INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD
MA5020 - Functional Analysis
Problem Sheet 3 (To be updated)
Autumn 2025

Problem 1. Let X be the vector space of all ordered pairs of complex numbers. Can we obtain the norm $\|x\| = |x_1| + |x_2|$, where $x = (x_1, x_2)$, from an inner product?

Problem 2. Show that for a sequence $\{x_n\}$ in an inner product space the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ imply that $x_n \to x$.

Problem 3. Let H be a Hilbert space. Show that $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all scalars α.

Problem 4. Let $Y = \{x \in l^2 : x_{2n} = 0, n \in \mathbb{N}\}$. Show that Y is a closed subspace of l^2, and compute $Y^\perp$.

Problem 5. Consider $X = C[0,1]$ the Banach space with the sup norm. Define

$$A = \{f \in X : \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1\}.$$

Show that

(i) A is closed and convex.

(ii) $\inf\{\|f\| : f \in A\} = 1$.

(iii) There does not exist any $f \in A$ such that $\|f\| = 1$.

Problem 6. Let H be a Hilbert space and $A \subset H$. Then:

(i) Prove that $(A^\perp)^\perp = \overline{\text{span}(A)}$.

(ii) If M_1 and M_2 are two closed subspaces of H, then show that $(M_1 \cap M_2)^\perp = \overline{M_1^\perp + M_2^\perp}$.

Problem 7. Let $M = \{(x, y, z) \in \mathbb{R}^3 : x + y + 2z = 0\}$, a subspace of $\mathbb{R}^3$. Determine the orthogonal projection $P_M : \mathbb{R}^3 \to M$.

Problem 8. Let H be a Hilbert space. Show that a subspace Y of H is closed if and only if $Y = (Y^\perp)^\perp$

Definition 0.1. *A subset M of a normed linear space X is total, if $\overline{\operatorname{span} M} = X$. An orthonormal set in an inner product space X which is total in X is called total orthonormal set in X.*

Problem 9. Let H be a Hilbert space. Then, a subset M of H is total if and only if $x \perp M$ implies $x = 0$.

Problem 10. Let H be a Hilbert space. If T_n is a sequence of bounded linear operator such that $T_n \to T$, then $T_n^* \to T^*$.

Problem 11. Let T_1 and T_2 be bounded linear operators on a complex Hilbert space H into itself. If $\langle T_1 x, x\rangle = \langle T_2 x, x\rangle$ for all $x \in H$, show that $T_1 = T_2$.

Problem 12. If $T : H \to H$ is a bounded self-adjoint linear operator and $T \neq 0$, then show that $T^n \neq 0$.

Problem 13. If $T_n : H \to H$ are normal linear operators and $T_n \to T$, then show that T is normal.
