$\mathcal{M}$-join of graphs and its spectra

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1.1 Preliminaries

Notations in graph theory

- K_n - The complete graph on n vertices
- $K_{p,q}$ - The complete bipartite graph whose partite sets having p and q vertices
- C_n - Cycle of length n
- P_n - Path on n vertices
- $\overline{G}$ - The complement graph of a graph G
- $N_G(v)$ - Set of all neighbors of v in a graph G

Notations in Matrix theory

- $J_{n\times m}$ - The $n \times m$ matrix in which all the entries are 1
- $\sigma(M)$ - The spectrum of a matrix M
- $\mathcal{R}_{n\times m}(s) := \{[m_{ij}] \in M_{n\times m}(\mathbb{C}) | \sum_{j=1}^{m} m_{ij} = s \text{ for } i = 1, 2, \ldots, n\}$
- $\mathcal{C}_{n\times m}(c) := \{[m_{ij}] \in M_{n\times m}(\mathbb{C}) | \sum_{i=1}^{n} m_{ij} = c \text{ for } j = 1, 2, \ldots, m\}$
- $\mathcal{RC}_{n\times m}(s, c) := \mathcal{R}_{n\times m}(s) \cap \mathcal{C}_{n\times m}(c)$.

Matrices associated to graphs

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$.

- The **adjacency matrix of G**, denoted by $A(G) = [a_{ij}]$, is the $n \times n$ matrix defined as $a_{ij} = 1$, if $i \neq j$ and v_i and v_j are adjacent in G; 0, otherwise.
- The **vertex-edge incidence matrix of G** is the $n \times m$ matrix $B(G) = [b_{ij}]$ is defined as $b_{ij} = 1$, if the vertex v_i is incident with the edge e_j; 0, otherwise.
- The **degree matrix of G**, denoted by $D(G)$, is the diagonal matrix $diag(d_1, d_2, \ldots, d_n)$, where d_i denotes the degree of v_i in G.
- The **Laplacian matrix of G** is $L(G) = D(G) - A(G)$.
- The **signless Laplacian matrix of G** is $Q(G) = D(G) + A(G)$.
- The **normalized Laplacian matrix of G** is $\hat{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$.

- The multi set of eigenvalues of $A(G)$, $L(G)$, $Q(G)$ and $\hat{L}(G)$ are said to be **the $A-$spectrum, $L-$spectrum, $Q-$spectrum and $\hat{L}$-spectrum of G**, respectively.
- The characteristic polynomial of $A(G)$, $L(G)$, $Q(G)$ and $\hat{L}(G)$ are denoted by $P_G(x)$, $L_G(x)$, $Q_G(x)$ and $\hat{L}_G(x)$, respectively.
- The $A-$spectrum, $L-$spectrum and $Q-$spectrum of G are denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G), \tag{1.1}$$
$$0 = \mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G), \tag{1.2}$$
$$\nu_1(G) \geq \nu_2(G) \geq \ldots \geq \nu_n(G), \tag{1.3}$$

respectively.
- In 1995, Cvetković et al. [16] introduced **the generalized characteristic polynomial** $\phi_G(x, \beta)$ of G, which is defined as

$$\phi_G(x, \beta) = |xI_n - (A(G) - \beta D(G))|.$$

Notice that $P_G(x)$, $L_G(x)$ and $Q_G(x)$ are equal to $\phi_G(x, 0)$, $(-1)^n\phi_G(-x, 1)$ and $\phi_G(x, -1)$, respectively.

Cospectral graphs:

In 2003, Van Dam and Haemers [58] asked "Which graphs are determined by their spectra ?".

- Two graphs are said to be $A-$**cospectral** (resp. $L-$**cospectral**, $Q-$**cospectral**, $\hat{L}$-**cospectral)** if they have same $A-$spectrum (resp. $L-$spectrum, $Q-$spectrum, $\hat{L}$-spectrum).

What is the significance of constructing the cospectral graphs?

- Several structural properties are same for cospectral graphs.

In 2010, Butler [8] asked the following question: Is there an example of two non-regular graphs which are simultaneously A-cospectral, L-cospectral, Q-cospectral and $\widehat{L}$-cospectral ?

What is the need of graph operations?

- A natural question arise is **"How far the spectrum of a given graph can be expressed in terms the spectrum of some other graphs ?"**.
- In this point of view, to construct graphs from the given graphs, several graph operations were defined in literature such as the union, the complement, the subdivision, the Cartesian product, the Kronecker product, the NEPS, the corona, the join, deletion of a vertex, insertion/deletion of an edge, etc.

Some unary graph operations in the literature

- ⋆ $\mathcal{L}(G)$ - The line graph of G
- ⋆ $S(G)$ - The subdivision graph of G
- ⋆ $R(G)$ - The $R-$graph of G
- ⋆ $\mathcal{Q}(G)$ - The $\mathcal{Q}-$ graph of G
- ⋆ $T(G)$ - The total graph of G
- ⋆ $C(G)$ - The central graph of G
- ⋆ $QT(G)$ - The quasitotal graph of G
- ⋆ $Du(G)$ - The duplication graph of G

In 2017, M. Somodi et al. [55] defined the following graph operation which generalizes the constructions of the middle, total, and quasitotal graphs:

Overlay of G and G'

Let G and G' be two graphs having n vertices with same vertex labeling $\{v_1, v_2, \ldots, v_n\}$. Then the **overlay of G and G'**, denoted by $G \ltimes G'$ is the graph obtained by taking one copy of $\mathcal{Q}(G)$, and joining the vertices v_i and v_j of G if and only if v_i and v_j are adjacent in G'.

Join of graphs

The **join of two graphs G and H** is the graph obtained by taking one copy of G and H, and joining all the vertices of G to all the vertices in H.

Variants of join of graphs

Year	Authors	Definitions
2012	Indulal	S-vertex join and S-edge join of graphs
	Schwenk	H-generalized join of graphs
		H-generalized join of graphs constrained by vertex subsets
2015	Liu et.al	R-vertex join and R-edge join of graphs
	Varghese et.al	DG-vertex join and DG-add vertex join of graph
2017	Lu et.al	Subdivision vertex-vertex join and subdivision vertex-edge join of graphs
	Tian et. al	subdivision double join, R-graph double join, Q-graph double join, total double join of graphs
2018	S. Paul	Generalized subdivision vertex join of graphs

Corona of graphs

In 1970, the corona of two graphs was first introduced by Frucht and Harary to construct a graph whose automorphism group is the wreath product of the automorphism group of their components [22].

corona of G and H

Let G and H be two graphs with $|V(G)| = n$. The **corona of G and H** is the graph obtained by taking one copy of G and n copies of H, and joining the $i-$th vertex of G to all the vertices in the $i-$th copy of H for $i = 1, 2, \ldots, n$.

In 2007, Barik et al. [4] determined the A-spectrum (resp. the L-spectrum) of the corona of arbitrary graph G and a regular graph H (resp. for any graph G and H), in terms of the A-spectrum (resp. the L-spectrum) of G and H.

Variants of corona of graphs:

Year	Authors	Definitions
2010	Y. Hou and W- C. Shiu	Edge corona
2011	G. Indulal	Neighbourhood corona
2013	X. Liu and P. Lu	Subdivision vertex corona and subdivision edge corona of two graphs
	P.L. Lu and Y.F. Miao	Subdivision vertex neighbourhood corona and subdivision edge neighbourhood corona of two graphs
2014	P. L. Lu and Y. F. Miao	Corona-vertex of subdivision graph and corona-edge of subdivision graph of two graphs
2015	J. Lan et al.	$R-$vertex corona, the $R-$edge corona, the $R-$vertex neighbourhood corona and the $R-$edge neighbourhood corona of two graphs
	C. Adiga and B.R. Rakshith	$C-$vertex neighbourhood corona, the $N-$vertex corona, $C-$edge corona, $N-$edge corona of two graphs
2016	X. Q. Zhu et al.	Total corona
	S. Barik and G. Sahoo	Subdivision double corona of graphs, R-graph double corona of graphs, Q-graph double corona of graphs, total graph double corona of graph, subdivision double neighbourhood corona, R-graph double neighbourhood corona, Q-graph double neighbourhood corona, total graph double neighbourhood corona of graphs

2017	C. Adiga et al.	Extended neighborhood corona, extended corona of graphs
2018	C. Adiga et al.	The duplication vertex corona, the duplication edge corona of graphs
	W. Wen et al.	Subdivision vertex-edge neighbourhood vertex-corona (short for SVEV- corona), subdivision vertex-edge neighbourhood edge-corona (short for SVEE- corona)
	Q. Liu	Generalized R-vertex corona, generalized R-edge corona of graphs

(H_1, H_2)-merged subdivision graph of a graph

First we define the ternary graph operation as follows:

Definition 2.1.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let H_1 and H_2 be two graphs with $V(H_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(H_2) = \{w_1, w_2, \ldots, w_m\}$. Then the **$(H_1, H_2)$-merged subdivision graph of G**, denoted by $[S(G)]^{H_1}_{H_2}$, is the graph obtained by taking one copy of $S(G)$, and joining the vertices v_i and v_j if and only if the vertices u_i and u_j are adjacent in H_1 for $i, j = 1, 2, \ldots, n$, and joining the new vertices which lie on the edges e_t and e_s if and only if w_t and w_s are adjacent in H_2 for $t, s = 1, 2, \ldots, m$.

