Adjacency algebra of a graph

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A talk in E-Seminar @ IITKGP (Oct 9,2020)

Adjacency Matrix of a graph

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Adjacency Matrix of a graph

Let X be a graph on n vertices and let us fix a labeling of the vertices of X. Then, the **adjacency matrix** of X, denoted $A(X) = [a_{ij}]$ (or A), is an $n \times n$ matrix with $a_{ij} = 1$, if the i-th vertex is adjacent to the j-th vertex and 0, otherwise.

Adjacency Matrix of a graph is symmetric

- Note that another labeling of the vertices of X gives rise to another matrix B such that $B = P^{-1}AP$, for some permutation matrix P (for a permutation matrix, recall that $P^t = P^{-1}$). Hence, we talk of the adjacency matrix of a graph X and we do not worry about the labeling of the vertices of X.

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- Clearly, the adjacency matrix A is a real symmetric matrix. Hence, A has n real eigenvalues, A is diagonalizable, and the eigenvectors can be chosen to form an orthonormal basis of $\mathbb{R}^n$.

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- Clearly, the adjacency matrix A is a real symmetric matrix. Hence, A has n real eigenvalues, A is diagonalizable, and the eigenvectors can be chosen to form an orthonormal basis of $\mathbb{R}^n$.
- The eigenvalues, eigenvectors, the minimal polynomial and the characteristic polynomial of a graph X are defined to be that of its adjacency matrix.

Adjacency algebra of a graph

If A is the adjacency matrix of a graph X, then $\mathbb{C}[A]$, the set of all polynomials in A with coefficients from $\mathbb{C}$ forms subalgebra of $\mathbb{M}_n(\mathbb{C})$ we denote it by $\mathcal{A}(X)$ and is called the **adjacency algebra** of X.

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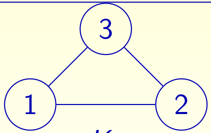
Graph	Adjacency matrix (A)	characteristic polynomial	minimal polynomial	$\mathcal{A}(X)$
 K_3	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$(x+1)^2(x-2)$	$(x+1)(x-2)$	$\{\alpha I + \beta A \mid \alpha, \beta \in \mathbb{C}\}$

$$\mathcal{A}(X) = \mathbb{C}[A] \cong \mathbb{C}[x]/\langle m_A(x) \rangle.$$

Hence $\dim \mathcal{A}(X) = \dim \left(\mathbb{C}[x]/\langle m_A(x) \rangle \right) =$ number of distinct eigenvalues of A.

A known result on $\mathbb{F}[A]$

Theorem

Let $m_A(x)$ be the minimal polynomial of $A \in \mathbb{M}_n(\mathbb{F})$. If $q(x)$ is a non-constant factor of $m_A(x)$ in $\mathbb{F}[x]$ then $\mathbb{F}[A]/\langle q(A)\rangle \cong \mathbb{F}[x]/\langle q(x)\rangle$.

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$$\mathbb{F}[A]/\langle q(A)\rangle \cong \mathbb{F}[x]/\langle q(x)\rangle \cong \mathbb{F}[\alpha].$$

That is, $\mathbb{F}[A]/\langle q(A)\rangle$ is a field.

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In this case we say that $(A, q(A))$ or simply A represents the field $\mathbb{F}(\alpha)$.

A represents the field $\mathbb{F}[\alpha] \Leftrightarrow \alpha$ is an eigenvalue of A.

A.Satyanarayana Reddy, Shashank K Mehta and A.K.Lal, *Representation of Cyclotomic Fields and their Subfields*, Indian J. Pure Appl. Math., 44(2)(2013),203–230.

Number of walks of length k from vertex v_i to vertex v_j

Lemma (Biggs [2])

Let X be a graph with adjacency matrix A. Then, for every positive integer k, $(A^k)_{ij}$ equals the number of walks of length k from the vertex v_i to the vertex v_j.

Proof

Proof.

Proof by induction on k.

Base Step: If $k = 1$, by definition $A_{ij} = \begin{cases} 1, & \text{if } v_i, v_j\ are adjacent \\ 0, & otherwise. \end{cases}$

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Base Step: If $k = 1$, by definition $A_{ij} = \begin{cases} 1, & \text{if } v_i, v_j\ are adjacent \\ 0, & otherwise. \end{cases}$

Assume the result is true for $k = L$ and let us consider the matrix A^{L+1}. Then,

$$(A^{L+1})_{ij} = \sum_{h=1}^{n} (A^L)_{ih}.(A)_{hj}.$$

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Therefore, $(A^{L+1})_{ij}$ equals the number of walks of length L from v_i to v_h and then a walk of length one (adjacency) from v_h to v_j, for all vertices $v_h \in V(X)$.

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□

$d + 1 \leq \dim(\mathcal{A}(X)) \leq n.$

Lemma (Biggs [2])

Let X be a connected simple graph on n vertices. If $d = dia(X)$ is the diameter of X, then $d + 1 \leq \dim(\mathcal{A}(X)) \leq n$.

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Then, from Lemma 2, for each $i \in \{1, 2, \ldots, d\}$, there is at least one path of length i from w_0 to w_i, but no shorter walk. Consequently, A^i has a non-zero entry in a position where the corresponding entries of $I, A, A^2, \ldots, A^{i-1}$ are zero.

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Proof.

Since the adjacency matrix is a real symmetric matrix, its minimal polynomial is the product of distinct linear polynomials. Hence, $\dim(\mathcal{A}(X))$ also equals the number of distinct eigenvalues of A. Thus, if the graph X has diameter d, then it has at least $d+1$ distinct eigenvalues. □

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$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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- A path graph on n vertices has n distinct eigenvalues.
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- Let X be a graph with two distinct eigenvalues. Then, X is a regular graph.

Proof.

Let X be a graph with two distinct eigenvalues, then $\dim \mathcal{A}(X) = 2$. Hence, I and A form a basis of $\mathcal{A}(X)$. Consequently $A^2 = aI + bA$, for some $a, b \in \mathbb{N}$. Thus, $(A^2)_{ii} = a$ for all i. □

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Proof.

