Graphs with Simply Structured Eigenvectors

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Introduction

Let $G = (V, E)$ be a simple graph with n vertices.

Adjacency matrix: $A(G) = [a_{ij}]_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } [i,j] \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Laplacian matrix: $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix of G.

Laplacian spectrum: $S(G) = \Big(\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)\Big)$, where $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ are the eigenvalues of $L(G)$.

• $\lambda_1(G) = 0$ and $\mathbb{1}$, the vector of all ones, is a corresponding eigenvector.

Introduction

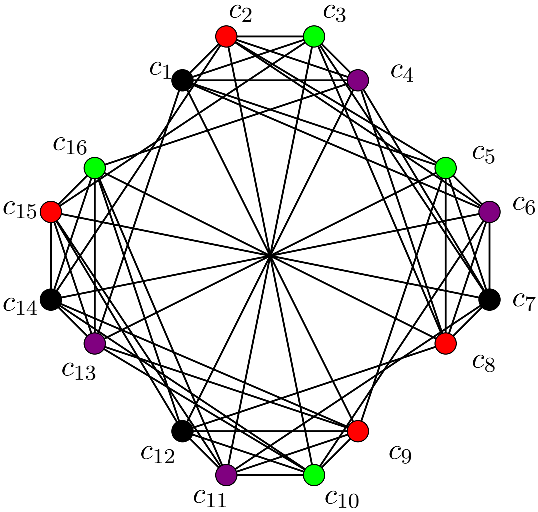


Figure: $G =$Sud(2), $\sigma(G) = (-3^{(4)}, -1^{(5)}, 1^{(4)}, 3^{(2)}, 7^{(1)})$.

Sudoku Graph: Sudoku graph, Sud(n) on n^4 vertices is obtained by taking each cell of the Sudoku as a vertex and two vertices are adjacent if and only if the corresponding cells in the Sudoku are situated in the same row, column or block.

1	2	3	4
3	4	1	2
2	3	4	1
4	1	2	3

Theorem (Sander[1]): All eigenvalues of Sud(n) are integers. All corresponding eigenspaces admit eigenvectors with the entries from the set $\{-1, 0, 1\}$.

[1] T. Sander, Sudoku graphs are integral, *The Electronic Journal of Combinatorics*, 16 (2009),#N25.

Introduction

Which graph does admit a simply structured eigenspace basis (eigenvectors entries come from the set $\{-1, 0, 1\}$)?

$\downarrow$

For a more straightforward eigenvector analysis it is desirable to achieve an eigenspace that is structurally simple.

Adjacency Case

- When G is regular of regularity r, $\mathbb{1}$ is an eigenvector $A(G)$ corresponding to the eigenvalue r.

Conjecture (Sander [2]): The null space of the adjacency matrix of every forest has a $\{-1, 0, 1\}$-basis.

- Partial answer was given when nullity is one.

[2] T. Sander, Eigenspace structure of certain graph classes, Ph.D Thesis, Technischen Universitat Clusthal (2004).

Introduction

Theorem (Akbari et. al [3]): For any forest F there exists a $\{-1,0,1\}$-basis for its null space. Indeed if F has n vertices with nullity s, then there exists an $n \times s$ $\{-1,0,1\}$-matrix, $\begin{bmatrix} X \\ I \end{bmatrix}$ whose columns form a basis for the null space of F.

Question: For which graphs is there a $\{-1,0,1\}$-basis for the null space?

[3] S. Akbari, A. Alipour, E. Ghorbani, G. Khosrovshahi, $\{-1,0,1\}$-Basis for the null space of a forest, *Linear Algebra Appl.* 414, 506–511, (2006).

Cographs

Definition: A graph that can be constructed from isolated vertices by a sequence of operations of unions and complements.

Join of G and H: $G \vee H = (G^c + H^c)^c$.

- Let $\mathcal{C}$ be a the class of graphs defined as follows:

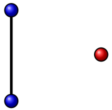
  1. $K_1 \in \mathcal{C}$.
  2. If $G, H \in \mathcal{C}$, then $G + H \in \mathcal{C}$, and
  3. If $G, H \in \mathcal{C}$, then $G \vee H \in \mathcal{C}$.

Then $\mathcal{C}$ is the class of cographs.