Notation 2.1.

We denote the graphs $[S(G)]^{\overline{K}_n}_{H}$ and $[S(G)]^{H}_{\overline{K}_m}$ simply by $[S(G)]_H$ and $[S(G)]^H$, respectively.

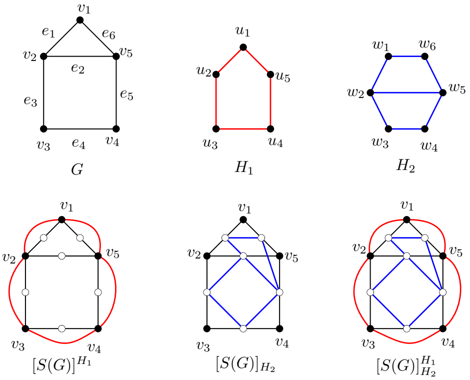


Figure 1: Examples for (H_1, H_2)-merged subdivision graph of a graph G

The construction used in Definition 2.1 generalizes many graph constructions: $S(G) \cong [S(G)]^{\overline{K_n}}_{\overline{K_m}}$, $R(G) \cong [S(G)]^{G}$ and $Ct(G) \cong [S(G)]^{\overline{G}}$. Also notice that the graph $[S(G)]^{H}_{\mathcal{L}(G)} \cong G \ltimes H$. Consequently, $\mathcal{Q}(G) \cong [S(G)]_{\mathcal{L}(G)}$, $T(G) \cong [S(G)]^{G}_{\mathcal{L}(G)}$, $QT(G) \cong [S(G)]^{\overline{G}}_{\mathcal{L}(G)}$ and the complete $\mathcal{Q}$-graph of G is isomorphic to $[S(G)]^{K_n}_{\mathcal{L}(G)}$.

Some of the special cases of $[S(G)]_{H_1}^{H_2}$ enable us to define some interesting unary graph operations:

Definition 2.2.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$.

(1) The **point complete subdivision graph of G** is the graph obtained by taking one copy of $S(G)$, and joining all the vertices $v_i, v_j \in V(G)$.

(2) The **$\mathcal{Q}$-complemented graph of G** is the graph obtained by taking one copy of $S(G)$, and joining the new vertices which lie on the non-adjacent edges of G.

(3) The **total complemented graph of G** is the graph obtained by taking one copy of $R(G)$, and joining the new vertices which lie on the non-adjacent edges of G.

(4) The **quasi-total complemented graph of G** is the graph obtained by taking one copy of $\mathcal{Q}$-complemented graph of G, and joining all the vertices $v_i, v_j \in V(G)$ which are not adjacent in G.

(5) The **complete $\mathcal{Q}$-complemented graph of G** is the graph obtained by taking one copy of $\mathcal{Q}$-complemented graph of G, and joining all the vertices of $v_i, v_j \in V(G)$.

(6) The **complete subdivision graph of G** is the graph obtained by taking one copy of $S(G)$, and joining all the new vertices which lie on the edges of G.

(7) The **complete R-graph of G** is the graph obtained by taking one copy of $R(G)$, and joining all the new vertices which lie on the edges of G.

(8) The **complete central graph of G** is the graph obtained by taking one copy of central graph of G, and joining all the new vertices which lie on the edges of G.

(9) The **fully complete subdivision graph of G** is the graph obtained by taking one copy of $S(G)$, and joining all the vertices of G and joining all the new vertices which lie on the edges of G.

Notice that the graphs mentioned in Definitions 2.2(1)-(9) are isomorphic to $[S(G)]^{K_n}$, $[S(G)]_{\overline{\mathcal{L}(G)}}$, $[S(G)]^{G}_{\overline{\mathcal{L}(G)}}$, $[S(G)]^{\overline{G}}_{\overline{\mathcal{L}(G)}}$, $[S(G)]^{K_n}_{\overline{\mathcal{L}(G)}}$, $[S(G)]_{K_m}$, $[S(G)]^{G}_{K_m}$, $[S(G)]^{\overline{G}}_{K_m}$, $[S(G)]^{K_n}_{K_m}$, respectively. The structures of these graphs for $G = C_4$ are shown in Figures 2(a)-(i), respectively. In these figures, the vertices colored with white represent the new vertices of $S(G)$.

Notation 2.2.

Let $\mathcal{U}_1$ be the collection of all unary graph operations defined in Definition 2.2 and the subdivision graph, the R-graph, the $\mathcal{Q}$-graph, the total graph, the central graph, the quasi-total graph, and the complete $\mathcal{Q}$-graph.

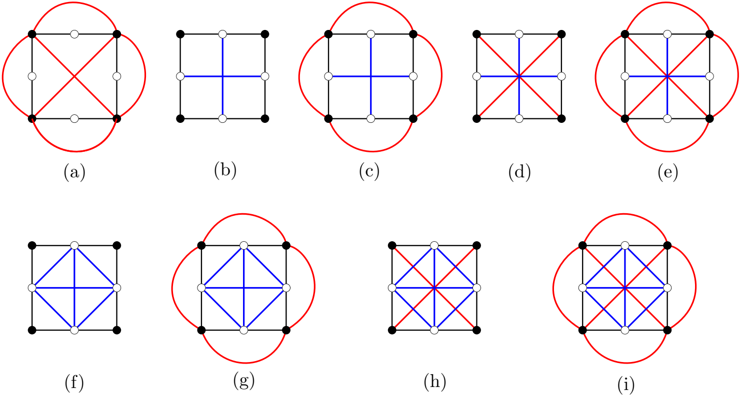


Figure 2: (a) The point complete subdivision graph of C_4, (b) The $\mathcal{Q}$-complemented graph of C_4, (c) The total complemented graph of C_4, (d) The quasi-total complemented graph of C_4, (e) The complete $\mathcal{Q}$-complemented graph of C_4, (f) The complete subdivision graph of C_4, (g) The complete R-graph of C_4, (h) The complete central graph of C_4, (i) The fully complete subdivision graph of C_4

Co-eigenvalues of matrices

Definition 2.3.

Let $A_1, A_2, \ldots, A_m$ be square matrices of order n with entries from $\mathbb{R}$. Then $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ are said to be **co-eigenvalues of** $A_1, A_2, \ldots, A_m$, if there exists a vector $X \in \mathbb{R}^n$ such that $A_i X = \lambda_i X$ for $i = 1, 2, \ldots, m$.

The following are some easy observations which will be used later.

Observation 2.1.

(1) If $A_1, A_2 \in M_n(\mathbb{R})$, then for each eigenvalue λ_1 of A_1, there need not exist an eigenvalue λ_2 of A_2 such that λ_1, λ_2 are co-eigenvalues of A_1, A_2.

(2) If $A_1, A_2, \ldots, A_m$ are symmetric and commutes with each other, then for each eigenvalue λ_1 of A_1, Proposition 1.3 ensures the existence of $\lambda_2, \lambda_3, \ldots, \lambda_m$ such that they are co-eigenvalues of $A_1, A_2, \ldots, A_m$.

(3) If λ is an eigenvalue of a matrix $A \in M_n(\mathbb{R})$, then $\lambda, 1$ are co-eigenvalues of A, I_n.

Continued...

(4) Let $A \in M_n(\mathbb{R})$ and $f(x) \in \mathbb{R}[x]$. If λ is an eigenvalue of A, then $\lambda, f(\lambda)$ are co-eigenvalues of A, $f(A)$.

(5) If G is an r-regular graph with n vertices, then $\lambda_i(G), \mu_i(G), \nu_i(G)$ are co-eigenvalues of $A(G)$, $L(G)$, $Q(G)$ for $i = 1, 2, \ldots, n$.

(6) If $f(x), g(x) \in \mathbb{R}[x]$ and λ_1, λ_2 are co-eigenvalues of A_1, A_2, then $f(\lambda_1), g(\lambda_2)$ are co-eigenvalues of $f(A_1), g(A_2)$. In particular, if G is an r-regular graph, $M \in M_n(\mathbb{R})$ and $\lambda(G), \lambda(M)$ are co-eigenvalues of $A(G)$, M, then $\mu(G), \lambda(M)$ are co-eigenvalues of $L(G)$, M, where $\mu(G) = r - \lambda(G)$; $\nu(G), \lambda(M)$ are co-eigenvalues of $Q(G), M$, where $\nu(G) = r + \lambda(G)$.

(7) If $f(x), g(x) \in \mathbb{R}[x]$ and λ_1, λ_2 are co-eigenvalues of A_1, A_2, then $\lambda_1, f(\lambda_1) + g(\lambda_2)$ are co-eigenvalues of $A_1, f(A_1) + g(A_2)$.

(8) If λ_1, λ_2 are co-eigenvalues of $A_1, A_2 \in M_n(\mathbb{R})$, then $\lambda_1 + \lambda_2$ is an eigenvalue of $A_1 + A_2$; $\lambda_1\lambda_2$ is an eigenvalue of A_1A_2.

Lemma 3.1

If $M \in \mathcal{RC}_{n\times n}(s, s)$, then s, n are co-eigenvalues of M, J_n. Also, $\lambda, 0$ are co-eigenvalues of M, J_n, where λ is an eigenvalue of M with an eigenvector X such that $X, J_{n\times 1}$ are linearly independent.

Corollary 2.1.

