Title: Inverses of graphs and reciprocal eigenvalue property.

Speaker: Swarup Panda.

Institute: IIT Kharagpur

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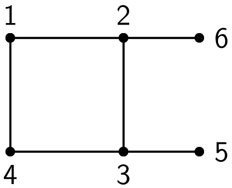

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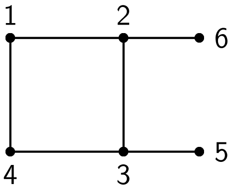

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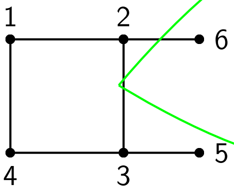

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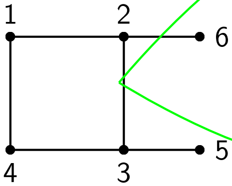

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- $A(G)$ is a (0,1)-symmetric matrix of size $n \times n$.

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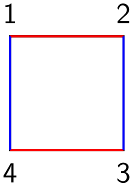

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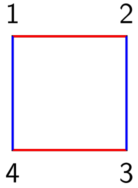

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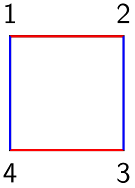

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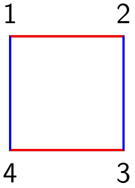

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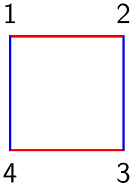

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- The edge $[u, v]$ is called matching edge (resp. nonmatching)if $[u, v]$ in $\mathcal{M}$ (resp. $[u, v]$ in $\mathcal{M}$).

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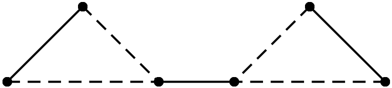

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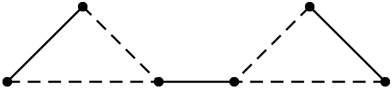

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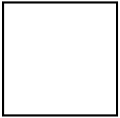

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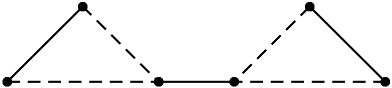

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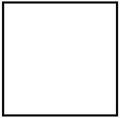

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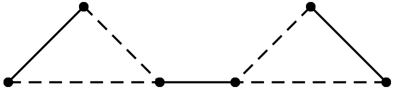

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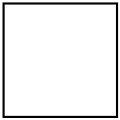

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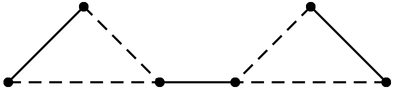

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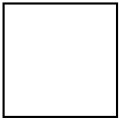

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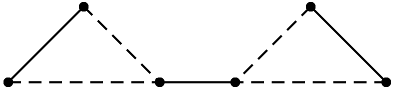

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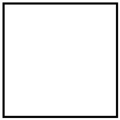

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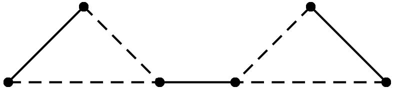

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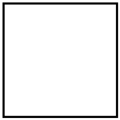

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Harary & Minc, 1976 Let G be a nonsingular graph. Then $A(G)^{-1}$ is nonnegative if and if $G = P_2$.

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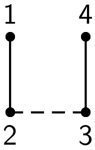

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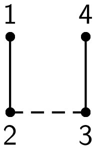

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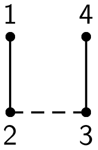

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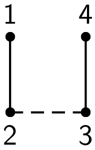

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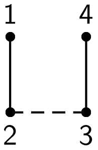

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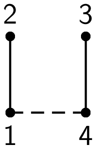
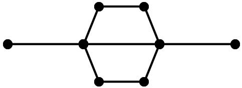
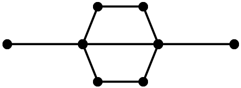
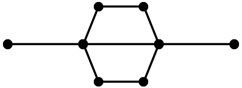
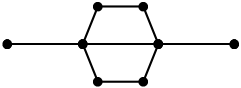
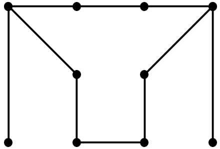
$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \longrightarrow$$



a) Inverse of  is 

a) Inverse of  is 

b) A graph without inverse: 

Hückel Graph

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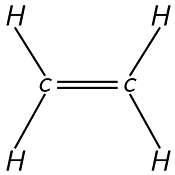

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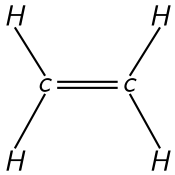

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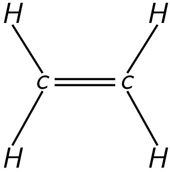

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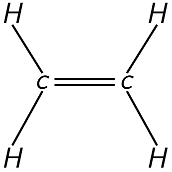

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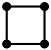
- The amount of energy to remove an electron from a hydrocarbon is correlated with the least positive eigenvalue of the corresponding Hückel graph.

