The Sensitivity Conjecture and its Resolution

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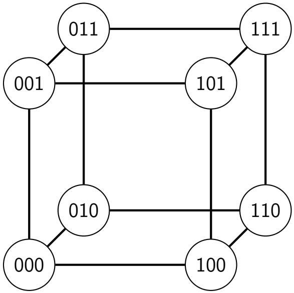
Graphs, Matrices and Applications
12 Nov 2021

Hao Huang

"Induced subgraphs of hypercubes and a proof of the Sensitivity Conjecture". *Annals of Mathematics*. 190 (3) Nov 2019: pp. 949–955.

Some slides are adapted from Huang's TCS+ talk slides.

A Combinatorial Question



Q^3

- The boolean hypercube Q^n has vertex set $\{0, 1\}^n$.
- Two vertices are adjacent iff they differ in exactly one coordinate.
- The 2^2 red points in Q^3 form an independent set.
- In Q^n, we can select 2^{n-1} points that form an independent set.
- We are interested in the max degree of the graph induced by $2^{n-1} + 1$ selected points.

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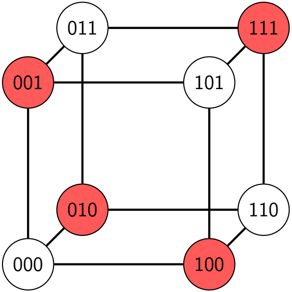

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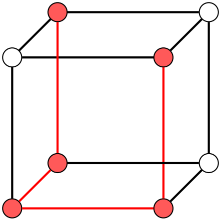
$2^{n-1} + 1$ points of Q^3



- The red vertices give an induced path on 5 vertices.
- We can even form an induced cycle on 6 vertices.
- In any combination of 5 vertices, there exists a vertex of degree ≥ 2.

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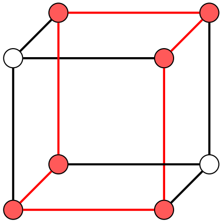

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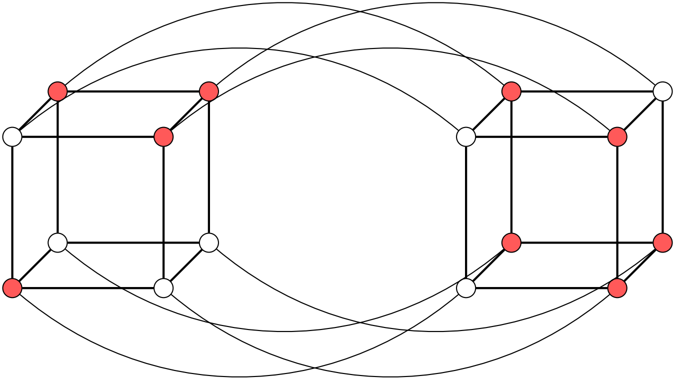
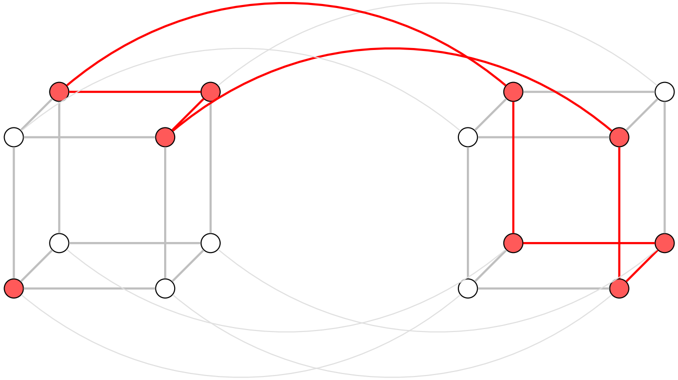


- The nine red vertices give an induced graph with maximum degree 2.
- In any combination of 9 vertices, there exists a vertex of degree ≥ 2.

$2^{n-1} + 1$ points of Q^4



- The nine red vertices give an induced graph with maximum degree 2.
- In any combination of 9 vertices, there exists a vertex of degree ≥ 2.