(1) *If G is a graph with n vertices, then the pair $0, n$, and for each $i = 2, 3, \dots, n$ the pairs $\mu_i(G), 0$ are co-eigenvalues of $L(G)$, J_n.*

(2) *If G is r-regular, then the pair r, n, and for each $i = 2, 3, \dots, n$, the pairs $\lambda_i(G), 0$, are co-eigenvalues of $A(G)$, J_n.*

(3) *If G is r-regular, then the pair $2r$, n, and for each $i = 2, 3, \dots, n$, the pairs $\nu_i(G)$, 0 are co-eigenvalues of $Q(G)$, J_n.*

Proposition 2.1.

Let G be a spanning r-regular subgraph of $K_{p,p}$. Then we have the following:

(1) *The co-eigenvalues of $A(G)$ and $A(K_{p,p})$ are: r, p; $-r, -p$ and $\lambda_i(G), 0$ for $i = 2, 3, \dots, 2p-1$;*

(2) *The co-eigenvalues of $L(G)$ and $L(K_{p,p})$ are: $0, 0$; $2r, 2p$ and $\mu_i(G), p$ for $i = 2, 3, \dots, 2p-1$;*

(3) *The co-eigenvalues of $Q(G)$ and $Q(K_{p,p})$ are: $2r, 2p$; $0, 0$ and $\nu_i(G), p$ for $i = 2, 3, \dots, 2p-1$,*

where $\lambda_i(G)$, $\mu_i(G)$ and $\nu_i(G)$ for $i = 1, 2, \dots, 2p$ are as in (1.1)–(1.3), respectively.

Spectra of (H_1, H_2)-merged subdivision graph of a graph

Now we proceed to determine the A-spectra, the L-spectra, the Q-spectra and the $\widehat{L}$-spectra of $[S(G)]_{H_2}^{H_1}$ for some families of G, H_1 and H_2, and the graphs constructed by the unary graph operations in $\mathcal{U}_1$.

It can be seen that

$$A\left([S(G)]_{H_2}^{H_1}\right) = \begin{bmatrix} A(H_1) & B(G) \\ B(G)^T & A(H_2) \end{bmatrix}, \tag{2.1}$$

$$L\left([S(G)]_{H_2}^{H_1}\right) = \begin{bmatrix} L(H_1)+D(G) & -B(G) \\ -B(G)^T & L(H_2)+2I_m \end{bmatrix}, \tag{2.2}$$

$$Q\left([S(G)]_{H_2}^{H_1}\right) = \begin{bmatrix} Q(H_1)+D(G) & B(G) \\ B(G)^T & Q(H_2)+2I_m \end{bmatrix}. \tag{2.3}$$

If G is r-regular ($r > 1$) and H_i is r_i-regular for $i = 1, 2$, then

$$\widehat{L}\left([S(G)]_{H_2}^{H_1}\right) = \begin{bmatrix} \frac{1}{r_1+r}[L(H_1)+D(G)] & \frac{1}{\sqrt{(r_1+r)(r_2+2)}}B(G) \\ \frac{1}{\sqrt{(r_1+r)(r_2+2)}}B(G)^T & \frac{1}{r_2+2}[L(H_2)+2I_m] \end{bmatrix}. \tag{2.4}$$

In the rest of the slides, we assume that

$$\theta_i = \begin{cases} 1 & \text{for } i = 1; \\ 0 & \text{for } i = 2, 3, \ldots, n. \end{cases}$$

Proposition 2.2.

Let $A \in M_n(\mathbb{R})$, $B \in \mathcal{RC}_{n\times m}(r, c)$, $t_1, t_2, t_3 \in \mathbb{R}$ and $c \neq 0$. Then the characteristic polynomial of the matrix

$$M = \begin{bmatrix} A & B \\ B^T & t_1 I_m + t_2 J_m + t_3 B^T B \end{bmatrix} \tag{2.5}$$

is

$$(x - t_1)^{m-n} \times \left| \left\{ (x - t_1) I_n - t_3 BB^T - \frac{t_2}{c} r J_n \right\} (xI_n - A) - BB^T \right|.$$

Moreover, if A and BB^T commutes with each other and $m \geq n$, then the spectrum of M contains

(i) *t_1 with multiplicity $m - n$;*

(ii) $\frac{1}{2}\left(\alpha_i + \lambda_i(A) \pm \sqrt{(\alpha_i - \lambda_i(A))^2 + 4\lambda_i(BB^T)}\right)$,
where $\alpha_i = t_1 + \frac{1}{c}\theta_i t_2 nr + t_3 \lambda_i(BB^T)$;
$\lambda_i(A)$, $\lambda_i(BB^T)$ *are co-eigenvalues of* A, BB^T *for* $i = 1, 2, \ldots, n$.

Corollary 2.2.

Let G be an r-regular graph ($r \geq 2$) with n vertices and m $(= \frac{1}{2}nr)$ edges. Let H_i be an r_i-regular graph, where H_1 has n vertices, which commutes with G, and $H_2 \in \{\overline{K}_m, K_m, \mathcal{L}(G), \overline{\mathcal{L}(G)}\}$. Then the A-spectrum, the L-spectrum, the Q-spectrum and the $\widehat{L}$-spectrum of $[S(G)]^{H_1}_{H_2}$ are

(i) t_1 *with multiplicity* $m - n$;

(ii) $\frac{1}{2}\left(\alpha_i + \beta_i \pm \sqrt{(\alpha_i - \beta_i)^2 + 4\gamma_2\nu_i(G)}\right)$,

where $\alpha_i = t_1 + 2\gamma_1 + \theta_i m t_2 + t_3\gamma_2\nu_i(G)$,

$$\gamma_1 = \begin{cases} 0 & \text{for } A\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ 1 & \text{for } L\text{-spectrum}, Q\text{-spectrum and } \widehat{L}\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \end{cases}$$

$$\gamma_2 = \begin{cases} 1 & \text{for } A\text{-specrurm}, L\text{-spectrum}, Q\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ \dfrac{1}{(r_1+r)(r_2+2)} & \text{for } \widehat{L}\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \end{cases}$$

$$\beta_i = \begin{cases} \lambda_i(H_1) & \text{for } A\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ \mu_i(H_1) + r & \text{for } L\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ \nu_i(H_1) + r & \text{for } Q\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ \dfrac{1}{r_1+r}(\mu_i(H_1) + r) & \text{for } \widehat{L}\text{-spectrum of } [S(G)]^{H_1}_{H_2}, \end{cases}$$

with $\nu_i(G)$, $\lambda_i(H_1)$, $\mu_i(H_1)$ and $\nu_i(H_1)$ are co-eigenvalues of $Q(G)$, $A(H_1)$, $L(H_1)$ and $Q(H_1)$ for $i = 1, 2, \ldots, n$ and t_1, t_2, t_3 are such that

$$t_1 I_m + t_2 J_m + t_3 B(G)^T B(G) = \begin{cases} A(H_2) & \text{for } A\text{-spectrum of } [S(G)]_{H_2}^{H_1}; \\ L(H_2) & \text{for } L\text{-spectrum of } [S(G)]_{H_2}^{H_1}; \\ Q(H_2) & \text{for } Q\text{-spectrum of } [S(G)]_{H_2}^{H_1}; \\ \dfrac{1}{r_2+2} L(H_2) & \text{for } \widehat{L}\text{-spectrum of } [S(G)]_{H_2}^{H_1}, \end{cases}$$

which can be obtained from Table 4.

S. No	Matrices
1.	$A(G) = -rI_n + B(G)B(G)^T$
2.	$L(G) = 2rI_n - B(G)B(G)^T$
3.	$Q(G) = (r-1)I_n + J_n - B(G)B(G)^T$
4.	$A(\overline{G}) = I_n + J_n - B(G)B(G)^T$
5.	$L(\overline{G}) = (m-2r)I_n - J_n + B(G)B(G)^T$
6.	$Q(\overline{G}) = (m-2r+2)I_m - J_m + B(G)B(G)^T$
7.	$A(\mathcal{L}(G)) = -2I_m + B(G)^T B(G)$
8.	$L(\mathcal{L}(G)) = 2rI_m - B(G)^T B(G)$
9.	$Q(\mathcal{L}(G)) = (2r-4)I_m + B(G)^T B(G)$
10.	$A(\overline{\mathcal{L}(G)}) = I_m + J_m - B(G)^T B(G)$
11.	$L(\overline{\mathcal{L}(G)}) = (m-2r)I_m - J_m + B(G)^T B(G)$

12.	$Q(\overline{\mathcal{L}(G)}) = (m-2r+2)I_m - J_m + B(G)^T B(G)$
13.	$A(K_m) = J_m - I_m$
14.	$L(K_m) = mI_m - J_m$
15.	$Q(K_m) = (m-2)I_m + J_m$

Table 4: Various matrices of the graphs expressed in terms of their incidence matrix, the identity matrix and the all-ones matrix

Corollary 2.3.