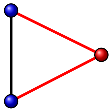
Example

Example

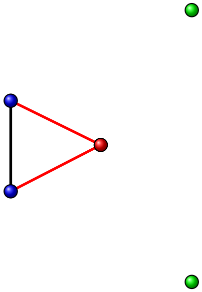
Example



Example



Example



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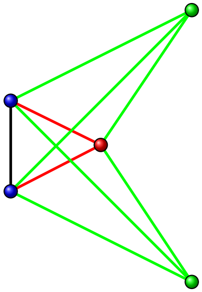

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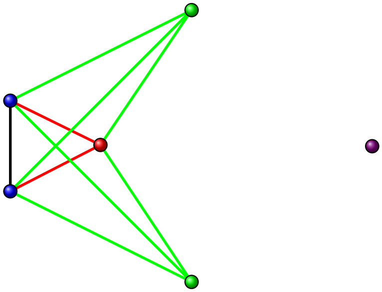
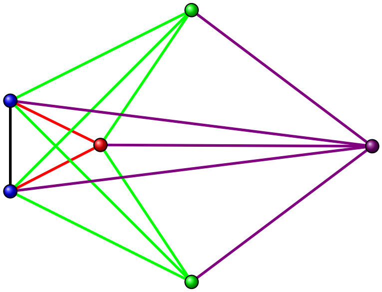


Example



Cograph

Theorem (Merris [4]): A graph is cograph if and only if it does not have an induced subgraph isomorphic to P_4.

Lemma (Merris): Let G be graph on $n \geq 2$ vertices. If G and G^c are both connected, then G has an induced subgraph isomorphic to P_4.

Note: If G is a connected cograph, then $G = G_1 \vee \cdots \vee G_k$, where $G_1, \ldots, G_k$ are cographs of lower order. (By induction)

[4] R. Merris, Laplacian graph eigenvectors, *Linear Algebra Appl.*, 278: 221–236, (1998).

Theorem (Royle[5]): The rank of a cograph G is equal to the number of distinct non-zero rows of $A(G)$.

(**Question:** Are there any other natural classes of graphs for which this rank property holds?)

Theorem (Sander[6]): Every cograph admits a simply structured eigenspace basis for the eigenvalues 0 and -1 with respect to the adjacency matrix.

- Sudoku Graphs, Hamming graphs, unitary Cayley graphs, GCD-graphs

[5] G. F. Royle, The rank of a cograph, *Electron. J. Combin.*, 10 (2003), #N11.

[6] T. Sander, On certain eigenspaces of cographs, *Electron. J. Combin.*, 15 (2008), #R140.

Laplacian Case

Theorem[7]: Let G be a cograph. Then there exists a basis B of eigenvectors of $L(G)$ such that each vector of B has **at most two** distinct nonzero entries.

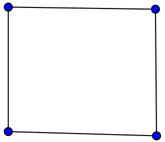
Converse is not true.

Lemma: If G is connected graph on $n \geq 3$ vertices, then $G\Box K_2$ is not a cograph.

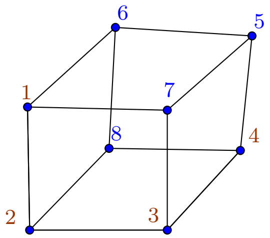
Cartesian product of G and H, $G\Box H$:
$V(G\Box H) = V(G) \times V(H)$ and two vertices (u, v) and (u', v') are adjacent in $G\Box H$ if either $u = u'$ and $v \sim v'$ in H or $u \sim u'$ in G and $v = v'$.

[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Introduction



C_4



$C_4 \square K_2$

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$x_i \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x_i \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Introduction

Special Case: We know $\mathbb{1}$ is an eigenvector of $L(G)$. Let us exclude 0 to be an entry in eigenvector.

Look for characterizations of all graphs G for which there exists a basis B of eigenvectors of $L(G)$ such that each vector of B has entries either 1 or -1.

Graph with property E: A graph G has property E if, for each eigenvalue λ of $L(G)$ there is a corresponding eigenvector with every entry equal to either 1 or -1.

Introduction

Hadamard matrix: An $n \times n$ matrix $H = [h_{ij}]$ is a Hadamard matrix of order n if the entries of H are either $+1$ or -1 and such that $HH^T = nI$.

Normalized Hadamard matrix: It is always possible to arrange to have the first row and first column of a Hadamard matrix contain only $+1$ entries. A Hadamard matrix in this form is said to be normalized.

Examples:

$$H_1 = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Introduction

The necessary condition for the existence of an $n \times n$ Hadamard matrix is that $n = 1, 2, 4k$ for some integer k.