Question

What is the smallest possible value of the maximum degree of H, where H is an induced subgraph of Q^n, with $|V(H)| = 2^{n-1} + 1$?

In other words

We want to determine the following:

$$\min_{\{H:|V(H)|=2^{n-1}+1\}} \max_{\{v \in V(H)\}} \deg_H v.$$

Question

$$\text{What is } \min_{\{H:|V(H)|=2^{n-1}+1\}} \max_{\{v\in V(H)\}} \deg_H v? \qquad (\star)$$

Theorem (Chung, Füredi, Graham, Seymour 1988)

- Every $(2^{n-1}+1)$-vertex induced subgraph of Q^n has maximum degree at least $(1/2 - o(1)) \log n$. Ans of $(\star) = \Omega(\log n)$.
- Q^n has a $(2^{n-1}+1)$-vertex induced subgraph of maximum degree $\lceil \sqrt{n} \rceil$. Ans of $(\star) \leq \sqrt{n}$.

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Upper Bound: Let $[n] = F_1 \cup F_2 \cup \ldots \cup F_{\sqrt{n}}$, with each $|F_i| = \sqrt{n}$. Let X be defined as the following set of points of $\{0,1\}^n$.

$$\{\text{even sets that contain some } F_i\} \cup \{\text{odd sets that don't contain any } F_i\}.$$

It can be verified that $|X| = 2^{n-1} \pm 1$ while $\Delta(X) = \Delta(X^C) = \sqrt{n}$.

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Theorem (Huang 2019)

Every $(2^{n-1}+1)$-vertex induced subgraph of Q^n contains a vertex of degree at least $\sqrt{n}$. Ans of $(\star) = \sqrt{n}$.

Proof of Huang's Result

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Every $(2^{n-1}+1)$-vertex induced subgraph of Q^n contains a vertex of degree at least $\sqrt{n}$.

Lemma

Let G be a graph. Let λ_1 be the largest eigenvalue of A, the adjacency matrix of G. Then

$$\lambda_1 \leq \Delta(G).$$

Proof: Let $\mathbf{v}$ be an eigenvector corresponding to λ_1. Let v_i be the entry of $\mathbf{v}$ with the largest absolute value. Then

$$|\lambda_1 v_i| = |(A\mathbf{v})_i| = |\sum_{j\sim i} v_j| \leq \Delta(G) \cdot |v_i|.$$

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EIgenvalue Interlacing

Cauchy's Interlacing Theorem

Let A be a symmetric matrix of size n, and B is a principal submatrix of A of size $m \leq n$. Suppose the eigenvalues of A are

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n,$$

and the eigenvalues of B are

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m.$$

Then for $1 \leq i \leq m$, we have

$$\lambda_{i+n-m} \leq \mu_i \leq \lambda_i.$$

The ith largest eigenvalue of B is at most the ith largest eigenvalue of A, and the jth smallest eigenvalue of B is at least the jth smallest eigenvalue of A.

Applying Interlacing on Q^n

- Let H be an induced subgraph of Q^n on $2^{n-1}+1$ vertices.
- Then $\lambda_1(H) \geq \lambda_{2^{n-1}}(Q^n)$.
- The eigenvalues of Q^n are

$$n^{\binom{n}{0}}, (n-2)^{\binom{n}{1}}, \ldots, (n-2i)^{\binom{n}{i}}, \ldots, (-n)^{\binom{n}{n}}.$$

Depending on the parity of n, we get $\Delta(H) \geq \lambda_1(H) \geq 0$ or $\Delta(H) \geq \lambda_1(H) \geq 1$.

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Signed Adjacency Matrix

Lemma

For every graph, and M is a symmetric signed adjacency matrix of G with largest eigenvalue λ_1,

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The proof is exactly the same as before!

$$|\lambda_1 v_i| = |(A\mathbf{v})_i| = |\sum_{j\sim i} v_j| \leq \Delta(G) \cdot |v_i|.$$

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The matrix M

We can view the adjacency matrix of Q^n as follows:

$$Q^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Q^n = \begin{bmatrix} Q^{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & Q^{n-1} \end{bmatrix}.$$

- There are two copies of Q^{n-1} and the identity matrix denotes the edges that connect the corresponding vertices.
- Huang considers the following matrix for obtaining the bound.

