# Expander Graphs and Ramanujan Graphs

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Consider large undirected graphs (networks). Can you construct an undirected graph $G$ on $n$ vertices (nodes) having two competing properties, (i) $G$ is very sparse, that is, the number of edges (connections) is very less (say $<< O(n^2)$), and (ii) $G$ is very well-connected, that is, for any nonempty subset $S$ of the vertices, the number of edges between $S$ and its complement $S'$ should be at least $c \min(|S|, |S'|)$ where $c > 0$ ? Their existence (and construction) is by no means obvious. Due to apparently conflicting features ((i), (ii)), on one hand the existence of such graphs is counterintuitive and on the other hand makes them extremely useful. Such a graphs are called expanders, our brain is speculated as an example of expander (neurons as vertices, synapses as edges). The application of expanders covers large spectrum in Computer science and Mathematics in directions like:

1. Error-correcting codes
2. Complexity theory
3. Explicit design of robust computer networks
4. Derandomization of random algorithms
5. Analysis of algorithms in computational group theory

We will mainly focus on expanders which are $d$-regular graphs (each vertex has exactly $d$ incident edges ). Let us define them more formally. Let $G = (V, E)$ be a $d$-regular graph on $n$ vertices with vertex set $V$ and edge set $E$. For any nonempty subset $S \subset V$, let $E(S, S')$ denote the set of edges where each edge has one end vertex in $S$ and other in $S'$. Often $E(S, S')$ is called the edge-boundary of $S$. The expansion ratio of $G$ (also called Cheeger constant or isoperimetric number) is defined as

$$h(G) = \min_{0<|S|\leq \frac{n}{2}} \frac{|E(S, S')|}{|S|}.$$

**Family of Expander Graphs**: A sequence of $d$-regular graphs $\{G_i\}i \in \mathbb{N}$ of size increasing with $i$ is a *Family of Expander Graphs* if there exists $\epsilon > 0$ such that $h(G_i) \geq \epsilon$ for all $i$.
A $d$-regular graph $G$ is ***Ramanujan graph*** if adjacency matrix of $G$ has second largest eigenvalue in modulus at most $2\sqrt{d-1}$. It turns out that there cannot be better expander graphs than Ramanujan graphs. WHY ? we will see in the talk..

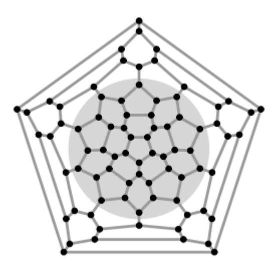

Figure 1: A Ramanujan graph on 80 vertices, $d = 3$. Expansion ratio is $\frac{1}{4}$ and $\lambda = 2.81811\ldots$