Let G be an r-regular graph ($r \geq 2$) with n vertices and m ($= \frac{1}{2}nr$) edges. Let H_i be an r_i-regular graph. Let $H_1 \in \{\overline{K}_n, K_n, G, \overline{G}\}$ and $H_2 \in \{\overline{K}_m, K_m, \mathcal{L}(G), \overline{\mathcal{L}(G)}\}$. Then the A-spectrum, the L-spectrum, the Q-spectrum and the $\widehat{L}$-spectrum of $[S(G)]^{H_1}_{H_2}$ can be obtained by taking $\beta_i = s_1 + r\gamma_1 + \theta_i n s_2 + s_3\nu_i(G)$ for $i = 1, 2, \ldots, n$ in Corollary 2.2, where s_1, s_2, s_3 are such that

$$s_1 I_n + s_2 J_n + s_3 B(G)B(G)^T = \begin{cases} A(H_1) & \text{for the } A\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ L(H_1) & \text{for the } L\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ Q(H_1) & \text{for the } Q\text{-spectrum of } [S(G)]^{H_1}_{H_2}; \\ \dfrac{1}{r_1 + r} L(H_1) & \text{for the } \widehat{L}\text{-spectrum of } [S(G)]^{H_1}_{H_2}, \end{cases}$$

which can be obtained from Table 4.

Corollary 2.4.

If G and G' are regular cospectral graphs, then $U(G)$ and $U(G')$ are simultaneously A-cospectral, L-cospectral, Q-cospectral and $\widehat{L}$-cospectral for $U \in \mathcal{U}_1$.

Remark 2.1.

Corollary 2.4 gives an affirmative answer to the question raised by Butler in ([8]).

(H_1, H_2)-merged subdivision graph of $K_{p,p}$

Corollary 2.5.

Let H be a spanning r_1-regular subgraph of $K_{p,p}$ and $H_2 \in \{\overline{K}_{p^2}, K_{p^2}, \mathcal{L}(K_{p,p}), \overline{\mathcal{L}(K_{p,p})}\}$. Then we have the following.

(1) *The A-spectrum, the L-spectrum, the Q-spectrum and the $\widehat{L}$-spectrum of $[S(K_{p,p})]^H_{H_2}$ can be obtained by taking $m = p^2$, $n = 2p$, $r = p$,*

$$\lambda_i(H_1) = \begin{cases} r_1 & \text{for } i = 1; \\ -r_1 & \text{for } i = 2p; \\ \lambda_i(H) & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases}$$

$$\mu_i(H_1) = \begin{cases} 0 & \text{for } i = 1; \\ 2r_1 & \text{for } i = 2p; \\ \mu_i(H) & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases}$$

$$\nu_i(H_1) = \begin{cases} 2r_1 & \text{for } i = 1; \\ 0 & \text{for } i = 2p; \\ \nu_i(H) & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases}$$

$$\nu_i(K_{p,p}) = \begin{cases} 2p & \text{for } i = 1; \\ 0 & \text{for } i = 2p; \\ p & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases}$$

in Corollary 2.2.

(2) The A-spectrum, the L-spectrum, the Q-spectrum and the $\widehat{L}$-spectrum of $[S(H)]_{K_{p^2}}^{K_{p,p}}$ can be obtained by replacing G, H_1, r by H, $K_{p,p}$, r_1, respectively, and substituting

$$\lambda_i(H_1) = \begin{cases} p & \text{for } i = 1; \\ -p & \text{for } i = 2p; \\ 0 & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases} \qquad \mu_i(H_1) = \begin{cases} 0 & \text{for } i = 1; \\ 2p & \text{for } i = 2p; \\ p & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases}$$

$$\nu_i(H_1) = \begin{cases} 2p & \text{for } i = 1; \\ 0 & \text{for } i = 2p; \\ p & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases} \qquad \nu_i(G) = \begin{cases} 2r_1 & \text{for } i = 1; \\ r_1 & \text{for } i = 2p; \\ \nu_i(H) & \text{for } i = 2, 3, \ldots, 2p-1, \end{cases}$$

in Corollary 2.2.

(H_1, H_2)-merged subdivision graph of $K_{1,m}$

Theorem 2.1.

If H is a graph with m vertices, then we have the following.

(1) *If H is r-regular, then the A-spectrum of $[S(K_{1,m})]_H$ is*

$$0, \frac{1}{2}\left(r \pm \sqrt{r^2 + 4m + 4}\right), \frac{1}{2}\left(\lambda_i(H) \pm \sqrt{\lambda_i(H)^2 + 4}\right) \text{ for } i = 2, 3, \ldots, m.$$

(2) The L-spectrum of $[S(K_{1,m})]_H$ is

$$0, \frac{1}{2}\left(m+3 \pm \sqrt{(m-1)^2+4}\right), \frac{1}{2}\left(\mu_i(H)+3 \pm \sqrt{[\mu_i(H)+1]^2+4}\right)$$

for $i = 2, 3, \ldots, m$.

(3) If H is r-regular $(r > 1)$, then the Q-spectrum of $[S(K_{1,m})]_H$ is
   - (i) $\frac{1}{2}\left(\nu_i(H)+3 \pm \sqrt{[\nu_i(H)+1]^2+4}\right)$ for $i = 2, 3, \ldots, m$,
   - (ii) the roots of the polynomial $x^3 - (m+2r+3)x^2 + (2mr+2m+2r+1)x - 2rm$.

(4) If H is r-regular $(r > 1)$, then the $\widehat{L}$-spectrum of $[S(K_{1,m})]_H$ is

$$0, 1, \frac{r+4}{r+2}, \frac{2r-\lambda_i(H)+4 \pm \sqrt{\lambda_i(H)^2+4r+8}}{2(r+2)} \text{ for } i = 2, 3, \ldots, m.$$

(H_1, H_2)-merged subdivision graph of P_n

Theorem 2.2.

([7, Theorem 3.2]) Let $n \geq 3$ and let $p(x)$ be a polynomial of degree less than n. Then $p(A(P_n))$ is the adjacency matrix of a graph if and only if $p(x) = P_{P_{2i+1}}(x)$, for some i, $0 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$.

Corollary 2.6.

Let $n \geq 3$ be an integer. If H is a graph with $A(H) = P_{P_{2i+1}}(A(P_{n-1}))$, for some i, with $0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor - 1$, then the A-spectrum of $[S(P_n)]_H$ is

$$0, \frac{c_j \pm \sqrt{c_j^2 + 8\left(\cos\frac{\pi j}{n} + 1\right)}}{2},$$

where $c_j = \sum_{k=0}^{i} (-1)^k \binom{2i+1-k}{k} \left(2\cos\frac{\pi j}{n}\right)^{2(i-k)+1}$ *and* $j = 1, 2, \ldots, n-1$.

$\mathcal{Q}$-complemented graph of a graph

Theorem 2.3.

Let G be a graph with n vertices and m edges. Then the characteristic polynomial of the adjacency matrix of the $\mathcal{Q}$-complemented graph of G is

$$(-1)^n(x-1)^m\left(1-\frac{x}{1-x}\Gamma_{\mathcal{L}(G)}\left(\frac{x^2+x-2}{1-x}\right)\right)Q_G(-x).$$

Corollary 2.7.

Let G be a graph with n vertices and m edges whose line graph is r-regular $(r \geq 1)$. Then the A-spectrum of the $\mathcal{Q}$-complemented graph of G is

$$1^{m-1}, -\nu_i(G), \frac{1}{2}\left(m-r-1 \pm \sqrt{(m-r-1)^2+4r+8}\right)$$

for $i = 2, 3, \ldots, n$.

Corollary 2.8.

The A-spectrum of the $\mathcal{Q}$-complemented graph of $K_{p,q}$ is

$$0, 1^{pq-1}, (-p)^{q-1}, (-q)^{p-1}, \frac{1}{2}\left(pq-p-q+1 \pm \sqrt{(pq-p-q+1)^2+4(p+q)}\right).$$

Complete subdivision graph of a graph

Theorem 2.4.

Let G be a graph with n vertices and m edges. Then the characteristic polynomial of the adjacency matrix of the complete subdivision graph of G is

$$(x+1)^{m-n}\left(1 - x\Gamma_{\mathcal{L}(G)}(x^2+x-2)\right) Q_G(x^2+x).$$

Corollary 2.9.

(1) *The A-spectrum of the complete subdivision graph of $tK_{1,2}$ $(t \geq 1)$ is*

$$0^t, \left(\frac{-1 \pm \sqrt{5}}{2}\right)^t, \left(\frac{-1 \pm \sqrt{13}}{2}\right)^{t-1}, \frac{1}{2}\left(2t-1 \pm \sqrt{(2t-1)^2+12}\right).$$

(2) Let G be a graph with n vertices and m edges whose line graph is r-regular ($r \geq 2$). Then the A-spectrum of the complete subdivision graph of G is

$$(-1)^{m-n}, \frac{1}{2}\left(m-1 \pm \sqrt{(m-1)^2+4r+8}\right), \frac{1}{2}\left(-1 \pm \sqrt{4\nu_i(G)+1}\right)$$

for $i = 2, 3, \dots, n$.

Corollary 2.10.

Let $(p,q) \neq (1,2), (2,1)$. Then the A-spectrum of the complete subdivision graph of $K_{p,q}$ is

$$0, (-1)^{\alpha}, \left(\frac{-1 \pm \sqrt{4p+1}}{2}\right)^{q-1}, \left(\frac{-1 \pm \sqrt{4q+1}}{2}\right)^{p-1}, \frac{1}{2}\left(\beta \pm \sqrt{\beta^2+4(p+q)}\right),$$

where $\alpha = pq - p - q + 1$; $\beta = pq - 1$.

$\mathcal{M}$-join of graphs

In 1969, Hedetniemi [26] introduced the following generalization of the join of two graphs, and studied its several graph theoretical properties.