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad M_n = \begin{bmatrix} M_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -M_{n-1} \end{bmatrix}.$$

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Eigenvalues of M_n

$$
\begin{aligned}
M_n^2 &= \begin{bmatrix} M_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -M_{n-1} \end{bmatrix} \begin{bmatrix} M_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -M_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} M_{n-1}^2 + I_{2^{n-1}} & 0 \\ 0 & M_{n-1}^2 + I_{2^{n-1}} \end{bmatrix} = nI_{2^n}.
\end{aligned}
$$

- By induction, $M_n^2 = nI$.
- This means that all the eigenvalues of M_n are $\pm\sqrt{n}$.
- M_n is a signed adjacency matrix of Q^n, hence $\mathrm{trace}(M_n) = 0$.
- The eigenvalues are $\sqrt{n}$ and $-\sqrt{n}$, each with multiplicty 2^{n-1}.

- In particular, the 2^{n-1}-th largest eigenvalue is $\sqrt{n}$, completing the proof!

Avoiding the Interlacing Theorem

- M_n has eigenvalue $\sqrt{n}$ with multiplicity 2^{n-1}.
- Let B be the $2^n \times 2^{n-1}$ matrix where each column is an eigenvector with eigenvalue $\sqrt{n}$. That is, $M_n B = \sqrt{n} B$.
- Let B^* be a $2^{n-1} - 1 \times 2^{n-1}$ matrix consisting of the $2^{n-1} - 1$ rows of B that correspond to vertices that **don't** belong to H.
- $\exists$ a $2^{n-1} \times 1$ vector $x \neq 0$ such that $B^* x = 0$.

- Then $y = Bx$ is a $2^n \times 1$ vector that is zero outside H.
- $M_n y = \sqrt{n} y$, since y is a linear combination of columns of B.
- Then $A(H) y = \sqrt{n} y$ since y is zero outside H.
- Therefore $\Delta(H) \geq \lambda_1(H) \geq \sqrt{n}$.

Exposition by Don Knuth of a comment by Shalev Ben-David on Scott Aaronson's blog.

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How was M_n determined?

Theorem (Hadamard's Inequality)

For an $m \times m$ matrix M with row vectors $\mathbf{v}_i$,

$$|\det(M)| \leq \prod_{i=1}^{m} \|\mathbf{v}_i\|.$$

Equality is achieved if and only if all the row vectors are orthogonal.

- Since M_n is a signed adjacency matrix of Q^n, Hadamard's Inequality implies $|\det(M_n)| \leq (\sqrt{n})^{2^n}$.
- The 2^{n-1}-th largest eigenvalue of M_n is at least $\sqrt{n}$. Since the matrix is the adjacency matrix of a bipartite graph, the eigenvalues are symmetric about 0. Thus $|\det(M_n)| \geq (\sqrt{n})^{2^n}$.

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We need $M_n^T M_n = nI$. Let $M_n = \begin{bmatrix} B & K \\ K & C \end{bmatrix}$.

Here B and C are signed adjacency matrices of Q^{n-1} and K is a diagonal matrix with ± 1 entries.

$$M_n^2 = \begin{bmatrix} B^2 + K^2 & BK + KC \\ KB + CK & C^2 + K^2 \end{bmatrix} = \begin{bmatrix} B^2 + I & BK + KC \\ KB + CK & C^2 + I \end{bmatrix}.$$

- $B^2 = C^2 = (n-1)I$. So we have $B^2 + I = C^2 + I = nI$.
- We want $BK + KC = 0$, hence $C = -KBK$.
- If we let $K = I$, we get

$$M_n = \begin{bmatrix} M_{n-1} & I \\ I & -M_{n-1} \end{bmatrix}.$$

Sensitivity of Boolean Functions

A boolean function $f : \{0,1\}^n \to \{0,1\}$ is an assignment of $\{0,1\}$ values to the vertices of the boolean hypercube.