From Lemma 2, we know that the ij-th entry of $I+A+A^2+A^3+\ldots+A^{n-1}$ equals the total number of walks of length less than or equal to $n-1$. As X is a connected graph on n vertices, $d(X)\leq n-1$. Hence, each entry in $I+A+A^2+A^3+\ldots+A^{n-1}$ is positive. Thus, the required result follows as
$(I+A)^{n-1}\geq I+A+A^2+A^3+\cdots+A^{n-1}$. □

k-th distance matrix of a graph

Definition

Let $X = (V, E)$ be a connected graph with diameter d. Then, for $0 \leq k \leq d$, the k-th distance matrix of X, denoted A_k, is defined as

$$(A_k)_{rs} = \begin{cases} 1, & \text{if } d(v_r, v_s) = k \\ 0, & \text{otherwise.} \end{cases}$$

From the above definition, it is clear that

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- A_k, for $0 \leq k \leq d$ is a symmetric matrix.
- the set $\{A_0, A_1, \ldots, A_d\}$ is a linearly independent set in $M_n(\mathbb{R})$.

Distance polynomial Graph

Definition (Paul M. Weichsel [2])

Let X be a connected graph with diameter d and let $A_k(X)$, for $0 \leq k \leq d$, be the k-th distance matrix of X. Then, X is said to be a **distance polynomial graph** if $A_k(X) \in \mathcal{A}(X)$, for $0 \leq k \leq d$.

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The complete graph K_n, Cycle graph C_n, Complete bipartite graph $K_{n,n}$ and Petersen graph are few examples of distance polynomial graphs.

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1. *k is an eigenvalue of X.*
2. *if X is connected, then the multiplicity of k is one.*
3. *for any eigenvalue λ of X, $|\lambda| \leq k$.*

Proof of Part 1

Proof.

Let $\mathbf{e} = [1, 1, \ldots, 1]^T$. Then $A\mathbf{e} = k\mathbf{e}$. Consequently, k is an eigenvalue with corresponding eigenvector $\mathbf{e}$. □

Proof of Part 2

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$$k\mathbf{a}_j = (A\mathbf{a})_j \sum_{\{v_i, v_j\} \in E} \mathbf{a}_i \leq k\mathbf{a}_j$$

as is vertex of X is adjacent to exactly k vertices and $\mathbf{a}_j \geq \mathbf{a}_i$, for all $i = 1, \ldots, n$.

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as is vertex of X is adjacent to exactly k vertices and $\mathbf{a}_j \geq \mathbf{a}_i$, for all $i = 1, \ldots, n$. Hence, $\mathbf{a}_i = \mathbf{a}_j$ for all vertices that are adjacent to the vertex v_j. Further, the condition that X is connected implies that we can recursively obtain $\mathbf{a}_i = \mathbf{a}_j$ for all i and j. Consequently, $\mathbf{a}$ is multiple of $\mathbf{e}$. □

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Let $A\mathbf{b} = \lambda\mathbf{b}$. As above, let b_j be an entry of $\mathbf{b}$ having the largest absolute value.

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$$|\lambda|\mathbf{b}_j = |(\lambda\mathbf{b})_j| = |(A\mathbf{b})_j| = |\sum_{\{v_i,v_j\}\in E} \mathbf{b}_i| \leq \sum_{\{v_i,v_j\}\in E} |\mathbf{b}_i| \leq k|\mathbf{b}_j|.$$

Thus, $|\lambda| \leq k$ □

$\mathbf{J} \in \mathcal{A}(X)$

Lemma 7 implies that if X is a connected k-regular graph then the minimal polynomial of X will have the form $(x - k)q(x)$ for some polynomial $q(x)$ with integer entries and $q(k) \neq 0$, as k is an eigenvalue of multiplicity 1. We use this idea in the next result.

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Lemma (Hoffman [3])

Let X be a connected k-regular graph on n vertices. Then, the matrix $\mathbf{J}$, consisting of all 1's, equals $\frac{n}{q(k)}q(A)$, i.e., $\mathbf{J} \in \mathcal{A}(X)$.

Proof

As X is a k-regular graph, its adjacency matrix A satisfies $A\mathbf{e} = k\mathbf{e}$. Let $(x-k)q(x) \in \mathbb{Z}[x]$ be the minimal polynomial of X. Hence,

$$\mathbf{J}A = A\mathbf{J} = k\mathbf{J} \quad \text{and} \quad q(A)\mathbf{e} = q(k)\mathbf{e}. \tag{1}$$

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Let $\left\{\frac{1}{\sqrt{n}}\mathbf{e}, \mathbf{x}_2, \ldots, \mathbf{x}_n\right\}$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A with corresponding eigenvalues $k, \lambda_2, \ldots, \lambda_n$. Thus, $\mathbf{x}_i^T\mathbf{e} = 0$, for $2 \leq i \leq n$. Hence, $\mathbf{J}\mathbf{x}_i = \mathbf{0}$.

Continuation of Proof

Now, Equation (1) gives

$$\mathbf{J}\frac{1}{\sqrt{n}}\mathbf{e} = \frac{n}{\sqrt{n}}\mathbf{e} = \left(\frac{n}{q(k)}q(k)\right)\frac{1}{\sqrt{n}}\mathbf{e} = \frac{n}{q(k)}q(A)\frac{1}{\sqrt{n}}\mathbf{e}. \tag{2}$$

As $(x-k)q(x)$ is the minimal polynomial of X, $q(\lambda_i) = 0$. So, $q(A)\mathbf{x}_i = q(\lambda_i)\mathbf{x}_i = \mathbf{0}$, *i.e.*, $\frac{n}{q(k)}q(A)\mathbf{x}_i = \mathbf{0}$.