- In 1978, Cvetkovic, Gutman and Simic have introduced the *pseudo-inverse* graph of a graph. Let G be a graph. The pseudo-inverse graph $PI(G)$ of G is a graph, defined on the same vertex set as G, and in which the vertices x and y are adjacent if and only if $G - x - y$ has a perfect matching.
  For example the graph

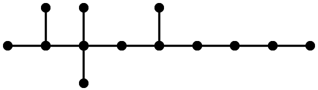
  

  for which $PI(G) = G$ and $\sigma(G) = \sigma(PI(G)) = \{-2, 0, 0, 2\}$, but $1/\lambda \in \sigma(PI(G))$ whenever $\lambda \in \sigma(G)$ is not true.

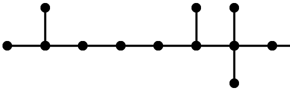
- In 1988, Buckley, Doty and Harary have introduced the *signed inverse* graph of a graph. A *signed graph* is a graph in which each edge has a positive or negative sign, see [?]. An adjacency matrix of a signed graph is symmetric and each entry is 0, 1, or -1. Let G be a nonsingular graph. The graph G has a signed inverse if $A(G)^{-1}$ is the adjacency matrix of some signed graph H.

• In 1990, Pavlikova and Jediny have introduced another notion of inverse graph of a graph. The *inverse* graph of a nonsingular graph with the spectrum $\lambda_1, \ldots, \lambda_n$ is a graph with the spectrum $1/\lambda_1, \ldots, 1/\lambda_n$. This type of inverse graph of a graph need not be unique.

One can construct a class of graphs which have more than one inverse graphs.



G



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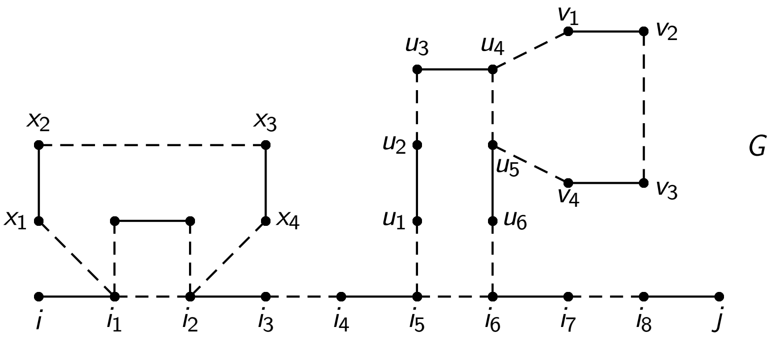

x_2
x_3
x_1
x_4
u_3
u_4
v_1
v_2
u_2
u_5
u_1
u_6
v_4
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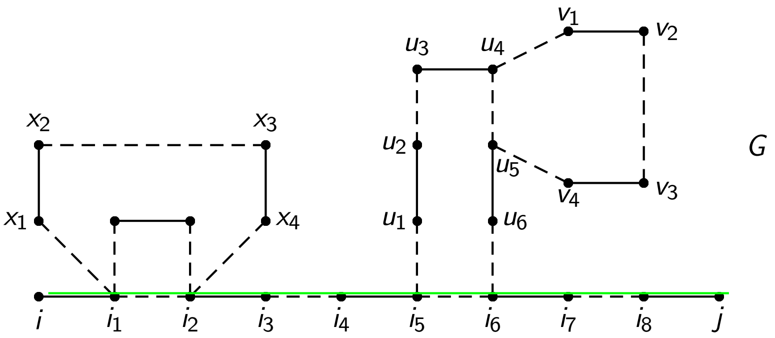
G
i
i_1
i_2
i_3
i_4
i_5
i_6
i_7
i_8
j



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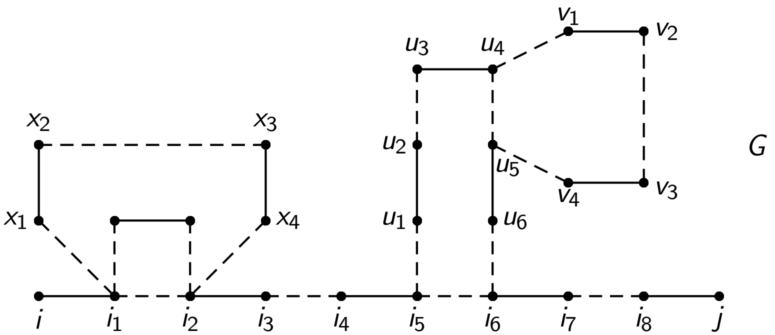

