Eigenvalue bounds for some classes of matrices associated with graphs

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An Observation

Let $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 0 & -3 \end{bmatrix} \end{aligned}$$

An Observation

Let $A = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ -1 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 & -1 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{bmatrix} \end{aligned}$$

The general case

Let $e = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ and $e' = \begin{bmatrix} 1 & -1 & \cdots & -1 \end{bmatrix}^T$. We define the matrix P as

$$P = \begin{bmatrix} e & e_2 & e_3 & \ldots & e_n \end{bmatrix}.$$

It is easy to verify that the matrix P is nonsingular and its inverse is equal to

$$P^{-1} = \begin{bmatrix} e' & e_2 & e_3 & \ldots & e_n \end{bmatrix}.$$

The general case

Let $A = [a_{ij}]_{n\times n}$ be a matrix with constant row sum r. Then
$$P^{-1}AP = \begin{bmatrix} r & \mathbf{x}^T \\ o_{n-1} & A(1) \end{bmatrix} \text{ where } \mathbf{x}^T = \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix},$$
and

$$A(1) = A(1|1) - \mathbf{j}_{n-1}\mathbf{x}^T = A(1|1) - \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{12} & a_{13} & \cdots & a_{1n} \end{bmatrix}$$

The general case

Let $P_k = \left[\begin{array}{cccccccc} e_2 & e_3 & \cdots & e_k & e_1 & e_{k+1} & \cdots & e_n \end{array}\right]$.
Therefore, the matrix A is similar to the matrix

$$A_k = P_k^{-1} A P_k = \left[\begin{array}{cc} a_{kk} & a(k)^T \\ y & A(k|k) \end{array}\right], \ \forall k = 1, 2, \ldots, n,$$

where $a(k)^T = \left[\begin{array}{cccccc} a_{k1} & \cdots & a_{k,k-1} & a_{k,k+1} & \cdots & a_{kn} \end{array}\right]$
Now,

$$P^{-1} A_k P = \left[\begin{array}{cc} r & a(k)^T \\ 0 & A(k) \end{array}\right], \quad \text{where}$$

$$\begin{aligned} A(k) &= A(k|k) - j_{n-1} a(k)^T \\ &= A(k|k) - \left[\begin{array}{cccccc} a_{k1} & \cdots & a_{k,k-1} & a_{k,k+1} & \cdots & a_{kn} \\ a_{k1} & \cdots & a_{k,k-1} & a_{k,k+1} & \cdots & a_{kn} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{k,k-1} & a_{k,k+1} & \cdots & a_{kn} \end{array}\right] \end{aligned}$$

Applicability in Spectral graph theory

- The adjacency A matrix of simple regular graph.
- The Laplacian matrix of connected graph: $L = D - A$, where D is the diagonal matrix of vertex degrees.
- The normalized adjacency matrix of connected graph: $\mathcal{A} = D^{-1}A$.

The Perron-Frobenious theorem

Definition: A square matrix A is *reducible* if there is a permutation matrix P such that

$$P^T A P = \begin{bmatrix} B & C \\ 0_{n-r \times r} & D \end{bmatrix} \text{ and } 1 \leq r \leq n-1.$$

A square matrix is *irreducible* if it is not reducible.

The Perron-Frobenious theorem

Theorem

Let $A_{n\times n}$ be nonnegative and irreducible. Then there is a (unique) positive real number ρ with the following properties:

(i) *There is a real vector $x > 0$ with $Ax = \rho x$.*

(ii) *ρ has algebraic multiplicity 1.*

(iii) *For each eigenvalue λ of A, we have $|\lambda| \leq \rho$.*

The general case

Theorem

Let A be nonnegative irreducible matrix with constant row sum r. Then any eigenvalue of A other than r is also an eigenvalue of

$$A(k) = A(k|k) - \boldsymbol{j}_{n-1}\boldsymbol{a}(k)^T, \; k = 1, 2, \ldots, n$$

where $\boldsymbol{a}(k)^T = \begin{bmatrix} a_{k1} & \cdots & a_{k,k-1} & a_{k,k+1} & \cdots & a_{kn} \end{bmatrix}$ *is the k-deleted row of A.*

Localizing the eigenvalues

Let $A = [a_{ij}]$ be an $n \times n$ complex matrix. The i-th Geršgorin disc of A is defined by

$$R_i(A) = \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}.$$

Theorem

*(**Geršgorin**) Let $A = [a_{ij}]$ be an $n \times n$ complex matrix. Then the eigenvalues of A lie in the region*

$$G_A = \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.$$

Example

$$S = \begin{bmatrix} 0.25 & 0.25 & 0.3 & 0.2 \\ 0 & 0.5 & 0.33 & 0.17 \\ 0.6 & 0.4 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{bmatrix}.$$

$$\begin{aligned} S(1) &= \begin{bmatrix} 0.5 & 0.33 & 0.17 \\ 0.4 & 0 & 0 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} - \begin{bmatrix} 0.25 & 0.3 & 0.2 \\ 0.25 & 0.3 & 0.2 \\ 0.25 & 0.3 & 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.25 & 0.03 & -0.03 \\ 0.15 & -0.3 & -0.2 \\ -0.05 & 0 & 0.2 \end{bmatrix}. \end{aligned}$$

Example

$$S(2) = \begin{bmatrix} 0.25 & -0.03 & 0.03 \\ 0.6 & -0.33 & -0.17 \\ 0.1 & -0.03 & 0.23 \end{bmatrix},$$

$$S(3) = \begin{bmatrix} -0.35 & -0.15 & 0.2 \\ -0.6 & 0.1 & 0.17 \\ -0.5 & -0.2 & 0.4 \end{bmatrix},$$

$$S(4) = \begin{bmatrix} 0.15 & 0.05 & 0 \\ -0.1 & 0.3 & 0.03 \\ 0.5 & 0.2 & -0.3 \end{bmatrix}.$$