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As $(x-k)q(x)$ is the minimal polynomial of X, $q(\lambda_i)=0$. So, $q(A)\mathbf{x}_i = q(\lambda_i)\mathbf{x}_i = \mathbf{0}$, *i.e.*, $\frac{n}{q(k)}q(A)\mathbf{x}_i = \mathbf{0}$. Thus, we see that the image of the two matrices $\mathbf{J}$ and $\frac{n}{q(k)}q(A)$ on the basis $\left\{\frac{1}{\sqrt{n}}\mathbf{e}, \mathbf{x}_2, \ldots, \mathbf{x}_n\right\}$ of $\mathbb{R}^n$ are same. Hence, the two matrices are equal. Therefore, $\mathbf{J} = \frac{n}{q(k)}q(A)$.

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for some positive integer r and $a_i \in \mathbb{R}$, $0 \leq i \leq r$. As each entry of $\mathbf{J}$ is non-zero, for each pair i, j, there exists the smallest power of A, say $t \leq r$, which has a non-zero entry. Hence, by definition there is a walk of length t from the vertex v_i to the vertex v_j. Thus, X is connected. By Equation (3), we see that $A\mathbf{J} = \mathbf{J}A$.

So, if d_i equals $\deg(v_i)$, for $1 \leq i \leq n$, then

$$\begin{bmatrix} d_1 & d_2 & \cdots & d_n \\ d_1 & d_2 & \cdots & d_n \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \cdots & d_n \end{bmatrix} = \mathbf{J}A = A\mathbf{J} = \begin{bmatrix} d_1 & d_1 & \cdots & d_1 \\ d_2 & d_2 & \cdots & d_2 \\ \vdots & \vdots & \ddots & \vdots \\ d_n & d_n & \cdots & d_n \end{bmatrix}.$$

Thus, $d_i = d_j$, for all i and j and hence X is a regular graph. Hence the proof.

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Observation

Let X be a connected regular graph and let $k = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ be the eigenvalues of X. Define, $h(x) = n \prod_{i=2}^{n} \frac{x-\lambda_i}{k-\lambda_i}$. Then, the eigenvalues of $h(A)$ are $\{h(k), h(\lambda_2), \ldots, h(\lambda_n)\} = \{n, 0\}$. Consequently, $h(A) - \mathbf{J}$ vanish at all eigenvectors of A or equivalently $h(A) = \mathbf{J} = \frac{n}{q(k)} q(A)$.

The eigenvalues of X^c, where X is regular.

Corollary

Let X be a connected k-regular graph on n vertices with eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. Then, the eigenvalues of X^c are $n - k - 1, -1 - \lambda_2, \ldots, -1 - \lambda_n$.

Proof

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Let A be the adjacency matrix of X. Then, $A(X^c) = \mathbf{J} - I - A$, the adjacency matrix of X^c. Now, using Lemma 29, the matrices $I, \mathbf{J}$ and A have the same set of eigenvectors.

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$$\begin{aligned} UA^cU^T &= U(\mathbf{J} - I - A)U^T = UJU^T - UIU^T - UAU^T \\ &= diag(n, 0, 0, \ldots, 0) - diag(1, 1, \ldots, 1) - diag(k, \lambda_2, \lambda_3, \ldots, \lambda_n) \\ &= diag(n - k - 1, -1 - \lambda_2, \ldots, -1 - \lambda_n). \end{aligned}$$

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Let X be a connected regular graph. Then X^c is connected if and only if $\mathcal{A}(X) = \mathcal{A}(X^c)$.

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As X^c is a $(n-k-1)$-regular connected graph, $\mathbf{J} \in \mathcal{A}(X^c)$. Hence, $A = \mathbf{J} - I - A(X^c) \in \mathcal{A}(X^c)$. Thus, $\mathcal{A}(X^c) \subset \mathcal{A}(X)$, Thus, the two sets are equal.

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Now, suppose that the two sets are equal. Then, $\mathbf{J} \in \mathcal{A}(X) = \mathcal{A}(X^c)$. Thus, by Lemma 9, the graph X^c is connected and regular. Hence, the required result follows. □

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Proof.

As X is a distance polynomial graph, by definition, X is already connected. If X has diameter d, then by definition, $A_k(X) \in \mathcal{A}(X)$, for $0 \leq k \leq d$. Consequently, $\mathbf{J} = \sum_{k=0}^{d} A_k(X) \in \mathcal{A}(X)$ and hence using Lemma 29, the result follows. □

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Examples of Strongly regular graphs

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- The line graphs of the complete bipartite graphs, $lg(K_{n,n})$ are strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$.

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$k(k - a - 1) = c(n - k - 1)$.

Theorem (Godsil and Royle [3])

Let A be the adjacency matrix of an (n, k, a, c)-strongly regular graph X. Then,

1. $A^2 = kI + aA + c(\mathbf{J} - I - A)$.
2. *the eigenvalues of X are k and roots of equation* $x^2 - (a - c)x - (k - c) = 0$.

Proof of first part

Proof.

To prove the first part, note that he $(i,j)^{th}$ entry of A^2 is the number of walks of length of 2 from the vertex i to the vertex j. Moreover, this number determined only by whether the vertices i and j are adjacent, non-adjacent or same.

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$$(A^2)_{ij} = \begin{cases} k & \text{whenever } i = j, \\ a & \text{if } i \neq j \text{ but } i \text{ and } j \text{ are adjacent,} \\ c & \text{if } i \neq j \text{ but } i \text{ and } j \text{ are not adjacent.} \end{cases}$$

Or equivalently, $A^2 = kI + aA + cA^c = kI + aA + c(\mathbf{J} - I - A)$. □

Proof of second part

Proof.

For the second part, note that k is indeed an eigenvalue of X with eigenvector $\mathbf{e}$. Now, let λ be an eigenvalue of X with corresponding eigenvector $\mathbf{x}$. Then, $\mathbf{e}^T\mathbf{x} = 0$. Hence, using the first part

$$a(\lambda\mathbf{x}) = a(A\mathbf{x}) = \left(A^2 - kI - c(\mathbf{J} - I - A)\right)\mathbf{x} = \lambda^2\mathbf{x} - k\mathbf{x} - c(0-1-\lambda)\mathbf{x} = (\lambda^2 + c\lambda - (k-c))\mathbf{x}.$$

As $\mathbf{x} \neq \mathbf{0}$, we must have $\lambda^2 - (a-c)\lambda - (k-c) = 0$. That is, λ satisfies the required equation. □

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Let X be a connected regular graph which is not a complete graph. Then,

1. *X is a strongly regular if and only if A^2 is linear combination of the matrices I, J and A.*

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1. *X is a strongly regular if and only if A^2 is linear combination of the matrices I, J and A.*
2. *X is a strongly regular if and only if it has exactly three distinct eigenvalues.*

Euler conjectured that no orthogonal Latin squares existed for oddly even numbers (even numbers not divisible by 4.).

