{th}$ row of P^{-1} .
- (3) implies (2).

Proof

- (1) implies (2). Permutation of D.
- (2) implies (1). Trivial.
- (2) implies (3). Set $\delta_i = i^{th}$ diagonal entry of Δ, $u_i = i^{th}$ column of P, and $v_i^T = i^{th}$ row of P^{-1} .
- (3) implies (2). Exercise!

Happy Tea !

Session - II

Symmetric and Hermitian matrices

Definition

An $n \times n$ matrix is said to be symmetric, if $A^T = A$.

Symmetric and Hermitian matrices

Definition

An $n \times n$ matrix is said to be symmetric, if $A^T = A$. An $n \times n$ matrix is said to be Hermitian, if $A^* = A$.

Symmetric and Hermitian matrices

Definition

An $n \times n$ matrix is said to be symmetric, if $A^T = A$. An $n \times n$ matrix is said to be Hermitian, if $A^* = A$.

Theorem

Eigenvalues of any Hermitian matrix are real numbers. (Eigenvalues of any real symmetric matrix are real numbers)

Symmetric and Hermitian matrices

Definition

An $n \times n$ matrix is said to be symmetric, if $A^T = A$. An $n \times n$ matrix is said to be Hermitian, if $A^* = A$.

Theorem

Eigenvalues of any Hermitian matrix are real numbers. (Eigenvalues of any real symmetric matrix are real numbers)

A real matrix A is said to be orthogonal if $AA^T = A^T A = I$, and a complex matrix A is said to be unitary if $AA^* = A^* A = I$.

Theorem (Spectral theorem for real matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

Theorem (Spectral theorem for real matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

- Proof by induction.

Theorem (Spectral theorem for real matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

- Proof by induction.
- $n = 1$, trivial case.

Theorem (Spectral theorem for real matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

- Proof by induction.
- $n = 1$, trivial case.
- Assume the result is true for matrices of order $n - 1$, and A be a symmetric $n \times n$ matrix.

Theorem (Spectral theorem for real matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

- Proof by induction.
- $n = 1$, trivial case.
- Assume the result is true for matrices of order $n - 1$, and A be a symmetric $n \times n$ matrix.
- Let λ be a real eigenvalue of A and x be a corresponding eigenvector with unit length.

Theorem (Spectral theorem for real matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

- Proof by induction.
- $n = 1$, trivial case.
- Assume the result is true for matrices of order $n - 1$, and A be a symmetric $n \times n$ matrix.
- Let λ be a real eigenvalue of A and x be a corresponding eigenvector with unit length.
- Let P be an orthogonal matrix with x as the first column.

Theorem (Spectral theorem for real matrices)

Any real symmetric matrix is orthogonally similar to a diagonal matrix.

- Proof by induction.
- $n = 1$, trivial case.
- Assume the result is true for matrices of order $n - 1$, and A be a symmetric $n \times n$ matrix.
- Let λ be a real eigenvalue of A and x be a corresponding eigenvector with unit length.
- Let P be an orthogonal matrix with x as the first column.
- Then $P^{-1}AP = \begin{pmatrix} \lambda & y^T \\ 0 & C \end{pmatrix}$, for some vector y^T and some $(n-1) \times (n-1)$ matrix C.

Proof continued...

- Since $P^{-1} = P^T$, we have $y^T = 0$ and C is symmetric.

Proof continued...

- Since $P^{-1} = P^T$, we have $y^T = 0$ and C is symmetric.
- By induction, we have $C = W^{-1}DW$, where D is a diagonal matrix and W is an orthogonal matrix.

Proof continued...

- Since $P^{-1} = P^T$, we have $y^T = 0$ and C is symmetric.
- By induction, we have $C = W^{-1}DW$, where D is a diagonal matrix and W is an orthogonal matrix.
- Set $Q = diag(1, W)$, then Q and PQ are diagonal matrices.

Proof continued...

- Since $P^{-1} = P^T$, we have $y^T = 0$ and C is symmetric.
- By induction, we have $C = W^{-1}DW$, where D is a diagonal matrix and W is an orthogonal matrix.
- Set $Q = diag(1, W)$, then Q and PQ are diagonal matrices.
- $(PQ)^{-1}A(PQ) = \begin{pmatrix} 1 & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & D \end{pmatrix}$
  .

□

Spectral theorem for Hermitian matrices and Schur's lemma

Theorem

Any Hermitian matrix is unitarily similar to a real diagonal matrix.

Spectral theorem for Hermitian matrices and Schur's lemma

Theorem

Any Hermitian matrix is unitarily similar to a real diagonal matrix.

Proof.