Definition 3.1.

([26]) For given graphs G and H, and a binary relation $\pi \subseteq V(G) \times V(H)$, the **$\pi$-graph of G and H** is the graph whose vertex set is $V(G) \cup V(H)$ and the edge set is $E(G) \cup E(H) \cup \pi$.

Notice that the binary relation π can be viewed as a matrix $M = [m_{ij}]$, where $m_{ij} = 1$ or 0, if the i-th vertex of G and the j-th vertex of H are related or not, respectively. So, we restate Definition 3.1 by using M as follows and call that graph as the M-join of G and H.

Definition 3.2.

Let G and H be graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$ and let M be a $0-1$ matrix of size $n \times m$. Then **the M-join of G and H** is the graph, denoted by $G \vee_M H$ and is obtained by taking one copy of G and H, and joining the vertices u_i and v_j if and only if the (i,j)-th entry of M is 1 for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$.

Example 4.1

Consider the graphs G and H as shown in Figure 3. Let $\pi = \{(u_2, v_1), (u_2, v_2), (u_2, v_4), (u_3, v_1), (u_4, v_1), (u_4, v_2), (u_4, v_4)\}$. Consequently π can be viewed as the matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $G \vee_M H$ is as shown in Figure 3.

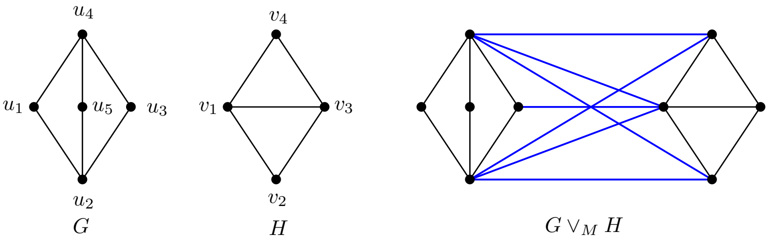


Figure 3: Example for M-join of two graphs

Next, we extend this definition for k graphs by using a sequence of matrices.

Definition 3.3.

Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of graphs with $|V(H_i)| = n_i$ for $i = 1, 2, \ldots, k$ and let $\mathcal{M} = (M_{12},\ M_{13},\ \ldots,\ M_{1k},\ M_{23},\ M_{24},\ \ldots,\ M_{2k},\ \ldots,\ M_{(k-1)k})$, where M_{ij} is a $0-1$ matrix of size $n_i \times n_j$. Then **the $\mathcal{M}$-join of the graphs in $\mathcal{H}_k$**, denoted by $\bigvee_{\mathcal{M}} \mathcal{H}_k$, is the graph $\displaystyle\bigcup_{\substack{i,j=1,\\ i<j}}^{k} (H_i \vee_{M_{ij}} H_j)$.

Notice that any given graph G can be viewed as a $\mathcal{M}$-join of $\mathcal{H}_k$, where $\mathcal{H}_k = (H_1,\ H_2,\ \ldots,\ H_k)$ is a sequence of k pairwise, vertex disjoint induced subgraphs of G and $V(H_i) = \{u_{i1}, u_{i2}, \ldots, u_{in_i}\}$ for $i = 1, 2, \ldots, k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$ and $\mathcal{M} = (M_{12},\ M_{13},\ \ldots,\ M_{1k},\ M_{23},\ M_{24},\ \ldots,\ M_{2k},\ \ldots,\ M_{(k-1)k})$ with $M_{ij} = [m_{rs}^{(ij)}]_{n_i \times n_j}$, where

$$m_{rs}^{(ij)} = \begin{cases} 1 & \text{if } u_{ir} \text{ and } u_{js} \text{ are adjacent in } G; \\ 0 & \text{otherwise} \end{cases}$$

for $r = 1, 2, \ldots, n_i$; $s = 1, 2, \ldots, n_j$; $i, j = 1, 2, \ldots, k$ and $i < j$. Consequently, the $\mathcal{M}$-join of graphs generalize all the variants of join of graphs.

Example 4.3

Let H_1, H_2, H_3 be the graphs as shown in Figure 4 and let

$$M_{12} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, M_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let $\mathcal{H}_3 = (H_1, H_2, H_3)$ and $\mathcal{M} = (M_{12}, M_{13}, M_{23})$. Then the $\mathcal{M}$-join of $\mathcal{H}_3$ is shown in Figure 4.

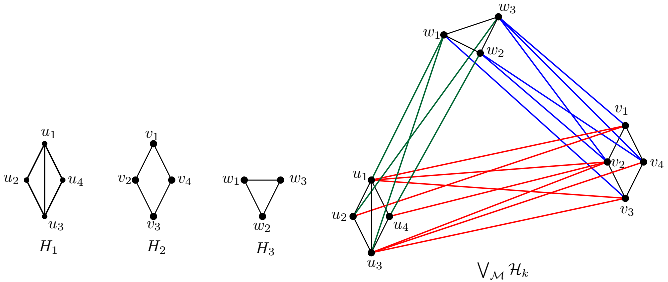


Figure 4: Example for $\mathcal{M}$-join of graphs

Unary graph operations as M-join of two graphs

As particular cases of the M-join of two graphs, we obtain some existing and new unary graph operations, which are described here. Let G be a graph with n vertices.

S. No	Description	Name of the unary graph operations
1.	$G \vee_{I_n} \overline{K}_n$	C-graph of G [1]
2.	$G \vee_{I_n} G$	Mirror graph of G [43]
3.	$G \vee_{I_n} \overline{G}$	V-complemented neighbourhood graph of G
4.	$G \vee_{I_n} K_n$	C-complete graph of G
5.	$G \vee_{J_n - I_n} \overline{K}_n$	VC-graph of G
6.	$G \vee_{J_n - I_n} G$	VC-neighbourhood graph of G
7.	$G \vee_{J_n - I_n} \overline{G}$	VC-complemented neighbourhood graph of G
8.	$G \vee_{J_n - I_n} K_n$	VC-complete graph of G
9.	$G \vee_{J_n} \overline{K}_n$	Join graph of G
10.	$G \vee_{J_n} G$	Join neighbourhood graph of G
11.	$G \vee_{J_n} \overline{G}$	Join complemented neighbourhood graph of G
12.	$G \vee_{J_n} K_n$	Join complete graph of G
13.	$G \vee_{A(G)} \overline{K}_n$	N-graph of G [1]
14.	$G \vee_{A(G)} G$	N-neighbourhood graph of G
15.	$G \vee_{A(G)} \overline{G}$	N-complemented neighbourhood graph of G
16.	$G \vee_{A(G)} K_n$	N-complete graph of G
17.	$G \vee_{A(G)+I_n} \overline{K}_n$	$\overline{N}$-graph of G
18.	$G \vee_{A(G)+I_n} G$	$\overline{N}$-neighbourhood graph of G
19.	$G \vee_{A(G)+I_n} \overline{G}$	$\overline{N}$-complemented neighbourhood graph of G
20.	$G \vee_{A(G)+I_n} K_n$	$\overline{N}$-complete graph of G
21.	$G \vee_{A(\overline{G})} \overline{K}_n$	NC-graph of G
22.	$G \vee_{A(\overline{G})} G$	NC-neighbourhood graph of G

Table 5

23.	$G \vee_{A(\overline{G})} \overline{G}$	NC-complemented neighbourhood graph of G
24.	$G \vee_{A(\overline{G})} K_n$	NC-complete graph of G
25.	$G \vee_{A(\overline{G})+I_n} \overline{K}_n$	$\overline{N}C$-graph of G
26.	$G \vee_{A(\overline{G})+I_n} G$	$\overline{N}C$-neighbourhood graph of G
27.	$G \vee_{A(\overline{G})+I_n} \overline{G}$	$\overline{N}C$-complemented neighbourhood graph of G
28.	$G \vee_{A(\overline{G})+I_n} K_n$	$\overline{N}C$-complete graph of G
29.	$\overline{G} \vee_{I_n} \overline{K}_n$	C-complement graph of G
30.	$\overline{G} \vee_{I_n} \overline{G}$	Mirror-complement graph of G
31.	$\overline{G} \vee_{I_n} K_n$	C-complete-complement graph of G
32.	$\overline{G} \vee_{J_n-I_n} \overline{K}_n$	VC-complement graph of G
33.	$\overline{G} \vee_{J_n-I_n} \overline{G}$	VC-neighbourhood-complement graph of G
34.	$\overline{G} \vee_{J_n-I_n} K_n$	VC-complete-complement graph of G
35.	$\overline{G} \vee_{J_n} \overline{K}_n$	Join-complement graph of G
36.	$\overline{G} \vee_{J_n} \overline{G}$	Join neighbourhood-complement graph of G
37.	$\overline{G} \vee_{J_n} K_n$	Join complete-complement graph of G
38.	$\overline{G} \vee_{A(G)} \overline{K}_n$	N-complement graph of G
39.	$\overline{G} \vee_{A(G)} \overline{G}$	N-neighbourhood-complement graph of G
40.	$\overline{G} \vee_{A(G)} K_n$	N-complete-complement graph of G
41.	$\overline{G} \vee_{A(G)+I_n} \overline{K}_n$	$\overline{N}$-complement graph of G
42.	$\overline{G} \vee_{A(G)+I_n} \overline{G}$	$\overline{N}$-neighbourhood-complement graph of G
43.	$\overline{G} \vee_{A(G)+I_n} K_n$	$\overline{N}$-complete-complement graph of G
44.	$\overline{G} \vee_{A(\overline{G})} \overline{K}_n$	NC-complete-complement graph of G
45.	$\overline{G} \vee_{A(\overline{G})} \overline{G}$	NC-neighbourhood-complement graph of G
46.	$\overline{G} \vee_{A(\overline{G})} K_n$	NC-complement graph of G