This conjecture by Euler was in 1782. In 1901, a French mathematician named Gaston Tarry $(1843 - 1913)$ proved that $n = 6$ was indeed impossible by laboriously checking all possible cases. But Eulers conjecture that orthogonality was impossible for all oddly even numbers remained to be resolved. Until 1959, when R.C. Bose, Shrikhande and E.T. Parker disproved the conjecture.

Once Shrikhande said:

"had the rare privilege of seeing our works reported on the front page of the Sunday Edition of the New York Times of April 26, 1959."

$\mathcal{A}(X)$ of SRG

If X is a connected strongly regular graph, then $\dim(\mathcal{A}(X)) = 3$ and

$$\{I, A, A^c\} = \{A_0, A_1, A_2\}$$

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A connected graph X is said to be a **distance regular graph** if for any two vertices u, v of X, the number of vertices at distance i from u and distance j from v depends only on i, j and $d(u, v)$, the distance between u and v.

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Theorem (Damerell [2])

Let X be a distance regular graph of diameter d. Then the set of distance matrices of X, $\{A_0(X), A_1(X), \ldots, A_d(X)\}$, forms a basis of the adjacency algebra $\mathcal{A}(X)$.

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A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance regular Graphs, Springer-Verlag*,(1989).

Adjacency matrix of a directed cycle

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The minimal polynomial of W_n is $x^n - 1$. Hence eigenvalues of W_n are the nth roots of unity.

Circulant Matrix

A matrix $A \in \mathbb{M}_n(\mathbb{F})$ is said to be a **circulant matrix** if $a_{ij} = a_{1j-i+1 \pmod{n}}$. That is, for each $i \geq 2$, the elements of the i-th row of A are obtained by cyclically shifting the elements of the $(i-1)$-th row of A, one position to the right. So, it is sufficient to specify its first row.

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Every circulant matrix is a polynomial in W_n

Lemma

Let $A \in M_n(\mathbb{F})$. Then A is a circulant matrix if and only if it is a polynomial in W_n. That is, the set of circulant matrices in $M_n(\mathbb{F})$ forms a commutative algebra. Note that as a vector space, its basis is $\{I = W_n^0, W_n^1, W_n^2, \ldots, W_n^{(n-1)}\}$.

Representer polynomial

Let $A \in M_n(\mathbb{Z})$ be a circulant matrix. Then, from Lemma 14, there exists a unique polynomial $\gamma_A(x) \in \mathbb{Z}[x]$ of degree $\leq n-1$, called the **representer polynomial** of A such that $A = \gamma_A(W_n)$.

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Eigenvalues of circulant graphs

Let ζ_n be the primitive nth root of unity, *i.e.*, $\zeta_n^n = 1$ but $\zeta_n^k \neq 1$, for $1 \leq k \leq n - 1$.

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Hence, verify that $W_n \begin{bmatrix} 1 \\ \zeta_n \\ \vdots \\ \zeta_n^{n-1} \end{bmatrix} = \zeta_n \begin{bmatrix} 1 \\ \zeta_n \\ \vdots \\ \zeta_n^{n-1} \end{bmatrix}$. Thus, one has the following result.

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Lemma

Let A be a circulant matrix with representer polynomial $\gamma_A(x)$. Then, A is diagonalizable with $\gamma_A(\zeta_n^k)$, for $0 \leq i \leq n-1$, as its eigenvalues.

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Theorem

The Cycle graph is a distance polynomial graph.

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Proof.

It is easy to check that for the cycle graph C_n, the distance matrices are $A_i = W_n^i + W_n^{n-i}$, for $1 \leq i < \lfloor \frac{n}{2} \rfloor$. For $\tau = \lfloor \frac{n}{2} \rfloor$,

$$A_\tau = \begin{cases} W_n^\tau, & \text{if n } \textit{is even}, \\ W_n^\tau + W_n^{n-\tau}, & \text{if n } \textit{is odd}. \end{cases}$$

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The identity $(x^k + x^{-k}) = (x + x^{-1})(x^{k-1} + x^{1-k}) - (x^{k-2} + x^{2-k})$ enables us to establish readily by mathematical induction that $x^k + x^{-k}$ is a monic polynomial in $x + x^{-1}$ of degree k with integral coefficients.

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The following result shows that every symmetric circulant matrix is a polynomial in the cycle graph. Hence, the eigenvalues of every circulant graph can be computed using the eigenvalues of C_n.

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Proof.

By the definition of the adjacency algebra of a graph, every element in $\mathcal{A}(C_n)$ is a symmetric circulant matrix. We now show that if B is a symmetric circulant matrix, then $B \in \mathcal{A}(C_n)$.

The following result shows that every symmetric circulant matrix is a polynomial in the cycle graph. Hence, the eigenvalues of every circulant graph can be computed using the eigenvalues of C_n.

Theorem

Let $B \in M_n(\mathbb{Q})$. Then B is symmetric circulant matrix if and only if $B \in \mathcal{A}(C_n)$.

Proof.

By the definition of the adjacency algebra of a graph, every element in $\mathcal{A}(C_n)$ is a symmetric circulant matrix. We now show that if B is a symmetric circulant matrix, then $B \in \mathcal{A}(C_n)$.

Let B be a symmetric circulant matrix with the representer polynomial $\gamma_B(x) = \sum_{i=0}^{n-1} b_i x^i$. Then $B = \sum_{i=0}^{n-1} b_i W_n^i$ and $B^T = \sum_{i=0}^{n-1} b_i W_n^{n-i}$. Consequently $b_i = b_{n-i}$, for $1 \leq i \leq n-1$. Thus, $B = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_i A_i$ and hence, the required result follows. □

A.K.Lal and A.Satyanarayana Reddy, *Non-singular circulant graphs and digraphs*, Electronic Journal of Linear Algebra, Volume 26,(2013), 248–257.