Similar to spectral theorem real symmetric matrices. □

Spectral decomposition

Theorem

Let A be an $n \times n$ Hermitian matrix with rank r. Then A can be represented in each of the following equivalent forms:

1. *There exists a unitary matrix P and a real diagonal nonsingular matrix Δ of rank r such that* $A = P \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} P^*$.

Spectral decomposition

Theorem

Let A be an $n \times n$ Hermitian matrix with rank r. Then A can be represented in each of the following equivalent forms:

1. *There exists a unitary matrix P and a real diagonal nonsingular matrix Δ of rank r such that* $A = P \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} P^*$.
2. *There exists non-zero real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ and orthogonal vectors $u_1, \ldots, u_r$ such that* $A = \sum_{i=1}^{n} \lambda_i u_i u_i^*$.

Spectral decomposition

Theorem

Let A be an $n \times n$ Hermitian matrix with rank r. Then A can be represented in each of the following equivalent forms:

1. *There exists a unitary matrix P and a real diagonal nonsingular matrix Δ of rank r such that* $A = P \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} P^*$.
2. *There exists non-zero real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ and orthogonal vectors $u_1, \ldots, u_r$ such that* $A = \sum_{i=1}^{n} \lambda_i u_i u_i^*$.
3. *There exists matrices R and Δ of orders $n \times r$ and $r \times r$, respectively, such that Δ is real, diagonal and non-singular,* $R^*R = I$ *and* $A = R\Delta R^*$.

Theorem (Schur, Jacobi)

Every complex matrix A is unitarily similar to an upper triangular matrix.

Theorem (Schur, Jacobi)

Every complex matrix A is unitarily similar to an upper triangular matrix.

Similar to the proof of invertible similarity.

Theorem (Schur, Jacobi)

Every complex matrix A is unitarily similar to an upper triangular matrix.

Similar to the proof of invertible similarity.

Remark

1. *True or False: Every real matrix is orthogonally similar to an upper triangular matrix.*

Theorem (Schur, Jacobi)

Every complex matrix A is unitarily similar to an upper triangular matrix.

Similar to the proof of invertible similarity.

Remark

1. *True or False: Every real matrix is orthogonally similar to an upper triangular matrix. Answer : False*

Theorem (Schur, Jacobi)

Every complex matrix A is unitarily similar to an upper triangular matrix.

Similar to the proof of invertible similarity.

Remark

1. *True or False: Every real matrix is orthogonally similar to an upper triangular matrix. Answer : False*
2. *Every real matrix A with real eigenvalues is orthogonally similar to an upper triangular matrix.*

Normal matrices

Suppose an $n \times n$ complex matrix A is unitarily similar to a diagonal matrix D i.e., $A = U^* D U$, where U is an unitary matrix. Then $AA^* = A^*A$.

Normal matrices

Suppose an $n \times n$ complex matrix A is unitarily similar to a diagonal matrix D i.e., $A = U^* D U$, where U is an unitary matrix. Then $AA^* = A^* A$.

Definition

An $n \times n$ matrix is said to be normal, if $AA^* = A^* A$.

Normal matrices

Suppose an $n \times n$ complex matrix A is unitarily similar to a diagonal matrix D i.e., $A = U^* DU$, where U is an unitary matrix. Then $AA^* = A^* A$.

Definition

An $n \times n$ matrix is said to be normal, if $AA^* = A^* A$.

Theorem

An upper triangular matrix is normal if and only if it is diagonal.

Normal matrices

Suppose an $n \times n$ complex matrix A is unitarily similar to a diagonal matrix D i.e., $A = U^* D U$, where U is an unitary matrix. Then $AA^* = A^* A$.

Definition

An $n \times n$ matrix is said to be normal, if $AA^* = A^* A$.

Theorem

An upper triangular matrix is normal if and only if it is diagonal.

Proof:

- Consider the k^{th} diagonal entry of TT^* and $T^* T$,

$$\sum_{i=1}^{k} |t_{ik}^2| = \sum_{j=k}^{n} |t_{kj}^2|.$$

Normal matrices

Suppose an $n \times n$ complex matrix A is unitarily similar to a diagonal matrix D i.e., $A = U^* D U$, where U is an unitary matrix. Then $AA^* = A^* A$.

Definition

An $n \times n$ matrix is said to be normal, if $AA^* = A^* A$.

Theorem

An upper triangular matrix is normal if and only if it is diagonal.

Proof:

- Consider the k^{th} diagonal entry of TT^* and T^*T,

$$\sum_{i=1}^{k} |t_{ik}^2| = \sum_{j=k}^{n} |t_{kj}^2|.$$

- By equating first diagonal entries of TT^* and T^*T, we can observe the first row of T is expect the diagonal entry.