47.	$\overline{G} \vee_{A(\overline{G})+I_n} \overline{K}_n$	$\overline{N}C$-complement graph of G
48.	$\overline{G} \vee_{A(\overline{G})+I_n} \overline{G}$	$\overline{N}C$-neighbourhood-complement graph of G
49.	$\overline{G} \vee_{A(\overline{G})+I_n} K_n$	$\overline{N}C$-complete-complement graph of G
50.	$\overline{K}_n \vee_{A(G)} \overline{K}_n$	Duplicate graph of G [53]
51.	$K_n \vee_{A(G)} \overline{K}_n$	Duplicate complete graph of G
52.	$K_n \vee_{A(G)} K_n$	Fully complete duplicate graph of G
53.	$\overline{K}_n \vee_{A(G)+I_n} \overline{K}_n$	$D\overline{N}$-graph of G
54.	$K_n \vee_{A(G)+I_n} \overline{K}_n$	$D\overline{N}$-complete graph of G
55.	$K_n \vee_{A(G)+I_n} K_n$	Fully complete $D\overline{N}$-graph of G
56.	$\overline{K}_n \vee_{A(\overline{G})} \overline{K}_n$	Complemented duplicate graph of G
57.	$K_n \vee_{A(\overline{G})} \overline{K}_n$	Complemented duplicate complete graph of G
58.	$K_n \vee_{A(\overline{G})} K_n$	Fully complete complemented duplicate graph of G
59.	$\overline{K}_n \vee_{A(\overline{G})+I_n} \overline{K}_n$	Closed duplicate graph of G
60.	$K_n \vee_{A(\overline{G})+I_n} \overline{K}_n$	Closed duplicate complete graph of G
61.	$K_n \vee_{A(\overline{G})+I_n} K_n$	Fully complete closed duplicate graph of G

Table 6: Some (existing and new) unary graph operations defined using M-join of two graphs

S. No	Description	Name of the unary graph operation
1.	$\overline{K}_n \vee_{J_{n\times m}-B(G)} \overline{K}_m$	*DEC*-graph of G
2.	$G \vee_{J_{n\times m}-B(G)} \overline{K}_m$	*EC*-graph of G
3.	$\overline{G} \vee_{J_{n\times m}-B(G)} \overline{K}_m$	Complemented *EC*-graph of G
4.	$K_n \vee_{J_{n\times m}-B(G)} \overline{K}_m$	Point complete *DEC*-graph of G
5.	$\overline{K}_n \vee_{J_{n\times m}-B(G)} \mathcal{L}(G)$	$\mathcal{Q}$-*DEC*-graph of G
6.	$G \vee_{J_{n\times m}-B(G)} \mathcal{L}(G)$	Total *DEC*-graph of G
7.	$\overline{G} \vee_{J_{n\times m}-B(G)} \mathcal{L}(G)$	Central *DEC*-graph of G
8.	$K_n \vee_{J_{n\times m}-B(G)} \mathcal{L}(G)$	Complete $\mathcal{Q} -$ *DEC*-graph of G
9.	$\overline{K}_n \vee_{J_{n\times m}-B(G)} \overline{\mathcal{L}(G)}$	$\mathcal{Q}$-complemented *DEC*-graph of G
10.	$G \vee_{J_{n\times m}-B(G)} \overline{\mathcal{L}(G)}$	Total complemented graph of G
11.	$\overline{G} \vee_{J_{n\times m}-B(G)} \overline{\mathcal{L}(G)}$	Double complemented total DEC graph of G
12.	$K_n \vee_{J_{n\times m}-B(G)} \overline{\mathcal{L}(G)}$	Complete $\mathcal{Q}$-complemented *DEC*-graph of G
13.	$\overline{K}_n \vee_{J_{n\times m}-B(G)} K_m$	Complete *DEC*-graph of G
14.	$G \vee_{J_{n\times m}-B(G)} K_m$	Complete *EC*-graph of G
15.	$\overline{G} \vee_{J_{n\times m}-B(G)} K_m$	Complemented *EC*-graph of G
16.	$K_n \vee_{J_{n\times m}-B(G)} K_m$	Fully complete *DEC*-graph of G

Table 7: Some new unary graph operations defined using M-join of two graphs

Notation 3.1.

(1) Let $\mathcal{U}_2$ and $\mathcal{U}_3$ be the set of all unary graph operations mentioned in Table 6 and 7, respectively.

1. Let $\mathcal{U} := \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$.

(2) The new vertices of $U(G)$ are denoted by $I(G)$ for $U \in \mathcal{U}$.

Definition 3.4.

Let G be a regular graph with n vertices and m edges.

(1) Let $U \in \mathcal{U}_1$. Then $U(G) = H_1 \vee_{B(G)} H_2$, where $H_1 \in \{G, \overline{G}, K_n, \overline{K}_n\}$ and $H_2 \in \{\mathcal{L}(G), \overline{\mathcal{L}(G)}, K_m, \overline{K}_m\}$. So, by using Table 4, we can write $A(H_1) = bI_n + b'J_n + b''B(G)B(G)^T$ and $A(H_2) = cI_m + c'J_m + c''B(G)^TB(G)$. Then we say that, the sequence (b, b', b'', c, c', c'') of scalars as *the scalars corresponding to U in $\mathcal{U}_1$* or **the sequence of scalars corresponding to $A(U(G))$ for $U \in \mathcal{U}_1$**.

(2) Let $U \in \mathcal{U}_2$. Then $U(G) = H_1 \vee_M H_2$, where $H_1, H_2 \in \{G, \overline{G}, K_n, \overline{K}_n\}$ and $M \in \{I_n, A(G), A(G) + I_n, A(\overline{G}), A(\overline{G}) + I_n, A(K_n), J_n, \mathbf{0}\}$. So, by using Table 4, we can write $A(H_1) = b_1 I_n + b_1' J_n + b_1'' B(G)B(G)^T$; $M = b_2 I_n + b_2' J_n + b_2'' B(G)B(G)^T$ and $A(H_2) = b_3 I_n + b_3' J_n + b_3'' B(G)B(G)^T$. Then we say that, the sequence $(b_1, b_1', b_1'', b_2, b_2', b_2'', b_3, b_3', b_3'')$ of scalars as *the scalars corresponding to U in $\mathcal{U}_2$* or **the sequence of scalars corresponding to $A(U(G))$ for $U \in \mathcal{U}_2$**.

(3) Let $U \in \mathcal{U}_3$. Then $U(G) = H_1 \vee_{J_{n\times m} - B(G)} H_2$, where $H_1 \in \{G, \overline{G}, K_n, \overline{K}_n\}$ and $H_2 \in \{\mathcal{L}(G), \overline{\mathcal{L}(G)}, K_m, \overline{K}_m\}$. So, by using Table 4, we can write $A(H_1) = bI_n + b' J_n + b'' B(G)B(G)^T$ and $A(H_2) = cI_m + c' J_m + c'' B(G)^T B(G)$. Then we say that, the sequence (b, b', b'', c, c', c'') of scalars as *the scalars corresponding to U in $\mathcal{U}_3$* or **the sequence of scalars corresponding to $A(U(G))$ for $U \in \mathcal{U}_3$**.

Similarly, we can define the **sequence of scalars corresponding to $L(U(G))$, $Q(U(G))$ for $U \in \mathcal{U}$**.

In the rest of the slides we assume the following, for a given graph G:

$$\alpha = \begin{cases} 1 & \text{for the } A\text{-spectrum and the } Q\text{-spectrum of } G; \\ -1 & \text{for the } L\text{-spectrum of } G; \end{cases}$$

$$\rho = \begin{cases} 0 & \text{for the } A\text{-spectrum of } G; \\ 1 & \text{for the } L\text{-spectrum and the } Q\text{-spectrum of } G \end{cases}$$

and $\theta_t = \begin{cases} 1 & \text{for } t = 1; \\ 0 & \text{for } t = 2, 3, \ldots, n. \end{cases}$

Next, we deduce the spectra of the graphs constructed by the unary graph operations in $\mathcal{U}$.

Theorem 3.1.

Let G be an r-regular graph ($r \geq 2$) with n vertices and $m(= \frac{1}{2}nr)$ edges. Then we have the following.