Automorphism group of a graph X

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$g \in \text{Aut}(X)$ if and only if $P_g A = AP_g$

Lemma

Let A be the adjacency matrix of a graph X. Then $g \in Aut(X)$ if and only if $P_g A = AP_g$.

Proof

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Let g be a permutation of $V(X) = \{v_1, v_2, \ldots, v_n\}$, and $g(v_i) = v_h, g(v_j) = v_k$.

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$$(P_g A)_{ik} = \sum_{t=1}^{n} (P_g)_{it} A_{tk} = (P_g)_{ih} A_{hk} = A_{hk}$$

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$$\begin{aligned} P_g A = AP_g \quad &\Leftrightarrow \quad A_{hk} = A_{ij} \Leftrightarrow \{v_h, v_k\} \in E \text{ if and only if } \{v_i, v_j\} \in E \\ &\Leftrightarrow \quad g \text{ is an automorphism of } X. \end{aligned}$$

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First note that a matrix B commutes with $\mathbf{J}$ if its every row sum is equal to its every column sum. Consequently every permutation matrix commutes with $\mathbf{J}$. Hence

$$P_g A = A P_g \Leftrightarrow P_g(\mathbf{J} - I - A) = (\mathbf{J} - I - A) P_g.$$

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Vertex transitive graphs

A graph $X = (V, E)$ is said to be a **vertex transitive** (**edge transitive**) graph if $\mathrm{Aut}(X)$ acts transitively on V (E).

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Lemma

Let $X = (V, E)$ be a k-regular vertex transitive graph. If λ is a simple eigenvalue of X then, λ equals k if $|V|$ is odd, and is contained in $\{-k, -k+2, \ldots, k-2, k\}$, if $|V|$ is even.

Cayley graph is vertex transitive

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Let $X = \text{Cay}(G, S)$ be a Cayley graph. Then, for every $g \in G$

$$\{x, y\} \in E(X) \Leftrightarrow xy^{-1} \in S \Leftrightarrow (xg)(yg)^{-1} \in S \Leftrightarrow \{xg, yg\} \in E(X).$$

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Thus, $G \subseteq \text{Aut}(X)$. Hence, if $a, b \in V = G$ then, the group element $a^{-1}b$ takes a to b. □

Circulant digraph

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Lemma

Let $\mathbb{Z}_n$ denote the cyclic group of order n. Then, every Cayley digraph $Cay(\mathbb{Z}_n, S)$ is a circulant digraph. Conversely, every circulant digraph is $Cay(\mathbb{Z}_n, S)$ for some non-empty subset S of $\mathbb{Z}_n$.

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Hence every circulant graph is vertex transitive. Every vertex transitive graph is not a Cayley graph. But vertex transitive graph of prime order is circulant graph.

Petersen graph is a vertex transitive but is not a Cayley graph

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Example

Show that the Petersen graph is a vertex transitive but is not a Cayley graph. Solution:The proof of the vertex transitivity is left as an exercise (use the first construction of the Petersen graph given in these notes). It is known that up to isomorphism there are only two groups of order 10*, namely, the Cyclic group and the Dihedral group. It is easy to verify that none of the cubic Cayley graphs obtained from these groups is isomorphic to the Petersen graph.*

Distance Transitive Graphs

Definition

A graph X is said to be **distance transitive** if for all vertices u, v, x, y of X with $d(u, v) = d(x, y)$, there is a $g \in \mathrm{Aut}(X)$ satisfying $g(u) = x$ and $g(v) = y$.

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Theorem

Let X be a distance transitive graph with diameter d. Then $\dim(\mathcal{A}(X)) = d + 1$.

Orbital Matrices

In fact, in case of distance transitive graphs something more is true and to state it, we need the following definition.

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Definition

Let G be a group acting on a non-empty set V. Then G also acts on $V \times V$, by $g(x, y) = (g(x), g(y))$. For each fixed element $(u, v) \in V \times V$, the set $\text{Orb}(u, v) = \{g(u, v) : g \in G\}$ is called the orbit of (u, v), under the action of G. The distinct orbits of $V \times V$ under the action of G are called *orbitals*.

In the context of a graph $X = (V, E)$, the orbitals of X are the distinct orbits of $E \subset V \times V$ under the action of $\mathrm{Aut}(X)$. That is, the *orbitals* are the orbits of the arcs/non-arcs of the graph X.

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If X is a distance transitive graph then orbital matrices and the distance matrices defined earlier will coincide. Moreover, they form a basis for adjacency algebra $\mathcal{A}(X)$.

Decomposition of K_p as isomorphic copies of circulant digraphs

Lemma

Let p be a prime number and let k be any factor of $p-1$. Then, the edge set of $K_p = (\mathbb{Z}_p, E)$, the complete graph on p vertices, can be partitioned into k subsets $E_1, E_2, \ldots, E_k$ such that the digraphs $X_i = (V, E_i)$, for $1 \leq i \leq k$ are r-regular circulant digraphs, where $r = \frac{p-1}{k}$. Moreover, the digraphs X_i and X_j for $1 \leq i < j \leq k$ are isomorphic.

Proof of first part

Proof.

Let α be a generator of $\mathbb{Z}_p^*$. Then $H = \langle \alpha^k \rangle = \{1, \alpha^k, \ldots, \alpha^{k(r-1)}\}$ is a subgroup of $\mathbb{Z}_p^*$ having r elements and let $H_j = \alpha^j H$ for $j = 0, 1, \ldots, k-1$ be the cosets of H in $\mathbb{Z}_p^*$ with $H_0 = H$.

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Since the cosets H_j, for $0 \leq j \leq k-1$, are disjoint, one has obtained k disjoint digraphs that are r-regular and this completes the proof of the first part. □

Proof of second part

Proof.