Normal matrices

Suppose an $n \times n$ complex matrix A is unitarily similar to a diagonal matrix D i.e., $A = U^*DU$, where U is an unitary matrix. Then $AA^* = A^*A$.

Definition

An $n \times n$ matrix is said to be normal, if $AA^* = A^*A$.

Theorem

An upper triangular matrix is normal if and only if it is diagonal.

Proof:

- Consider the k^{th} diagonal entry of TT^* and T^*T,

$$\sum_{i=1}^{k} |t_{ik}^2| = \sum_{j=k}^{n} |t_{kj}^2|.$$

- By equating first diagonal entries of TT^* and T^*T, we can observe the first row of T is expect the diagonal entry.
- By a similar argument, we can conclude T must be diagonal.

Spectral theorem for Normal matrices

Theorem

A matrix is unitarily similar to a diagonal matrix if and only if it is normal.

Spectral theorem for Normal matrices

Theorem

A matrix is unitarily similar to a diagonal matrix if and only if it is normal.

Proof:

- "If" part is clear.

Spectral theorem for Normal matrices

Theorem

A matrix is unitarily similar to a diagonal matrix if and only if it is normal.

Proof:

- "If" part is clear.
- Assume A is normal.

Spectral theorem for Normal matrices

Theorem

A matrix is unitarily similar to a diagonal matrix if and only if it is normal.

Proof:

- "If" part is clear.
- Assume A is normal.
- By Schur's lemma, A is unitarily similar to a upper triangular matrix T.

Spectral theorem for Normal matrices

Theorem

A matrix is unitarily similar to a diagonal matrix if and only if it is normal.

Proof:

- "If" part is clear.
- Assume A is normal.
- By Schur's lemma, A is unitarily similar to a upper triangular matrix T.
- Now, $T = U^*AU$, and T is normal.

Spectral theorem for Normal matrices

Theorem

A matrix is unitarily similar to a diagonal matrix if and only if it is normal.

Proof:

- "If" part is clear.
- Assume A is normal.
- By Schur's lemma, A is unitarily similar to a upper triangular matrix T.
- Now, $T = U^*AU$, and T is normal.
- So, T is diagonal, by previous theorem.

Positive Semidefinite Matrices(PSD)

Let $\mathcal{S}^n$ denote the subspace of symmetric matrices in $\mathbb{R}^{n \times n}$. $A \in \mathcal{S}^n$ is *positive semidefinite*(PSD) if $x^T A x \geq 0$ for every $x \in \mathbb{R}^n$.

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$*:*

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$*:*

1. A *is PSD*

Theorem

TFAE for $A \in \mathbb{R}^{n\times n}$:

1. *A is PSD*
2. *There exists $C \in \mathcal{S}^n$ such that $A = C^2$,*

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$*:*

1. A *is PSD*
2. *There exists* $C \in \mathcal{S}^n$ *such that* $A = C^2$*,*
3. *There exists an* $n \times n$ *lower triangular matrix* L *such that* $A = LL^T$*,*

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$*:*

1. A *is PSD*
2. *There exists* $C \in \mathcal{S}^n$ *such that* $A = C^2$*,*
3. *There exists an* $n \times n$ *lower triangular matrix* L *such that* $A = LL^T$*,*
4. *There exists an* $n \times k$ *real matrix* B *such that* $A = BB^T$*,*

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$:

1. *A is PSD*
2. *There exists $C \in \mathcal{S}^n$ such that $A = C^2$,*
3. *There exists an $n \times n$ lower triangular matrix L such that $A = LL^T$,*
4. *There exists an $n \times k$ real matrix B such that $A = BB^T$,*

Proof

- (1) implies (2), Spectral theorem.

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$:

1. A *is PSD*
2. *There exists* $C \in \mathcal{S}^n$ *such that* $A = C^2$,
3. *There exists an* $n \times n$ *lower triangular matrix* L *such that* $A = LL^T$,
4. *There exists an* $n \times k$ *real matrix* B *such that* $A = BB^T$,

Proof

- (1) implies (2), Spectral theorem.
- (2) implies (3), QR-decomposition

Theorem

TFAE for $A \in \mathbb{R}^{n \times n}$*:*

1. A *is PSD*
2. *There exists* $C \in \mathcal{S}^n$ *such that* $A = C^2$,
3. *There exists an* $n \times n$ *lower triangular matrix* L *such that* $A = LL^T$,
4. *There exists an* $n \times k$ *real matrix* B *such that* $A = BB^T$,

Proof

- (1) implies (2), Spectral theorem.
- (2) implies (3), QR-decomposition
- (3) implies (4),

Theorem

TFAE for $A \in \mathbb{R}^{n\times n}$:

1. *A is PSD*
2. *There exists $C \in \mathcal{S}^n$ such that $A = C^2$,*
3. *There exists an $n \times n$ lower triangular matrix L such that $A = LL^T$,*
4. *There exists an $n \times k$ real matrix B such that $A = BB^T$,*

Proof

- (1) implies (2), Spectral theorem.
- (2) implies (3), QR-decomposition
- (3) implies (4), proof? Exercise!