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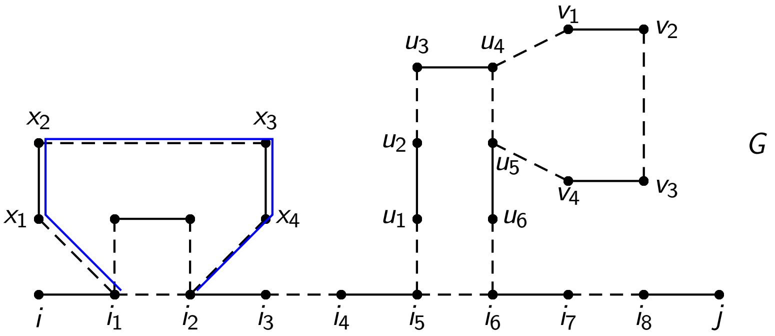

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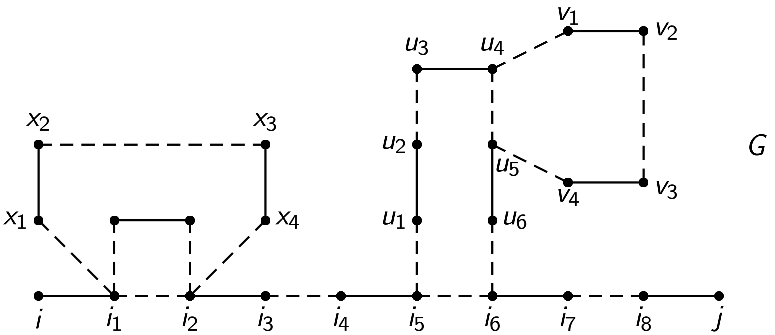

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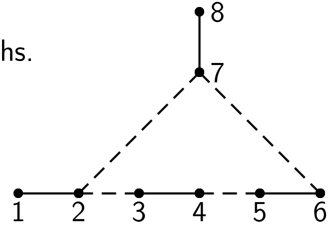

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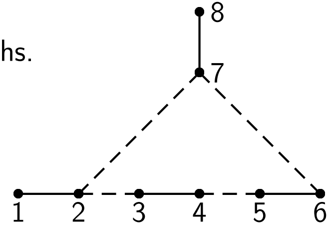

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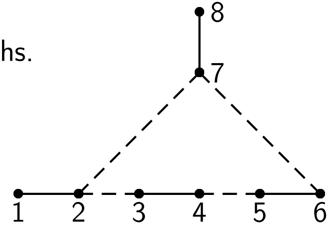

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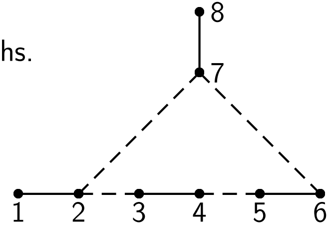

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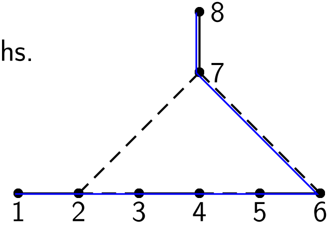

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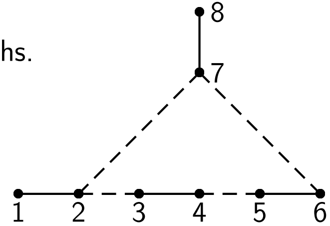

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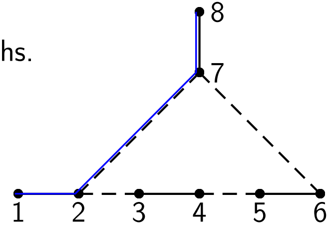

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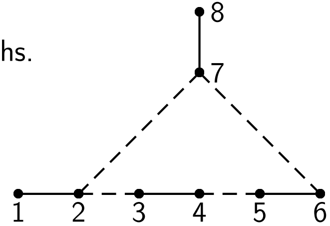

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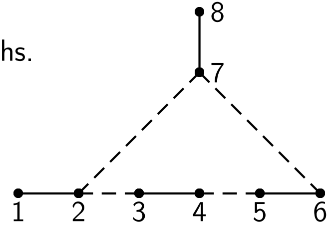