Sensitivity

Given a boolean function f, the local sensitivity $\mathrm{s}(f,x)$ on the input x is defined as the number of indices i, such that $f(x) \neq f(x^{\{i\}})$.
The sensitivity $\mathrm{s}(f)$ of f is $\max_x \mathrm{s}(f,x)$.
The vector $x^{\{i\}} \in \{0,1\}^n$ is the same as x, with bit i flipped.

- *AND* function over n bits.
- *OR* function over n bits.
- *XOR* function over n bits.
- $f(x) = x_1$.

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Block Sensitivity

Given a boolean function $f : \{0,1\}^n \to \{0,1\}$. The local block sensitivity $\mathrm{bs}(f,x)$ on the input x is defined as the maximum number of disjoint blocks $B_1, \ldots, B_k$ of $[n]$, such that for each B_i, $f(x) \neq f(x^{B_i})$. The block sensitivity $\mathrm{bs}(f)$ of f is $\max_x \mathrm{bs}(f,x)$. The vector $x^{B_i} \in \{0,1\}^n$ is the same as x, with bits in B_i flipped.

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- For any non constant f, $1 \leq \mathrm{s}(f) \leq \mathrm{bs}(f) \leq n$.
- This is because block sensitivity is a generalization of sensitivity.
- Hence $\mathrm{bs}(AND) = \mathrm{bs}(OR) = \mathrm{bs}(XOR) = n$
- Can we upper bound $\mathrm{bs}(f)$ in terms of $\mathrm{s}(f)$?

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Sensitivity Conjecture

Sensitivity Conjecture (Nisan, Szegedy 1992)

For every boolean function f,

$$\mathrm{bs}(f) \leq \mathrm{poly}(\mathrm{s}(f)).$$

In other words,

$$\exists \text{ a constant } c \text{ such that } \mathrm{bs}(f) = O(\mathrm{s}(f)^c).$$

- We know $\mathrm{s}(f) \leq \mathrm{bs}(f)$.

Relevance & History

- The study of sensitivity started from the works of Cook, Dwork and Reischuk (1986).
- They showed the lower bound $\mathrm{CREW}(f) = \Omega(\log \mathrm{s}(f))$
- $\mathrm{CREW}(f)$ is the minimum number of steps required to compute f on a CREW PRAM – Consecutive Read Exclusive Write Parallel RAM
- Later, Nisan (1989) showed $\mathrm{CREW}(f) = \Theta(\log \mathrm{bs}(f))$
- Nisan (1989) and Nisan and Szegedy (1992) showed the relations between many other parameters.

Relevance & History

Two complexity measures s_1 and s_2 of boolean functions are polynomially related if $\exists C_1, C_2 > 0$, such that for every boolean f:

$$s_2(f)^{C_1} \leq s_1(f) \leq s_2(f)^{C_2}.$$

Polynomially related parameters

Block sensitivity	Certificate complexity
Degree (as a real polynomial)	Approximate degree
Randomized query complexity	Quantum query complexity
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Sensitivity Conjecture

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For every boolean function f,

$$\mathrm{bs}(f) \leq \mathrm{poly}(\mathrm{s}(f)).$$

In other words,

$$\exists \text{ a constant } c \text{ such that } \mathrm{bs}(f) = O(\mathrm{s}(f)^c).$$

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The Rubinstein Function

Define $f : \{0,1\}^{n^2} \to \{0,1\}$ as

$$f(x_{11}, \dots, x_{nn}) = \bigvee_{i=1}^{n} g(x_{i1}, \dots, x_{in}),$$

where $g(x_1, \dots, x_n) = 1$ iff $x_j = x_{j+1} = 1$ for some $1 \leq j \leq n-1$ and all other $x_k = 0$.