Theorem

TFAE for $A \in \mathbb{R}^{n\times n}$:

1. A *is PSD*
2. *There exists* $C \in \mathcal{S}^n$ *such that* $A = C^2$,
3. *There exists an* $n \times n$ *lower triangular matrix* L *such that* $A = LL^T$,
4. *There exists an* $n \times k$ *real matrix* B *such that* $A = BB^T$,

Proof

- (1) implies (2), Spectral theorem.
- (2) implies (3), QR-decomposition
- (3) implies (4), proof? Exercise!
- (4) implies (1).

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ *matrix* A *is a representation of* A *in the following form:* $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$*, where* U *and* V *are unitary matrices and* Δ *is a diagonal matrix with positive diagonal entries.*

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ *matrix* A *is a representation of* A *in the following form:* $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$, *where* U *and* V *are unitary matrices and* Δ *is a diagonal matrix with positive diagonal entries.*

Theorem

Every matrix has a singular value decomposition.

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ *matrix* A *is a representation of* A *in the following form:* $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$, *where* U *and* V *are unitary matrices and* Δ *is a diagonal matrix with positive diagonal entries.*

Theorem

Every matrix has a singular value decomposition.

Proof:

- Let A be an $m \times n$ matrix with rank r.

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ matrix A is a representation of A in the following form: $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$, *where U and V are unitary matrices and Δ is a diagonal matrix with positive diagonal entries.*

Theorem

Every matrix has a singular value decomposition.

Proof:

- Let A be an $m \times n$ matrix with rank r.
- Then the matrix AA^* is Hermitian with rank r.

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ matrix A is a representation of A in the following form: $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$, *where U and V are unitary matrices and Δ is a diagonal matrix with positive diagonal entries.*

Theorem

Every matrix has a singular value decomposition.

Proof:

- Let A be an $m \times n$ matrix with rank r.
- Then the matrix AA^* is Hermitian with rank r.
- By Spectral theorem, we have $AA^* = R\Lambda R^*$, where $\Lambda = diag(d_1, \ldots, d_r)$ and $R^*R = I$.

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ matrix A is a representation of A in the following form: $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$, *where U and V are unitary matrices and Δ is a diagonal matrix with positive diagonal entries.*

Theorem

Every matrix has a singular value decomposition.

Proof:

- Let A be an $m \times n$ matrix with rank r.
- Then the matrix AA^* is Hermitian with rank r.
- By Spectral theorem, we have $AA^* = R\Lambda R^*$, where $\Lambda = diag(d_1, \ldots, d_r)$ and $R^*R = I$.
- Take $B = A^*R$, then $B^*B = \Lambda$.

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ matrix A is a representation of A in the following form: $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$, where U and V are unitary matrices and Δ is a diagonal matrix with positive diagonal entries.

Theorem

Every matrix has a singular value decomposition.

Proof:

- Let A be an $m \times n$ matrix with rank r.
- Then the matrix AA^* is Hermitian with rank r.
- By Spectral theorem, we have $AA^* = R\Lambda R^*$, where $\Lambda = diag(d_1, \ldots, d_r)$ and $R^*R = I$.
- Take $B = A^*R$, then $B^*B = \Lambda$.
- Define $S = BG$ where $G = diag(\frac{1}{d_1}, \ldots, \frac{1}{d_r})$.

Singular value decomposition(SVD)

Definition

A singular value decomposition of an $m \times n$ matrix A is a representation of A in the following form: $A = U \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} V^*$, where U and V are unitary matrices and Δ is a diagonal matrix with positive diagonal entries.

Theorem

Every matrix has a singular value decomposition.

Proof:

- Let A be an $m \times n$ matrix with rank r.
- Then the matrix AA^* is Hermitian with rank r.
- By Spectral theorem, we have $AA^* = R\Lambda R^*$, where $\Lambda = diag(d_1, \ldots, d_r)$ and $R^*R = I$.
- Take $B = A^*R$, then $B^*B = \Lambda$.
- Define $S = BG$ where $G = diag(\frac{1}{d_1}, \ldots, \frac{1}{d_r})$.
- Verify $RG^{-1}S^*$ is a singular value decomposition for A.

Applications!

Applications! Ask Swanand!