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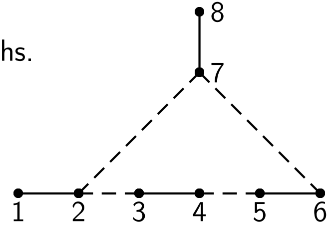

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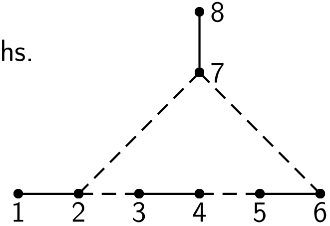

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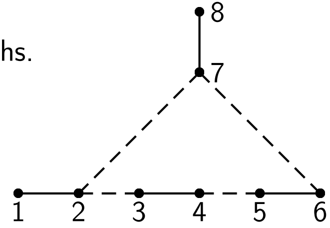

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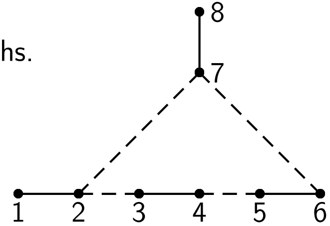

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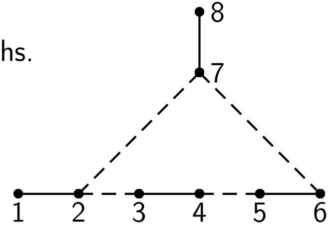

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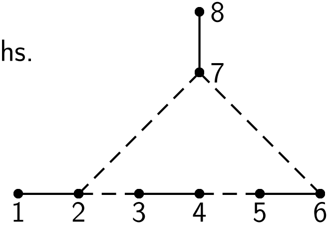

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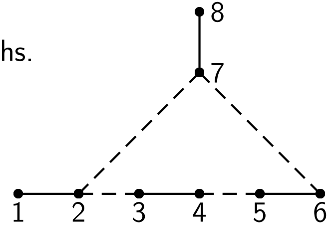

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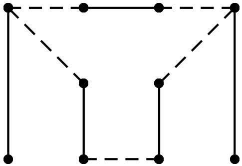

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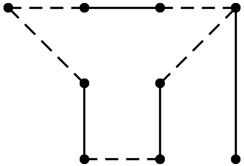

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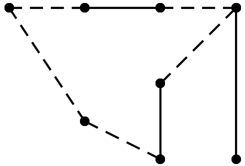

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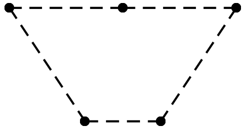

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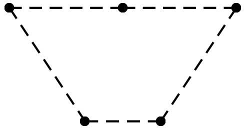

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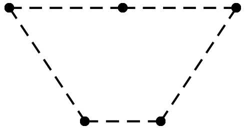

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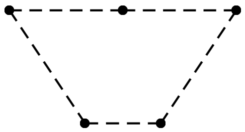

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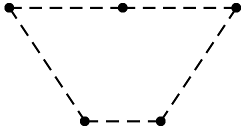

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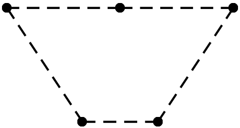

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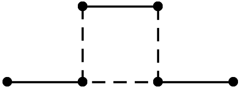

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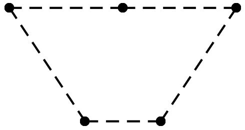

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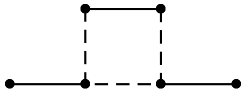

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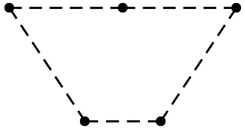

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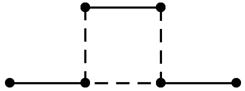

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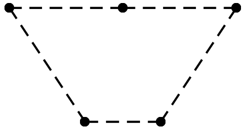

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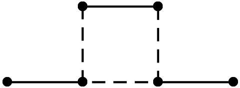

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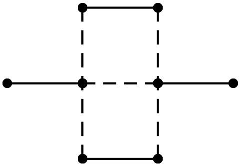

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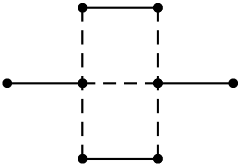

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- A satisfactory explanation remained to be found.