(1) *If $U \in \mathcal{U}_1$, then the A-spectrum, the L-spectrum and the Q-spectrum of $U(G)$ are*

(i) $b_2 + 2\rho$ *with multiplicity* $m - n$;

(ii) $\frac{1}{2}\left(\alpha_1^{(t)} + \alpha_2^{(t)} \pm \sqrt{\left(\alpha_1^{(t)} - \alpha_2^{(t)}\right)^2 - 4\nu_t(G)}\right)$ *for* $t = 1, 2, \ldots, n$,

where $\alpha_1^{(t)} = b_1 + r\rho + \theta_t n b_1' + b_1''\nu_t(G)$; $\alpha_2^{(t)} b_2 + 2\rho + \theta_t m b_2' + b_2''\nu_t(G)$ *for* $t = 1, 2, \ldots, n$; $(b_1, b_1', b_1'', b_2, b_2', b_2'')$ *is the sequence of scalars corresponding to* $A(U(G))$, $L(U(G))$ *and* $Q(U(G))$ *in* $\mathcal{U}_1$ *for A-spectrum, the L-spectrum and the the Q-spectrum of* $U(G)$, *respectively.*

(2) *If $U \in \mathcal{U}_2$, then the A-spectrum, the L-spectrum and the Q-spectrum of $U(G)$ are*

$$\frac{1}{2}\left(\alpha_1^{(t)} + \alpha_2^{(t)} \pm \sqrt{\left(\alpha_1^{(t)} - \alpha_2^{(t)}\right)^2 - 4\left(\alpha_3^{(t)}\right)^2}\right) \text{ for } t = 1, 2, \ldots, n,$$

where $\alpha_i^{(t)} = b_i + \rho(b_3 + nb_3' + 2rb_3'') + \theta_t n b_i' + b_i''\nu_t(G)$ *for* $i = 1, 2, 3$, $(b_1, b_1', b_1'', b_2, b_2', b_2'', b_3, b_3', b_3'')$ *is the sequence of scalars corresponding to* $A(U(G))$, $L(U(G))$ *and* $Q(U(G))$ *in* $\mathcal{U}_2$ *for the A-spectrum, the L-spectrum and the Q-spectrum of* $U(G)$, *respectively.*

(3) If $U \in \mathcal{U}_3$, then the A-spectrum, the L-spectrum and the Q-spectrum of $U(G)$ are

(i) $b_2 + \rho(n-2)$ with multiplicity $m - n$;

(ii) $\frac{1}{2}\left(\alpha_1^{(t)} + \alpha_2^{(t)} \pm \sqrt{\left(\alpha_1^{(t)} - \alpha_2^{(t)}\right)^2 - 4\alpha_3^{(t)}}\right)$ for $t = 1, 2, \ldots, n$,

where $\alpha_1^{(t)} = b_1 + \rho(m-r) + \theta_t n b_1' + b_1'' \nu_t(G)$
$\alpha_2^{(t)} = b_2 + \rho(n-2) + \theta_t m b_2' + b_2'' \nu_t(G)$ $\alpha_3^{(t)} = \theta_t(mn - nr - 2m) + \nu_t(G)$
for $t = 1, 2, \ldots, n$; $(b_1, b_1', b_1'', b_2, b_2', b_2'')$ is the sequence of scalars corresponding to $A(U(G))$, $L(U(G))$ and $Q(U(G))$ in $\mathcal{U}_3$ for the A-spectrum, the L-spectrum and the Q-spectrum of $U(G)$, respectively.

As a consequence of Theorem 3.1, we obtain the following result.

Corollary 3.1.

Let G and G' be two regular cospectral graphs and $U \in \mathcal{U}$. Then the graphs $U(G)$ and $U(G')$ are simultaneously A-cospectral, L-cospectral and Q-cospectral.

Spectra of $\mathcal{M}$-join of graphs

Now we study various spectra of the $\mathcal{M}$-join of some special graphs. It can be seen that

$$A\left(\bigvee_{\mathcal{M}} \mathcal{H}_k\right) = \begin{bmatrix} A(H_1) & M_{12} & \cdots & M_{1k} \\ M_{12}^T & A(H_2) & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1k}^T & M_{2k}^T & \cdots & A(H_k) \end{bmatrix}.$$

Theorem 3.2.

Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of pairwise commuting regular graphs each having n vertices and let $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$ be a sequence of symmetric pairwise commuting matrices such that each $M_{ij} \in \mathcal{RC}_{n\times n}(m_{ij}, c_{ij})$ commutes with $A(H_t)$ for $i, j, t = 1, 2, \ldots, k$ and $i < j$. Then the A-spectrum, the L-spectrum and the Q-spectrum of the $\mathcal{M}$-join of $\mathcal{H}_k$ are

$$\sum_{t=1}^{n} \sigma(A_t),$$

where

$$A_t = \begin{bmatrix} \lambda_t(M_{11}) + \rho d_1 & \alpha\lambda_t(M_{12}) & \cdots & \alpha\lambda_t(M_{1k}) \\ \alpha\lambda_t(M_{12}) & \lambda_t(M_{22}) + \rho d_2 & \cdots & \alpha\lambda_t(M_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha\lambda_t(M_{1k}) & \alpha\lambda_t(M_{2k}) & \cdots & \lambda_t(M_{kk}) + \rho d_k \end{bmatrix}$$

for $t = 1, 2, \ldots, n$, with $M_{ii} = \begin{cases} A(H_i) & \text{for the } A\text{-spectrum of } \bigvee_{\mathcal{M}} \mathcal{H}_k; \\ L(H_i) & \text{for the } L\text{-spectrum of } \bigvee_{\mathcal{M}} \mathcal{H}_k; \\ Q(H_i) & \text{for the } Q\text{-spectrum of } \bigvee_{\mathcal{M}} \mathcal{H}_k, \end{cases}$

$$d_i = \begin{cases} \sum_{j=2}^{k} m_{1j} & \text{for } i = 1; \\ \sum_{j=1}^{i-1} c_{ji} + \sum_{j=i+1}^{k} m_{ij} & \text{for } i = 2, 3, \ldots, k-1; \\ \sum_{j=1}^{k-1} c_{jk} & \text{for } i = k, \end{cases}$$

$\lambda_t(M_{ij})$s are co-eigenvalues of M_{ij}s for $i, j = 1, 2, \ldots, k$; $i \leq j$.

Corollary 3.2.

Let H_i and H_i' be regular commuting graphs for $i = 1, 2, \ldots, k$. Let $\mathcal{H}_k = (H_1,\ H_2,\ \ldots,\ H_k)$, $\mathcal{H}_k' = (H_1',\ H_2',\ \ldots,\ H_k')$ be sequence of pairwise commuting regular graphs each having n vertices and let $\mathcal{M} = (M_{12},\ M_{13},\ \ldots,\ M_{1k},\ M_{23},\ M_{24},\ \ldots,\ M_{2k},\ \ldots,\ M_{(k-1)k})$ be a sequence of symmetric pairwise commuting $0-1$ matrices such that each $M_{ij} \in \mathcal{RC}_{n\times n}(m_{ij}, c_{ij})$. If M_{ij}, $A(H_t)$, $A(H_t')$ are cospectral for $i, j, t = 1, 2, \ldots, k$ and $i < j$, then the $\mathcal{M}$-join of $\mathcal{H}_k$ and the $\mathcal{M}$-join of $\mathcal{H}_k'$ are simultaneously A-cospectral, L-cospectral and Q-cospectral.

Corollary 3.3.

Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a sequence of graphs each having n vertices and let $M \in \mathcal{RC}_{n\times n}(m,c)$ be a $0-1$ symmetric matrix which commutes with $A(H_i)$ for $i = 1, 2, \ldots, k$. If $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$ is a sequence of $0-1$ matrices, where $M_{ij} = M$ for $i, j = 1, 2, \ldots, k$; $i < j$. Then the characteristic polynomials of the adjacency, the Laplacian and the signless Laplacian matrices of $\bigvee_{\mathcal{M}} \mathcal{H}_k$ are

$$\prod_{t=1}^{n}\left[\left\{\prod_{i=1}^{k}(x-\lambda_i^{(t)}+m_t)\right\} - m_t\left\{\sum_{i=1}^{k}\left(\prod_{\substack{j=1,\\ j\neq i}}^{k}(x-\lambda_j^{(t)}+m_t)\right)\right\}\right],$$

with $\lambda_i^{(t)} = \begin{cases} \lambda_t(H_i) & \text{for the characteristic polynomial of } A\left(\bigvee_{\mathcal{M}}\mathcal{H}_k\right); \\ \mu_t(H_i)+d_i & \text{for the characteristic polynomial of } L\left(\bigvee_{\mathcal{M}}\mathcal{H}_k\right); \\ \nu_t(H_i)+d_i & \text{for the characteristic polynomial of } Q\left(\bigvee_{\mathcal{M}}\mathcal{H}_k\right), \end{cases}$

$m_t = \alpha\lambda_t(M)$;

$d_i = (i-1)c + (k-i)m$ *for* $i = 1, 2, \ldots, k$; $t = 1, 2, \ldots, n$,

$\lambda_t(M), \lambda_t(H_1), \lambda_t(H_2), \ldots, \lambda_t(H_k)$ *are co-eigenvalues of* $M, A(H_1), A(H_2), \ldots, A(H_k)$;

$\lambda_t(M), \mu_t(H_1), \mu_t(H_2), \ldots, \mu_t(H_k)$ *are co-eigenvalues of* $M, L(H_1), \ldots, L(H_k)$;

$\lambda_t(M), \nu_t(H_1), \nu_t(H_2), \ldots, \nu_t(H_k)$ *are co-eigenvalues of* M, $Q(H_1), \cdots, Q(H_k)$ *for* $t = 1, 2, \ldots, n$.

Some new variants the of join of graphs

Definition 3.5.