Conjecture: (Hadamard) An $n \times n$ Hadamard matrix exists for $n = 1, n = 2$, and $n = 4k$ for any $k \in \mathbb{N}$.

Hadamard diagonalizable graphs

Definition: A graph G is said to be **Hadamard diagonalizable** if there is a Hadamard matrix H that diagonalizes $L(G)$.

Examples:

- K_2
- Empty graph of order 4, $K_2 + K_2$, $K_{2,2}$ and K_4.

Note: Any Hadamard diagonalizable graph has property E.

Lemma: A graph G is Hadamard diagonalizable if and only if there is a normalized Hadamard matrix that diagonalizes $L(G)$.

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[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Hadamard diagonalizable graphs

Observation: Let H be a normalized Hadamard matrix of order $n = 4k, k \geq 1$. Then

1. K_n is diagonalizable by H.
2. there is a permutation matrix P such that $K_{2k,2k}$ is diagonalizable by the Hadamard matrix PH.

Theorem: Let G be a graph which is Hadamard diagonalizable. Then G is regular and all its Laplacian eigenvalues are even integers.

[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Hadamard diagonalizable graphs

Lemma: Let G_1 and G_2 be two nonempty graphs. If $G_1 + G_2$ is Hadamard diagonalizable, then

(i) G_1 and G_2 both are regular graphs of same order and same regularity,

(ii) G_1 and G_2 both have even eigenvalues,

(iii) G_1 and G_2 share the same eigenvalues.

Lemma: Let G be Hadamard diagonalizable graph. Then G^c, $G + G$, and $G \vee G$ are also Hadamard diagonalizable.

Lemma: Let G_1 and G_2 be Hadamard diagonalizable. Then $G_1 \Box G_2$ is Hadamard diagonalizable.

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[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Hadamard diagonalizable graphs

Observation:
$K_8^c, K_2 + K_2 + K_2 + K_2, (K_2 \vee K_2) + (K_2 \vee K_2),$
$(K_2 \vee K_2)\Box K_2, K_4 + K_4, K_4 \Box K_2, (K_4 + K_4)^c = K_{4,4},$
$((K_2 \vee K_2) + (K_2 \vee K_2))^c, (K_2 + K_2 + K_2 + K_2)^c$
and K_8 are the all 10 graphs of order 8 which are Hadamard diagonalizable.

Theorem: K_{12}, $K_{6,6}$, K_{12}^c and $K_6 + K_6$ are the only graphs of order 12, which are Hadamard diagonalizable.

[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Eigenspaces for regular cographs

Proposition: Let $G = G_1 \vee \cdots \vee G_k$ be a regular connected cograph with property E. Then $G_1, \ldots, G_k$ are regular cographs with same order and the same degree of regularity. Further, the graphs $G_1, \ldots, G_k$ all share the same eigenvalues.

Lemma: Let G_1, G_2 be two connected regular cographs with property E on $n \geq 2$ vertices. If $S(G_1) = S(G_2)$, then $G_1 = G_2$.

[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Eigenspaces for regular cographs

Theorem: Let $S_0 = \{K_m : m \geq 2,\ m \text{ is even}\}$. For $i \in \mathbb{N}$, let $S_i = \{G^c \vee \ldots \vee G^c : G \in S_{i-1}$ and the number of joined copies of G^c is even$\}$. Then, Γ is a regular cograph with property E on $n \geq 2$ vertices if and only if $\Gamma \in S_i$ for some $i = 0, 1, 2, \ldots$.

Example: Let $G_0 = K_2 \in S_0$. Consider the graph $G_1 = K_2^c \vee K_2^c = C_4 \in S_1$. Let $G_2 = C_4^c \vee C_4^c \vee C_4^c \vee C_4^c$. Then $G_2 \in S_2$.

Theorem: Let G be a connected regular cograph. Then there is a basis of $\{1, -1\}$ eigenvectors for G if and only if $G \in S_i$ for some $i = 0, 1, 2, \ldots$.

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[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Eigenspaces for regular cographs

Observe that if $\Gamma \in S_i$, then there is a unique $(i+1)$-tuple of even integers associated with Γ:

If $\Gamma \in S_0$, then $\Gamma = K_{m_0}$ for some even m_0. Write $\Gamma \equiv G(m_0)$.

If $\Gamma \in S_i$ for some $i \geq 1$, then $\Gamma = \Gamma_1^c \vee \cdots \vee \Gamma_1^c$ (m_i copies) where $\Gamma_1^c = G(m_0, m_1, \ldots, m_{i-1}) \in S_{i-1}$. So, $\Gamma \equiv G(m_0, m_1, \ldots, m_i)$.