We now need to show that the k digraphs, X_j, for $0 \leq j \leq k-1$, are mutually isomorphic. We will do so by proving that the digraphs X_0 and X_j are isomorphic, for $1 \leq j \leq k-1$.

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$$x - y \in H \Leftrightarrow \alpha^j(x-y) \in H_j \Leftrightarrow (\alpha^j x - \alpha^j y) \in H_j \Leftrightarrow \psi(x) - \psi(y) \in H_j.$$

This completes the proof of the lemma. □

Pattern Polynomial graphs

Hadamard Product

- Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then the *Hadamard product* of $A = [a_{ij}]$ and $B = [b_{ij}]$, denoted $A \odot B$, is defined as $(A \odot B)_{ij} = a_{ij} b_{ij}$, for $1 \leq i, j \leq n$.

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Theorem (Higman [2], Brouwer, Cohen & Neumaier [4])

Let $\mathcal{M}$ be a vector subspace of symmetric $n \times n$ matrices. Then $\mathcal{M}$ has a basis of mutually disjoint $0, 1$-matrices if and only if $\mathcal{M}$ is closed under Hadamard multiplication.

Definition

A subalgebra of $\mathbb{M}_n(\mathbb{C})$ containing the matrices I (Identity matrix) and $\mathbf{J}$ (matrix with all entries being 1) is called a coherent algebra if it is closed under conjugate-transposition and Hadamard product.

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- $\mathbb{M}_n(\mathbb{C})$ is the largest coherent algebra.
- The minimal polynomial of $\mathbf{J}$ is $p_{\mathbf{J}}(x) = x(x - n)$.

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- Note that $\mathbb{C}[\mathbf{J}] = \mathbb{C}[\mathbf{J} - I]$ which is same as $\mathcal{A}(K_n)$.

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We already observed that W_n is the adjacency matrix of a directed cycle.

Coherent closure of A

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$$\begin{bmatrix} p_{11}(\mathbf{y}) & p_{12}(\mathbf{y}) & \ldots & p_{1n}(\mathbf{y}) \\ p_{21}(\mathbf{y}) & p_{22}(\mathbf{y}) & \ldots & p_{2n}(\mathbf{y}) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(\mathbf{y}) & p_{n2}(\mathbf{y}) & \ldots & p_{nn}(\mathbf{y}) \end{bmatrix},$$

where $p_{ij}(\mathbf{y}) = \sum_k y_k (A^k)_{ij}$ can be viewed as a linear polynomial in ℓ indeterminates $y_0, y_1, \ldots, y_{\ell-1}$.

$\mathcal{L}(A)$

Let us assume that $S = \{q_1(\mathbf{y}), q_2(\mathbf{y}), \ldots, q_r(\mathbf{y})\}$ is the set of distinct polynomials appearing as elements in the matrix $B(\mathbf{y})$. We now use the set S to define r matrices, $P_1, P_2, \ldots, P_r$, called the **pattern matrices** of A, by

$$(P_j)_{s,t} = \begin{cases} 1, & \text{if } B(\mathbf{y})_{s,t} = p_{st}(\mathbf{y}) = q_j(\mathbf{y}), \\ 0, & \text{otherwise.} \end{cases}$$

Then, we define $\mathcal{L}(A)$ as the linear subspace $L(P_1, P_2, \ldots, P_r)$ of $\mathbb{M}_n(\mathbb{C})$.

Observation

Let the pattern matrices $P_1, P_2, \ldots, P_r$ be as defined above. Then

1. $P_i \odot P_j = \mathbf{0}$, *for* $1 \leq i \neq j \leq r$ *and* $P_i \odot P_i = P_i$, *for* $1 \leq i \leq r$. *Also, by definition,* $I \in \mathcal{L}(A)$ *and since* $\sum_{i=1}^{r} P_i = \mathbf{J}$, $\mathbf{J} \in \mathcal{L}(A)$.
2. *Let* $M, N \in \mathcal{L}(A)$. *Then* $M = \sum_{i=1}^{r} a_i P_i$ *and* $N = \sum_{i=1}^{r} b_i P_i$, *for some* $a_i, b_i \in \mathbb{C}$, $1 \leq i \leq r$. *Therefore, by definition,* $M \odot N = \sum_{i=1}^{r} a_i b_i P_i \in \mathcal{L}(A))$. *Thus,* $\mathcal{L}(A)$ *is closed under Hadamard product.*

Observation

1. *$\mathcal{L}(A)$ is the smallest subspace of $\mathbb{M}_n(\mathbb{C})$ closed under Hadamard product and contains all powers of A. Consequently, $\mathbb{C}[A] \subseteq \mathcal{L}(A) \subseteq \mathcal{CC}(A)$ and $l \leq r$.*
2. *Let $P_i^T \in \{P_1, P_2, \ldots, P_r\}$ for all $i, 1 \leq i \leq r$. Then $\mathcal{L}(A)$ is also closed under conjugate transposition. In particular, if A is symmetric, then all pattern matrices are symmetric and $\mathcal{L}(A)$ is closed under conjugate transposition.*

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Theorem

Let $A \in M_n(\mathbb{C})$ *be a symmetric matrix. Then* $\mathbb{C}[A] = \mathcal{CC}(A)$ *if and only if* $\ell = r$.

Recall the following result stated earlier.

Lemma (Hoffman [3])

A graph X is connected and k-regular if and only if $\mathbf{J} \in \mathcal{A}(X)$. Moreover, in this case, $\mathbf{J} = \frac{n}{q(k)} q(A)$, where $(x - k)q(x)$ is the minimal polynomial of A.

Properties of pattern polynomial graphs

If X is a Pattern Polynomial graph then X is a

- Connected regular graph.

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Properties of pattern polynomial graphs

If X is a Pattern Polynomial graph then X is a

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- Every pattern polynomial graph except K_2 has at least one multiple eigenvalue.In particular, if X is a pattern polynomial graph with odd number of vertices, then we show that $\dim(\mathcal{A}(X)) \leq \frac{n+1}{2}$.

Some graph classes which are pattern polynomial

We proved that the following classes of graphs are pattern polynomial graphs.