$\mathrm{bs}(f) \geq \mathrm{bs}(f, \vec{0}) = \Omega(n^2)$.

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & \longrightarrow & 0 \\ 0 & 0 & 0 & 0 & \longrightarrow & 0 \\ 0 & 0 & 0 & 0 & \longrightarrow & 0 \\ 0 & 0 & 0 & 0 & \longrightarrow & 0 \\ & & & & & \downarrow \\ & & & & & 0 \end{array}$$

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0 & 0 & 1 & 1 & \longrightarrow & 1 \\
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Sensitivity of Rubinstein Function

We will see that $\mathrm{s}(f) = O(n)$.

Case 1: $f(x) = 0$.
Every row must output 0. In such a case, each row has at most two sensitive coordinates, when the row looks like

$$0\ldots010\ldots0 \quad \text{or} \quad 0\ldots111\ldots0.$$

So $\mathrm{s}(f,x) \le 2n$.

Case 2: $f(x) = 1$.

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Back to sensitivity and block sensitivity

Upper bounds for $\mathrm{bs}(f)$ in terms of $\mathrm{s}(f)$:

- $\mathrm{bs}(f) = O(\mathrm{s}(f) 4^{\mathrm{s}(f)})$. (Simon 1983)
- $\mathrm{bs}(f) \leq (e/\sqrt{2\pi}) e^{\mathrm{s}(f)} \sqrt{\mathrm{s}(f)}$. (Kenyon, Kutin 2004)
- $\mathrm{bs}(f) \leq 2^{\mathrm{s}(f)-1} \mathrm{s}(f)$. (Ambainis, Gao, Mao, Sun, Zuo 2013)

Gaps between $\mathrm{bs}(f)$ and $\mathrm{s}(f)$:

- $\mathrm{bs}(f) = \frac{1}{2}\mathrm{s}(f)^2$. (Rubinstein 1995)
- $\mathrm{bs}(f) = \frac{1}{2}\mathrm{s}(f)^2 + \mathrm{s}(f)$. (Virza 2011)
- $\mathrm{bs}(f) = \frac{2}{3}\mathrm{s}(f)^2 - \frac{1}{2}\mathrm{s}(f)$. (Ambainis, Sun 2011)

All upper bounds are exponential,
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The Gotsman-Linial Equivalence

Theorem (Gotsman, Linial 1992)

The following are equivalent for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- For any induced subgraph of the n-dimensional boolean hypercube Q^n, with $|V(H)| \neq 2^{n-1}$, we have

$$\max\{\Delta(H), \Delta(Q^n \setminus H)\} \geq h(n).$$

- For any boolean function f, we have $\mathrm{s}(f) \geq h(\deg(f))$.

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Theorem (Huang 2019)

Every $(2^{n-1}+1)$-vertex induced subgraph of Q^n contains a vertex of degree at least $\sqrt{n}$.

With the Gotsman-Linial equivalence, we get:

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For every boolean function f, $\mathrm{s}(f) \geq \sqrt{\deg(f)}$. Tight!

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- We only need the direction that $1 \Rightarrow 2$.
- We show $1 \Rightarrow 3 \Rightarrow 2$.
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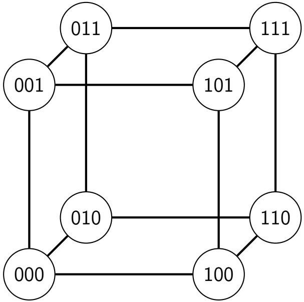

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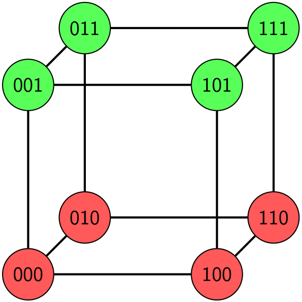

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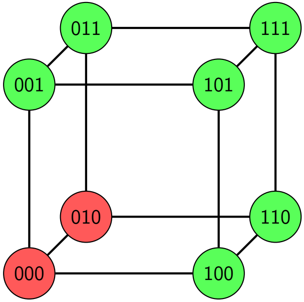