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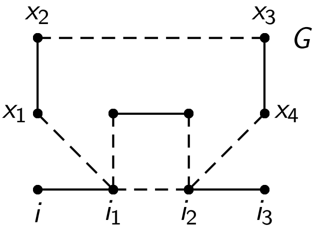

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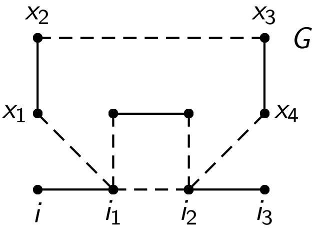

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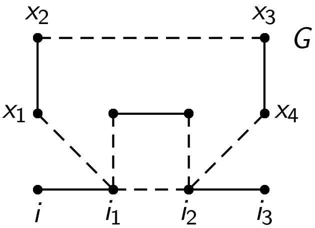

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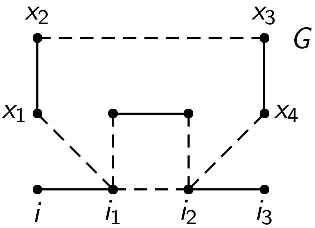

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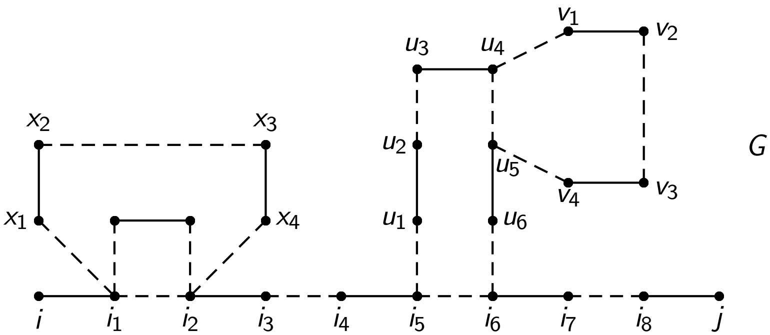
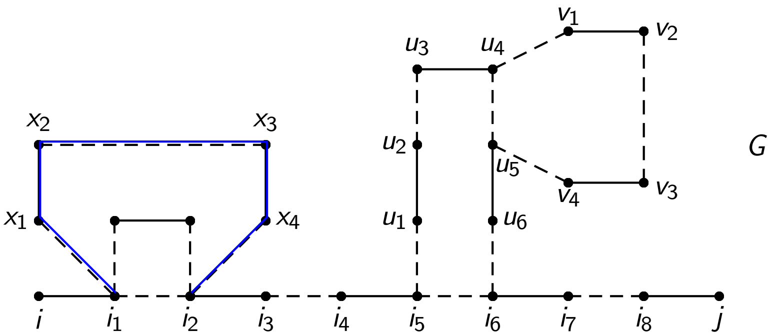


v_1
v_2
u_3
u_4
x_2
x_3
u_2
G
u_5
v_4
v_3
x_1
x_4
u_1
u_6
i
i_1
i_2
i_3
i_4
i_5
i_6
i_7
i_8
j



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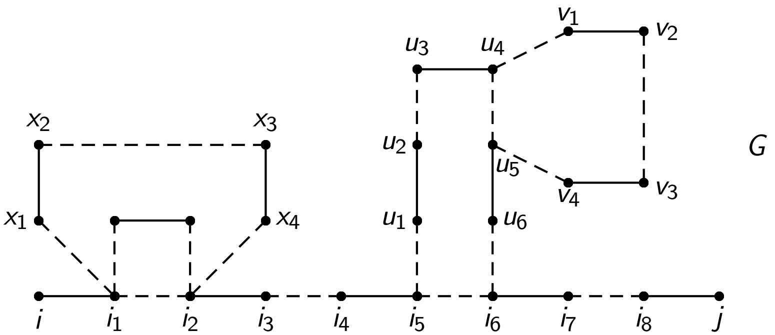

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$[i_1, x_1, x_2, x_3, x_4, i_2]$	odd type extension
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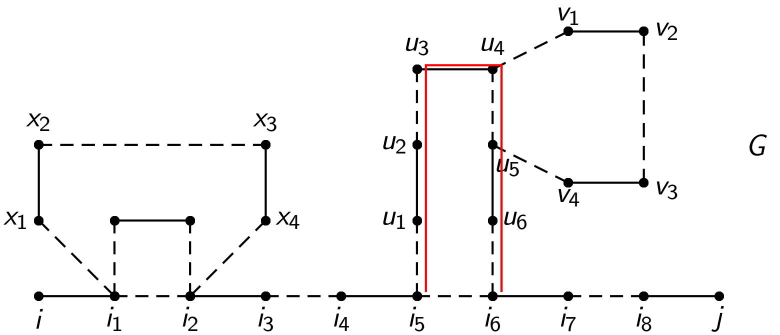