- Orbit polynomial graphs.

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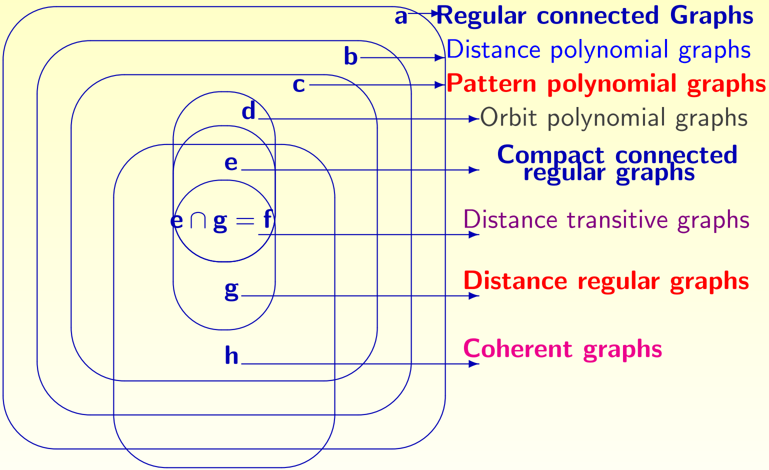

Figure: Graph classes

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A graph Y is said to be a polynomial in a graph X if $A(Y) \in \mathcal{A}(X)$.For an arbitrary graph X it seems difficult to find whether a given graph is polynomial in X or not. But the problem is tractable in case when X is a pattern polynomial graph.The following result gives a necessary and sufficient condition for a given graph to be a polynomial in a pattern polynomial graph.

A graph Y is said to be a polynomial in a graph X if $A(Y) \in \mathcal{A}(X)$.For an arbitrary graph X it seems difficult to find whether a given graph is polynomial in X or not. But the problem is tractable in case when X is a pattern polynomial graph.The following result gives a necessary and sufficient condition for a given graph to be a polynomial in a pattern polynomial graph.Which is also an extension of the results stated in [Robert A.Beezer [6]] and [Paul M.Weichsel [2]] on the polynomial of a graph.

Theorem

Let X be a pattern polynomial graph and let $\{P_1, P_2, \ldots, P_r\}$ where $P_1 = I$ be the standard basis of $\mathcal{A}(X)$. Then a graph Y is a polynomial in X if and only if $A(Y) = \sum_{i=2}^{r-1} a_i P_i$ where $a_i \in \{0, 1\}$.

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Corollary

There are 2^{r-1} graphs in the adjacency algebra of a pattern polynomial graph X, where r is the degree of the minimal polynomial of $A(X)$.

Lemma

Let a graph Y be a polynomial in a pattern polynomial graph X, then $\mathcal{CC}(Y) \subseteq \mathcal{CC}(X)$.

If a graph Y is a polynomial in a pattern polynomial graph X, then $\mathcal{CC}(Y)$ is a symmetric (every matrix in $\mathcal{CC}(Y)$ is symmetric) commutative algebra. Hence

- Y is a walk regular graph.
- Y is a strongly distance-balanced graph.
- Y has a multiple eigenvalue, whenever $Y \neq K_2$.
- $\dim(\mathcal{CC}(Y)) \leq n$, . Further if the number of vertices in Y is odd, then $\dim(\mathcal{CC}(Y)) \leq \frac{n+1}{2}$.

From the design theory point of view, a graph is a pattern polynomial graph, if its adjacency algebra is a Bose-Mesner algebra see the definition of Bose-Mesner algebra in the book by Brouwer, Cohen & Neumaier [4] or in the original paper by Bose & Mesner [3]. Consequently pattern polynomial graphs can be used to construct partially balanced incomplete block designs, for the definition of partially balanced incomplete block designs refer the book by Raghavarao [4].

In the above Figure 1, the sets **a, b, c, . . . , h** represent connected regular graphs, distance polynomial graphs, pattern polynomial graphs, . . . , coherent graphs, respectively.

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- Recall the cycle graph C_4 on four vertices and the matrix W_4, the companion matrix of $x^4 - 1$. Then $\{I, W_4^2, J - I - W_4^2 = W_4 + W_4^3\}$ is the standard basis of $\mathcal{CC}(C_4) = \mathcal{CC}(C_4^c)$. Hence, C_4^c is an example of a coherent graph that is not connected. Also, it can be easily checked that C_4^c is neither a distance polynomial graph nor a pattern polynomial graph. Similarly, one can verify that C_6^c is an example of a pattern polynomial graph that is not a coherent graph.

- Let X be a connected circulant graph of prime order. Then X is an orbit polynomial graph (see Beezer [4]). But X need not be a compact graph (see Lemma 2.2 in [1]). Similarly all connected circulant graphs of prime order are not distance transitive (see Theorem 1.2 in [4]).

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- Let X be a compact graph. Then using the fact that $\text{Aut}(X) = \text{Aut}(X^c)$ it is easy to verify that X is compact if and only if X^c is compact. Thus, C_6^c is an example of a compact connected regular graph that is not a distance transitive graph.

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- Let X be the line graph of the complete graph K_n, for $n \geq 7$. Then X is a distance transitive graph but not a compact graph for details refer Godsil [1].

- Let X be a distance transitive graph. Then it is easy to see that X is a distance regular graph. But, the well known Shrikhande graph (see Figure 3) is a distance regular graph that is not an orbit polynomial graph. Hence, the Shrikhande graph is also not a distance transitive graph. In fact, there are many distance regular graphs whose automorphism group is trivial (see Spence [3] or Weisfeiler [4]).

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- The truncated tetrahedron graph (see Figure 82) is an example of a connected regular graph that is not a distance polynomial graph (for details, see Weichsel [2]). But, if we assume that X is a k-regular connected graph with diameter 2 then X is clearly a distance polynomial graph ($A_2(X) = \mathbf{J} - I - A$).

- Let X be the truncated tetrahedron graph (see Figure 82). Then observe that X^c is a connected regular graph of diameter 2. Hence, X^c is an example of a distance polynomial graph, that is not a pattern polynomial graph.