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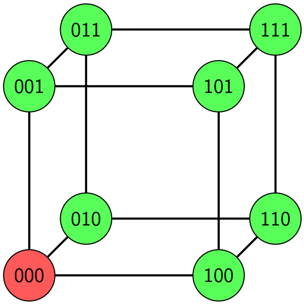

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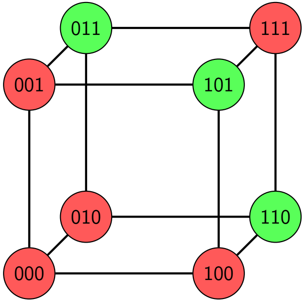

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Note: For this slide alone, we consider $g : \{0,1\}^n \to \{+1,-1\}$.

- Suppose there exists g such that $\mathrm{s}(g) < h(n)$ and $\deg(g) = n$.
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- Consider the induced subgraph H of Q^n with vertex set $V(H) = \{x : g'(x) = 1\}$.
- We have $\max\{\Delta(H), \Delta(Q^n \setminus H)\} = \mathrm{s}(g) < h(n)$.
- $|V(H)| - |V(Q^n \setminus H)| = \mathbb{E}[g(x)p(x)] = \langle g, p\rangle = \hat{g}([n])$.
- Since $\deg(g) = n$, we have $\hat{g}([n]) \neq 0$.
- Hence $|V(H)| \neq |V(Q^n \setminus H)|$. Contradiction.

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How did he come up with this proof? In Huang's words

Nov 2012: I was introduced to this problem by Michael Saks when I was a postdoc at the IAS, and got immediately attracted by the induced subgraph reformulation. And of course, in the next few weeks, I exhausted all the combinatorial techniques that I am aware of, yet I could not even improve the constant factor from the Chung-Füredi-Graham-Seymour paper.

Around mid-year 2013: I started to believe that the maximum eigenvalue is a better parameter to look at, actually it is polynomially related to the max degree, i.e
$\sqrt{\Delta(G)} \leq \lambda(G) \leq \Delta(G)$.

2013-2018: I revisited this conjecture every time when I learn a new tool, without any success though. But at least thinking about it helps me quickly fall asleep many nights.

Excerpts from Huang's comment in Scott Aaronson's blog:
https://www.scottaaronson.com/blog/?p=4229#comment-1813116

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Late 2018: After working on a project and several semesters of teaching a graduate combinatorics course, I started to have a better understanding of eigenvalue interlacing, and believe that it might help this problem.

June 2019: In a Madrid hotel when I was painfully writing a proposal and trying to make the approaches sound more convincing, I finally realized that the maximum eigenvalue of any pseudo-adjacency matrix of a graph provides lower bound on the maximum degree. The rest is just a bit of trial-and-error and linear algebra.

Excerpts from Huang's comment in Scott Aaronson's blog:
https://www.scottaaronson.com/blog/?p=4229#comment-1813116

Open Questions

- We saw that $\mathrm{bs}(f) = O(\mathrm{s}(f)^4)$. We saw an f where $\mathrm{bs}(f) = \Omega(\mathrm{s}(f)^2)$. It will be interesting to find the best bound possible.
- Let $c > 1/2$. What is the smallest t such that every t-vertex induced subgraph of Q^n has maximum degree at least n^c?
- For a given graph G, can we get similar bounds on the degrees of $(\alpha(G) + 1)$-vertex induced subgraphs of G?

Ryan O'Donnell
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Hao Huang@Emory:

Ex.1: ∃edge-signing of n-cube with 2^{n-1} eigs each of +/-sqrt(n)

Interlacing=>Any induced subgraph with >2^{n-1} vtcs has max eig >= sqrt(n)

Ex.2: In subgraph, max eig <= max valency, even with signs

Hence [GL92] the Sensitivity Conj, s(f) >= sqrt(deg(f))

5:02 AM · Jul 2, 2019 · Twitter Web Client

Thank You