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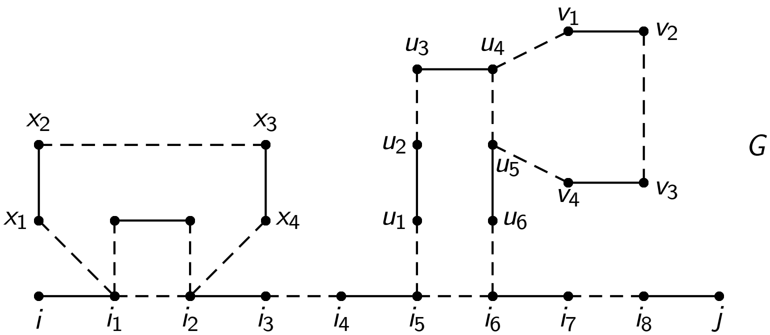

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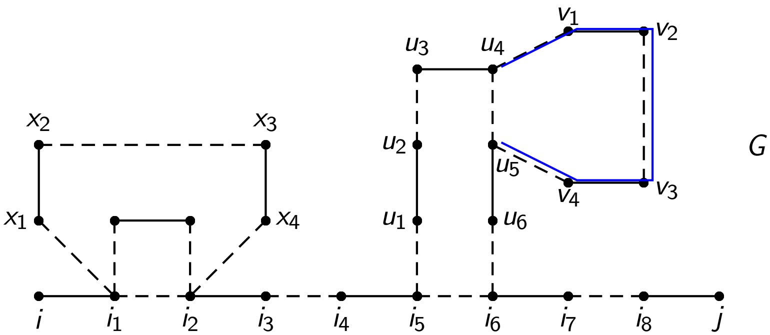

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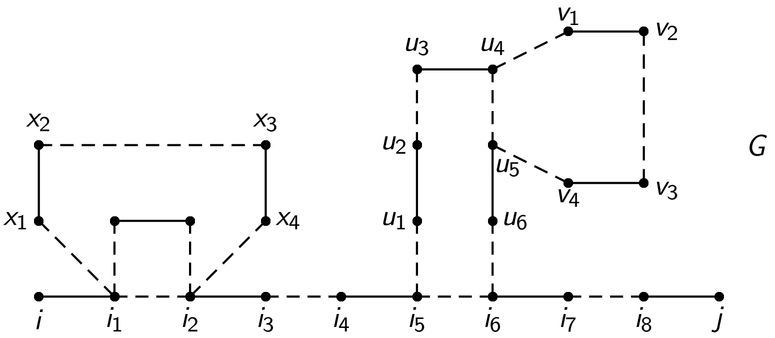

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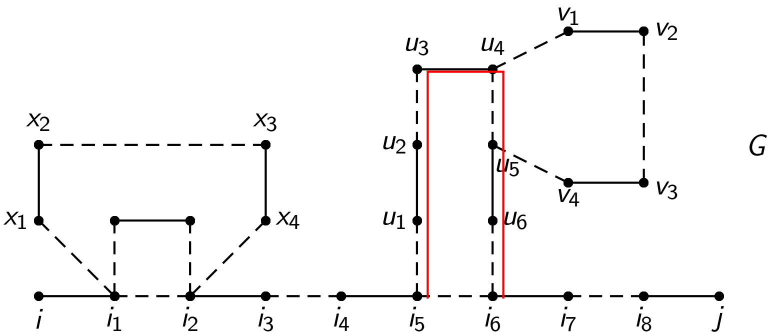

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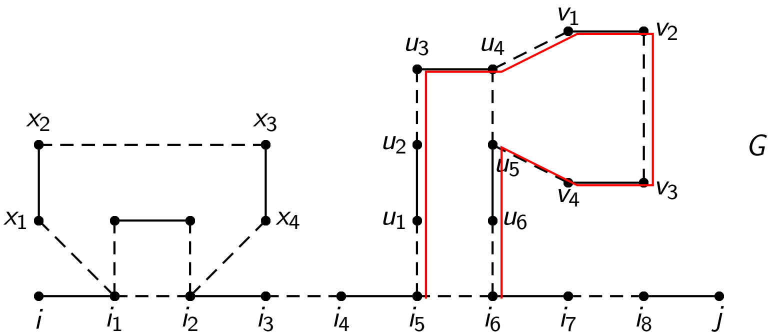

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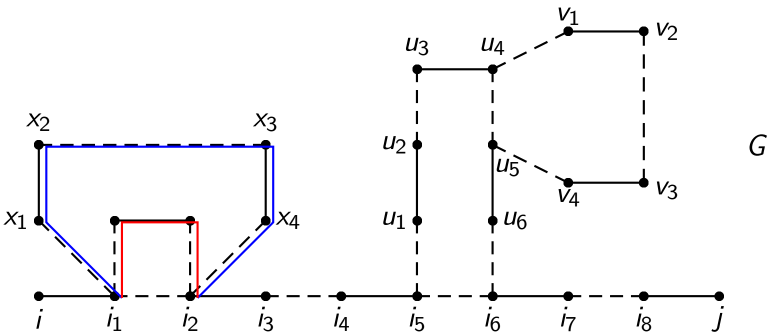