Example

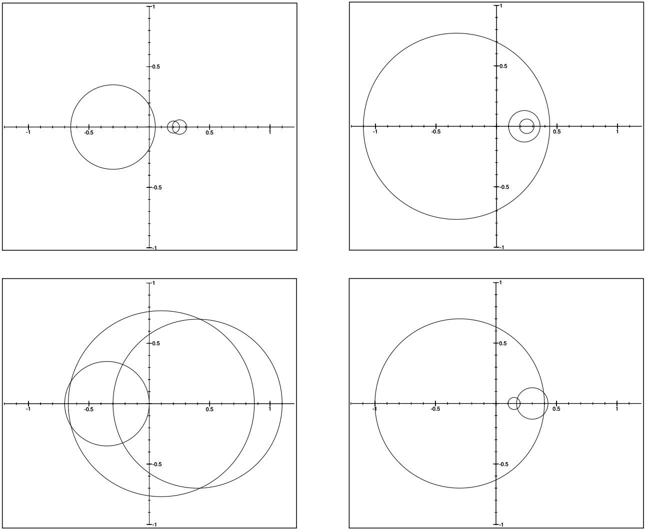

Figure: The regions $G_S(k)$, $k = 1, 2, 3, 4$.

An eigenvalue Localization theorem

Theorem

Let $A_{n\times n}$ be a matrix with constant row sum r. Then the eigenvalues of A lie in the region

$$\bigcap_{i=1}^{n}\left[G_{A(i)}\cup\{r\}\right],$$

where $G_{A(i)}=\bigcup_{k\neq i}\{z\in\mathbb{C}:|z-a_{kk}+a_{ik}|\leq\sum_{j\neq k}|a_{kj}-a_{ij}|\}$.

Bounding adjacency eigenvalues of regular graphs

Let $G = (V, E)$ be a simple r-regular graph with vertex set $V = \{1, 2, \ldots, n\}$ and A be the adjacency matrix of G. Then

$$A(1) = \begin{bmatrix} a_{22} - a_{12} & a_{23} - a_{13} & \cdots & a_{2n} - a_{1n} \\ a_{32} - a_{12} & a_{33} - a_{13} & \cdots & a_{3m} - a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n2} - a_{12} & a_{n3} - a_{13} & \cdots & a_{nn} - a_{1n} \end{bmatrix}$$

Here,

$$a_{jj} - a_{1j} = \begin{cases} -1, & \text{if } j \sim 1, \\ 0, & \text{if } j \nsim 1. \end{cases}$$

Bounding adjacency eigenvalues of regular graphs

$$a_{kj} - a_{1j} = \begin{cases} -1, & \text{if } j \sim 1, k \nsim j \\ 1, & \text{if } j \nsim 1, k \sim j \\ 0 & \text{otherwise.} \end{cases}$$

The Gershgorin discs for $A(1)$ are given by

$$\begin{cases} |z+1| \leq 2r - 2N(1,j) - 2 & \text{if } j \sim 1 \\ |z| \leq 2r - 2N(1,j) & \text{if } j \nsim 1. \end{cases}$$

Bounding adjacency eigenvalues of regular graphs

Theorem

Let G be a connected r-regular graph on n vertices. Then

$$-2r + \max_{i \in G}\{\min_{k \neq i}\{\alpha_{ik}\}, r\} \leq \lambda_n \leq \lambda_2 \leq 2r - \max_{i \in G}\{\min_{k \neq i}\{\beta_{ik}\}, r\},$$

where, for $k \neq i$, α_{ik} and β_{ik} are given by

$$\alpha_{ik} = \begin{cases} 1 + 2N(i,k), & \text{if } k \sim i \\ 2N(i,k), & \text{if } k \nsim i \end{cases}$$

and

$$\beta_{ik} = \begin{cases} 3 + 2N(i,k), & \text{if } k \sim i \\ 2N(i,k), & \text{if } k \nsim i. \end{cases}$$

The normalized adjacency matrix

Let $G = (V, E)$ be a finite, simple, connected, undirected graph with vertex set $V = \{1, 2, \ldots, n\}$. The normalized adjacency matrix $\mathcal{A} = [a_{ij}]$ is defined by

$$a_{ij} = \begin{cases} \frac{1}{d_i}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

- $\mathcal{A}$ is similar to symmetric matrix $D^{\frac{1}{2}} \mathcal{A} D^{-\frac{1}{2}}$. So all eigenvalues of $\mathcal{A}$ are real.
- $\sum_{j=1}^{n} a_{ij} = 1$ for all $i = 1, 2, \ldots, n$. So 1 is an eigenvalue of $\mathcal{A}$.
- All eigenvalues other than 1 lies in $[-1, 1)$.
- -1 is an eigenvalue of G iff G is bipartite.

Bound for normalized adjacency eigenvalues

Theorem

Let G be a simple connected graph of order n. Then

$$-2 + \max_{i \in G}\{\min_{k \neq i}\{\alpha_{ik}\}, 1\} \leq \lambda \leq 2 - \max_{i \in G}\{\min_{k \neq i}\{\beta_{ik}\}, 1\},$$

where, for $k \neq i$, α_{ik} and β_{ik} are given by

$$\alpha_{ik} = \begin{cases} \frac{1}{d_k} + \frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \sim i \\ \frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \nsim i \end{cases}$$

and

$$\beta_{ik} = \begin{cases} \frac{1}{d_k} + \frac{2}{d_i} + \frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \sim i \\ \frac{2N(i,k)}{\max\{d_i, d_k\}}, & \text{if } k \nsim i. \end{cases}$$

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Thank You