Lemma: $G(m_0, m_1, \ldots, m_i)$ has exactly $i+2$ distinct eigenvalues.

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[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Eigenspaces for regular cographs

Theorem: Label the distinct eigenvalues of $G(m_0, \ldots, m_i)$ as $0 = \mu_1 < \mu_2 < \cdots < \mu_{i+2}$. Then

(a) The dimension of the eigenspace corresponding to $\mu_{\lfloor \frac{i}{2} \rfloor + 2}$ is $m_i m_{i-1} \ldots m_1(m_0 - 1)$.

(b) For each $l = 1, \ldots, \lfloor \frac{i}{2} \rfloor + 1$, the dimension of the eigenspace corresponding to μ_l is $m_i m_{i-1} \ldots m_{i+4-2l}(m_{i+3-2l} - 1)$ (here we interpret this quantity as 1 when $l = 1$). Further, every $\{1, -1\}$ eigenvector corresponding to μ_l has the form $w \otimes \mathbb{1}_{m_0 m_1 \ldots m_{i+2-2l}}$ for some $\{1, -1\}$ vector $w \in \mathbb{R}^{m_{i+3-2l} m_{i+4-2l} \cdots m_i}$.

(c) For each $l = 1, \ldots, i - \lfloor \frac{i}{2} \rfloor$, the dimension of the eigenspace corresponding to μ_{i+3-l} is $m_i m_{i-1} \ldots m_{i+3-2l}(m_{i+2-2l} - 1)$ (here we interpret this quantity as $m_i - 1$ when $l = 1$). Further, every $\{1, -1\}$ eigenvector corresponding to μ_{i+3-l} has the form $w \otimes \mathbb{1}_{m_0 m_1 \ldots m_{i+1-2l}}$ for some $\{1, -1\}$ vector $w \in \mathbb{R}^{m_{i+2-2l} m_{i+3-2l} \cdots m_i}$.

Eigenspaces for regular cographs

Theorem: Suppose that we have even integers $m_0, m_1, \ldots, m_i$ such that $G(m_0, m_1, \ldots, m_i)$ is a Hadamard diagonalizable graph. Then for each $k = 0, \ldots, i$, there exists a Hadamard matrix of order $\prod_{j=k}^{i} m_j$.

- **We suspect that the converse of the above theorem is true. As a result, all Hadamard diagonalizable cographs would be completely described!**

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[7] S. Barik, S. Fallat and S. Kirkland, On the Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 435:1885–1902, (2011).

Recent Developments

Theorem: If n is even and there exists a Hadamard matrix of order n, then the graphs $K_n, K_{\frac{n}{2},\frac{n}{2}}, nK_1$ and $2K_{\frac{n}{2}}$ are Hadamard diagonalizable.

Theorem [8] Let G be a graph of order n. If $n = 8k + 4$ and G is Hadamard diagonalizable, then G is either K_n or $K_{\frac{n}{2},\frac{n}{2}}$ or nK_1 or $2K_{\frac{n}{2}}$.

[8] J. Breen, S. Butler, M. Fuentes, B. Lidický, M. Phillips, A. W. N. Riasanovsky, S.-Y. Song, R. R. Villagrán, C. Wiseman, X. Zhang, Hadamard diagonalizable graphs of order at most 36, arXiv:2007.09235v1 [math.CO], July 2020.

Recent Developments

Table: **Number of non-equivalent Hadamard matrices and number of Hadamard diagonalizable graphs**

Order	Hadamard matrices	Hadamard diagonalizable graphs
4	1	4
8	1	**10**
12	1	4
16	5	**50**
20	3	4
24	60	**26**
28	487	4
32	13,710,027	**10,196**
36	(unknown)	4

- Two Hadamard matrices are considered equivalent if one can be obtained from the other by negating rows or columns, or by interchanging rows or columns.

Recent Developments

- N. Johnston, S. Kirkland, S. Plosker, R. Storey and X. Zhang, Perfect quantum state transfer using Hadamard diagonalizable graphs, *Linear Algebra Appl.*, 531: 375–398, (2017).

- A. Chan, S. Fallat, S. Kirkland, J. C.-H. Lin, S. Nasserasr, and S. Plosker, Complex Hadamard diagonalizable graphs, arXiv:2001.00251v2 [math.CO] 19 Jul, 2020.

THANK YOU