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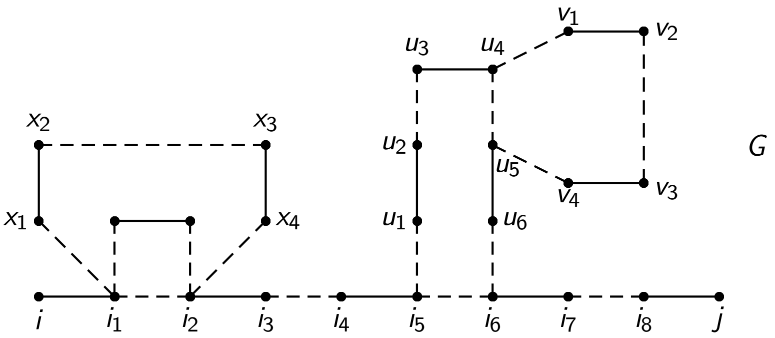

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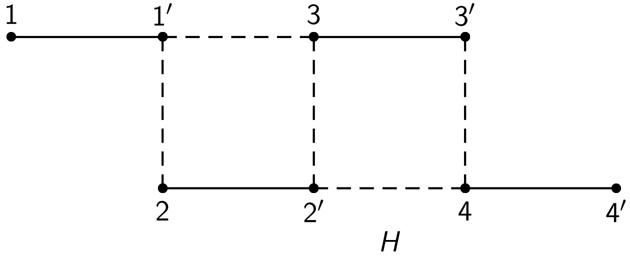
1
1′
3
3′
2
2′
4
4′
H

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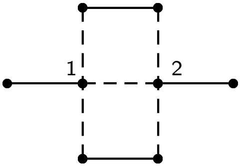

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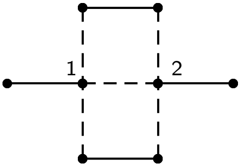

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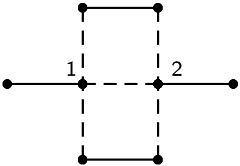

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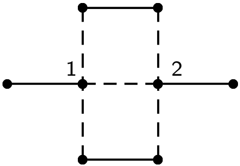

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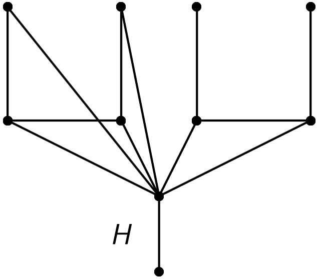
• Graphs in $\mathcal{H}$ that have inverses are now characterized in Yang & Ye 2017.

Godsil & Mckay, 1978

Property SR A nonsingular graph G is said to satisfy the strong reciprocal eigenvalue property or property SR if $1/\lambda$ is an eigenvalue of $A(G)$ whenever λ is an eigenvalue of $A(G)$ and both have the same multiplicity.

Property R When the multiplicity condition is relaxed, we say G has the reciprocal eigenvalue property or property R.


G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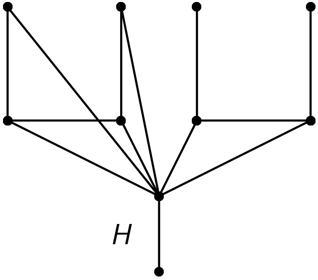

λ	Multiplicity	$1/\lambda$	Multiplicity
$-\frac{\sqrt{3-\sqrt{5}}}{\sqrt{2}}$	1	$-\frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}}$	1
$\frac{\sqrt{3-\sqrt{5}}}{\sqrt{2}}$	1	$\frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}}$	1

λ	Multiplicity	$1/\lambda$	Multiplicity
$\frac{-1-\sqrt{5}}{2}$	1	$\frac{1-\sqrt{5}}{2}$	2
$\frac{-1+\sqrt{5}}{2}$	1	$\frac{1+\sqrt{5}}{2}$	2
$\frac{\sqrt{2(13-\sqrt{41})}-(\sqrt{41}-1)}{4}$	1	$\frac{-\sqrt{2(13-\sqrt{41})}-(\sqrt{41}-1)}{4}$	1
$\frac{\sqrt{2(13+\sqrt{41})}+\sqrt{41}+1}{4}$	1	$\frac{-\sqrt{2(13+\sqrt{41})}+\sqrt{41}+1}{4}$	1

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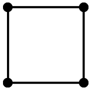

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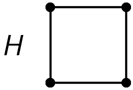

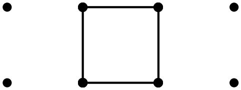

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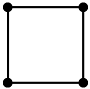

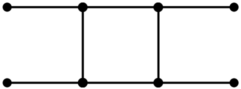

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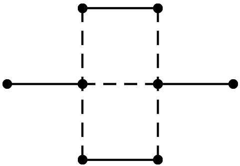

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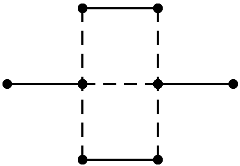

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- A satisfactory explanation remained to be found.