- Let X be the truncated tetrahedron graph (see Figure 82). Then observe that X^c is a connected regular graph of diameter 2. Hence, X^c is an example of a distance polynomial graph, that is not a pattern polynomial graph.
- Let X be a distance regular graph of diameter ≥ 3 having trivial automorphism group (for examples of such graphs, see Spence [3] or Weisfeiler [4]). Also assume that X^c is connected. Then, using X^c is a pattern polynomial graph. But then the diameter of X is ≥ 3 implies that X^c is not a coherent graph and thus X^c is not a distance regular graph. Also, X^c is not an orbit polynomial graph as automorphism group of X is trivial. Consequently, X^c is an example of a pattern polynomial graph that is neither a distance regular graph nor an orbit polynomial graph.

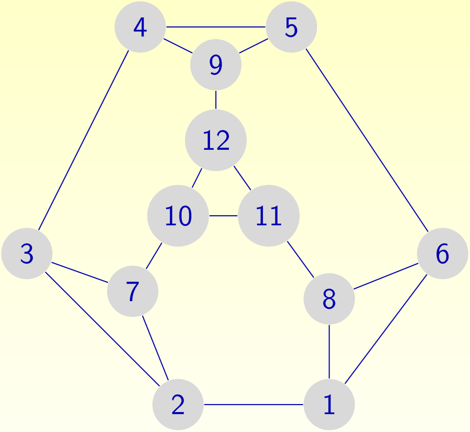

Figure: Truncated Tetrahedron Graph

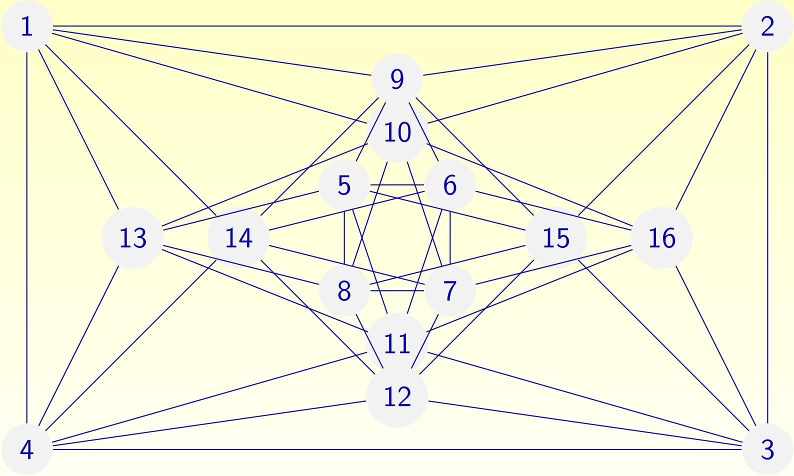

Figure: Shrikhande Graph

Now it is interesting to answer the following question: If Y is a graph such that $\mathcal{CC}(Y)$ is symmetric commutative algebra, then "does there exist a pattern polynomial graph X such that Y is a polynomial in X?". For example, if Y is a circulant graph (Cayley graph on cyclic group) with n vertices, then clearly $\mathcal{CC}(Y)$ is symmetric commutative algebra and it is also known that Y is a polynomial in cycle graph C_n, which is a pattern polynomial graph.

Let $X = (V, E)$ be a graph on n vertices and let A be its adjacency matrix.

1. Coherent Graph: A graph X is said to be a *coherent graph* if its adjacency matrix is a member of the standard basis of $\mathcal{CC}(X)$.
2. Compact Graph: A graph X is said to be a *compact graph* if every doubly stochastic matrix that commutes with A is a convex combination of matrices from $\text{Aut}(X)$.
3. Distance Polynomial Graph: Let X be a connected graph with diameter d and let $A_k(X)$, for $0 \leq k \leq d$, be the k-th distance matrix of X. Then X is said to be a *distance polynomial graph* if $A_k(X) \in \mathcal{A}(X)$, for $0 \leq k \leq d$.

1. Distance Regular Graph: A connected graph X is said to be a *distance regular graph* if for any two vertices u, v of X, the number of vertices at distance i from u and distance j from v depends only on i, j and $d(u, v)$, the distance between u and v.
2. Distance Transitive Graph: A graph X is said to be a *distance transitive graph* if for any four vertices u, v, x and y of X with $d(u, v) = d(x, y)$, there exists an element $g \in \text{Aut}(X)$, such that $g(u) = x$ and $g(v) = y$.
3. Edge Regular Graph: A graph X is said to be an *edge-regular graph* if every pair of adjacent vertices of X have the same number of common neighbors.

1. Orbit Polynomial Graph: A graph X is said to be an *orbit polynomial graph* if each orbital matrix is a member of $\mathcal{A}(X)$.
2. Pattern Polynomial Graph: A graph X is said to be a *pattern polynomial graph* if $\mathcal{A}(X) = \mathcal{CC}(X)$.
3. Walk Regular Graph: A graph X is said to be a *walk-regular graph* if for each s, the number of closed walks of length s, starting at a vertex v, is independent of the choice of v.

List of papers related to this work

1. A.K.Lal and A.Satyanarayana Reddy, *Non-singular circulant graphs and digraphs*, Electronic Journal of Linear Algebra, Volume 26,(2013), 248–257.
2. A.Satyanarayana Reddy, Shashank K Mehta and A.K.Lal, *Representation of Cyclotomic Fields and their Subfields*, Indian J. Pure Appl. Math., 44(2)(2013),203–230.
3. A.Satyanarayana Reddy, *Adjacency Algebra of Unitary Cayley Graph*, Journal of Global Research in Mathematical Archives, Vol 1, Issue 1(2013), 77–84.
4. A.Satyanarayana Reddy, *Few Non-derogatory Directed Graphs from Directed Cycles*, International Journal of Graph Theory, Vol 1,Issue 2(2013), 41–53.
5. A.Satyanarayana Reddy *Respectable Graphs*, International Journal of Mathematical Combinatorics, Vol.2 (2011),104–110.

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Thank you

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