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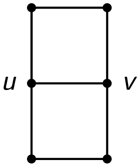
Let boxminus corona $H_S^{\boxminus}$ be the graph created from H by adding two even type extensions of length 3 at each edge $e \in S$.
This is same as replacing each $[u, v] \in S$ with the the following *boxminus graph*.



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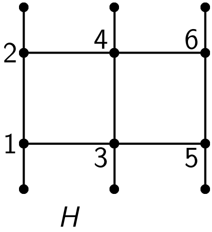

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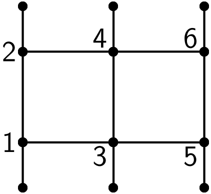

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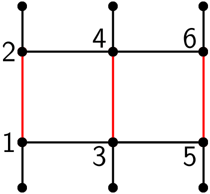

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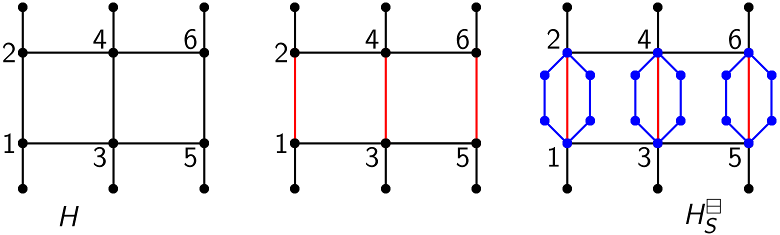

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- **Open Problems**

- Characterize the bipartite graphs with unique perfect matching which are self inverse.

- Is there any bipartite graphs with unique perfect matching which satisfies property R but not SR.

- Characterize the bipartite graphs with unique perfect matching which satisfy property R.

- Characterize the bipartite graphs with unique perfect matching which satisfy property SR.

- Characterize the self-inverse bipartite graphs with unique perfect matching which satisfy property SR.

- S. Akbari and S. J. Kirkland. *On unimodular graphs*. Linear Algebra and its Applications, 421 (2007), pp. 3–15.

- R. Frucht and F. Harary. *On the corona of two graphs*. Aequationes Mathematicae, 4 (1970), pp. 322–325.

- C. D. Godsil and B. D. Mckay. *A new graph product and its spectrum*. Bull. Austral. Math. Soc., 18(1978), pp. 21–28.

- C. D. Godsil. *Inverses of trees*. Combinatorica, 5(1) (1985), pp. 33–39.

- F. Harary and H. Minc. *Which nonnegative matrices are self-inverses?*. Math. Mag, 49 (1976), pp. 91–92.

- S. K. Panda and S. Pati. *On The Inverse Of A Class Of Bipartite Graphs With Unique Perfect Matchings*. Electronic Journal of Linear Algebra, 29 (2015), pp. 89–01.

- S. K. Panda and S. Pati. *On some graphs which possess inverses*. Linear Multilinear Algebra, 64(7) (2016), pp. 1445–1459.

- R. B. Bapat, S. K. Panda and S. Pati. *Strong reciprocal eigenvalue property of a class of weighted graphs*. Linear Algebra and its Applications, 511(2016), pp. 460–475.

- S. K. Panda and S. Pati. *Inverses of weighted graphs*. Linear Algebra and its Applications, 532 (2017), pp. 222–230.

- R. B. Bapat, S. K. Panda and S. Pati. *Self-inverse unicyclic graphs and strong reciprocal eigenvalue property*. Linear Algebra and its Applications, 531(2017), pp. 459–478.

- Y. Yang and D. Ye. *Inverses of bipartite graphs*. Combinatorica, online 2017.

- R. M. Tifenbach, S. J. Kirkland. *Directed intervals and dual of a graph*. Linear Algebra and its Applications, 431 (2009), pp. 792–807.

- M. Neumann and S. Pati. *On reciprocal eigenvalue property of weighted trees*. Linear Algebra and its Applications, 438 (2013), pp. 3817–3828.

Thank You.