Let G, H_1, H_2, $\ldots$, H_k be graphs, each having n vertices and let H be a graph having k vertices with $A(H) = [h_{ij}]$, $i, j = 1, 2, \ldots, k$. Let M be one of the matrix as given in Table 8. Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ and $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, where $M_{ij} = h_{ij}M$ for $i, j = 1, 2, \ldots, k$. Then we call the $\bigvee_{\mathcal{M}} \mathcal{H}_k$ as in Table 8.

S. No	M	Name of the graph operation
1.	I_n	The identity join of $\mathcal{H}_k$ with respect to H
2.	$A(G)$	The G-neighbourhood join of $\mathcal{H}_k$ with respect to H
3.	$I_n + A(G)$	The G-closed neighbourhood join of $\mathcal{H}_k$ with respect to H
4.	$J_n - I_n$	The vertex complemented join of $\mathcal{H}_k$ with respect to H
5.	$J_n - A(G)$	The G-neighbourhood complemented join of $\mathcal{H}_k$ with respect to H
6.	$J_n - I_n - A(G)$	The G-closed neighbourhood complemented join of $\mathcal{H}_k$ with respect to H

Table 8: Some new variants of join of graphs constructed as particular cases of $\mathcal{M}$-join of graphs

Example 4.3

Consider the graphs G, H, H_1, H_2, H_3 as shown in Figure 5. Let $\mathcal{H}_3 = (H_1, H_2, H_3)$. Then the graphs constructed by using these graphs and the graph operations mentioned in Table 8 are shown in Figure 5.

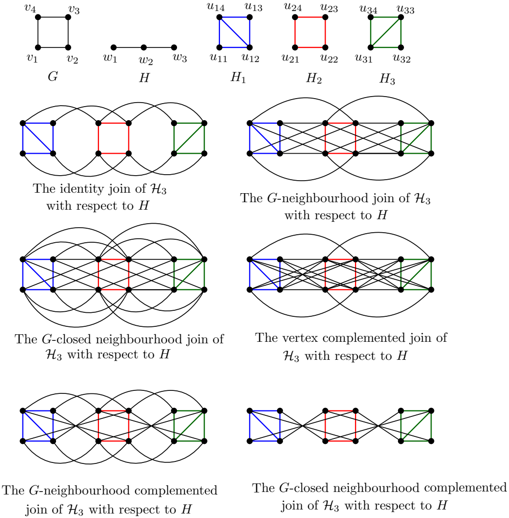


Figure 5: Examples for new variants of join of graphs defined in Table 8

Theorem 3.3.

Let G be an r-regular graph with n vertices and m edges, $H_i \in \{G, \overline{G}, K_n, \overline{K}_n\}$, $M_{ij} \in \{I_n,\ A(G),\ A(G)+I_n,\ A(\overline{G}),\ A(\overline{G})+I_n,\ A(K_n),\ J_n,\ \mathbf{0}\}$ for $i, j = 1,\ 2,\ \ldots,\ k_1$ and $i < j$, $G_i \in \{\mathcal{L}(G),\ \overline{\mathcal{L}(G)},\ K_m,\ \overline{K}_m\}$, $N_{ij} \in \{I_m,\ A(\mathcal{L}(G)),\ A(\mathcal{L}(G))+I_m,\ A(\overline{\mathcal{L}(G)}),\ A(\overline{\mathcal{L}(G)})+I_m,\ A(K_m),\ J_m,\ \mathbf{0}\}$ for $i, j = 1, 2, \ldots, k_2$ and $i < j$, and $P_{ij} \in \{\mathbf{0}, B(G), J_{n\times m}, J_{n\times m} - B(G)\}$ for $i = 1, 2, \ldots, k_1$ and $j = 1, 2, \ldots, k_2$. Let $\mathcal{H}_{k_1} = (H_1, H_2, \ldots, H_{k_1})$, $\mathcal{G}_{k_2} = (G_1, G_2, \ldots, G_{k_2})$, $\mathcal{M} = (M_{12},\ M_{13},\ \ldots,\ M_{1k_1},\ M_{23},\ M_{24},\ \ldots,\ M_{2k_1},\ \ldots,\ M_{(k_1-1)k_1})$, $\mathcal{N} = (N_{12},\ N_{13},\ \ldots,\ N_{1k_2},\ N_{23},\ N_{24},\ \ldots,\ N_{2k_2},\ \ldots,\ N_{(k_2-1)k_2})$ and

$$\mathcal{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1k_2} \\ P_{21} & P_{22} & \cdots & P_{2k_2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k_1 1} & P_{k_1 2} & \cdots & P_{k_1 k_2} \end{bmatrix}.$$

Let m_{ij}, r_{iq} and n_{hq} be the sum of the entries in a row of M_{ij}, P_{iq}, N_{hq}, respectively, and let r'_{iq} be the sum of the entries in a column of P_{iq} for $i, j = 1, 2, \ldots, k_1$ and $h, q = 1, 2, \ldots, k_2$. Then the A-spectrum, the L-spectrum and the Q-spectrum of $\left(\bigvee_{\mathcal{M}} \mathcal{H}_{k_1}\right) \vee_{\mathcal{P}} \left(\bigvee_{\mathcal{N}} \mathcal{G}_{k_2}\right)$ can be obtained by substituting $p_1 = r$, $p_2 = 2$ and $\lambda_t = \nu_t(G)$ for $t = 1, 2, \ldots, n$ and the values b_{ij}, b'_{ij}, b''_{ij}, p_{iq}, p'_{iq}, c_{hq}, c'_{hq}, c''_{hq} for $i, j = 1, 2, \ldots, k_1$ and $h, q = 1, 2, \ldots, k_2$ in Corollary, which can be obtained by the following procedure:

(i) Take

$$B_{ii} = \begin{cases} A(H_i) & \text{for the } A\text{-spectrum of } \left(\bigvee_{\mathcal{M}} \mathcal{H}_{k_1}\right) \vee_{\mathcal{P}} \left(\bigvee_{\mathcal{N}} \mathcal{G}_{k_2}\right); \\ L(H_i) + d_i I_n & \text{for the } L\text{-spectrum of } \left(\bigvee_{\mathcal{M}} \mathcal{H}_{k_1}\right) \vee_{\mathcal{P}} \left(\bigvee_{\mathcal{N}} \mathcal{G}_{k_2}\right); \\ Q(H_i) + d_i I_n & \text{for the } Q\text{-spectrum of } \left(\bigvee_{\mathcal{M}} \mathcal{H}_{k_1}\right) \vee_{\mathcal{P}} \left(\bigvee_{\mathcal{N}} \mathcal{G}_{k_2}\right), \end{cases}$$

with $d_i = \sum_{\substack{j=1, \\ j \neq i}}^{k_1} m_{ij} + \sum_{h=1}^{k_2} r_{ih}$ for $i = 1, 2, \ldots, k_1$;

$B_{ij} = \alpha M_{ij} = B_{ji}^T$ for $i, j = 1, 2, \ldots, k_1$; $i < j$;

$$C_{hh} = \begin{cases} A(G_h) & \text{for the } A\text{-spectrum of } \left(\bigvee_{\mathcal{M}} \mathcal{H}_{k_1}\right) \vee_{\mathcal{P}} \left(\bigvee_{\mathcal{N}} \mathcal{G}_{k_2}\right); \\ L(G_h) + d'_h I_m & \text{for the } L\text{-spectrum of } \left(\bigvee_{\mathcal{M}} \mathcal{H}_{k_1}\right) \vee_{\mathcal{P}} \left(\bigvee_{\mathcal{N}} \mathcal{G}_{k_2}\right); \\ Q(G_h) + d'_h I_m & \text{for the } Q\text{-spectrum of } \left(\bigvee_{\mathcal{M}} \mathcal{H}_{k_1}\right) \vee_{\mathcal{P}} \left(\bigvee_{\mathcal{N}} \mathcal{G}_{k_2}\right), \end{cases}$$

with $d'_h = \sum_{\substack{s=1, \\ h \neq s}}^{k_2} n_{hs} + \sum_{j=1}^{k_1} r'_{jh}$ for $h = 1, 2, \ldots, k_2$;

$C_{hq} = \alpha N_{hq} = C_{qh}^T$ for $h, q = 1, 2, \ldots, k_2$; $h < q$;

$Q_{hi} = P_{ih}^T$ for $i = 1, 2, \ldots, k_1$; $h = 1, 2, \ldots, k_2$.

(ii) Then $B_{ij} = b_{ij} I_n + b'_{ij} J_n + b''_{ij} B(G)B(G)^T$, $P_{ih} = p_{ih} J_{n\times m} + p'_{ih} B(G)$ and $C_{hq} = c_{hq} I_m + c'_{hq} J_m + c''_{hq} B(G)^T B(G)$, where the values b_{ij}, b'_{ij}, b''_{ij}, c_{hq}, c'_{hq}, c''_{hq} for $i, j = 1, 2, \ldots, k_1$ and $h, q = 1, 2, \ldots, k_2$ can be obtained by using Table 4;

Remark 3.1.

If G is an r-regular graph and $H_i \in \{G,\ \overline{G},\ K_n,\ \overline{K}_n\}$, $M_{ij} \in \{I_n,\ A(G),\ A(G) + I_n,\ A(\overline{G}),\ A(\overline{G}) + I_n,\ A(K_n),\ J_n,\ \mathbf{0}\}$ for $i, j = 1, 2, \ldots, k$, then by using Theorem 3.3, the A-spectrum, the L-spectrum and the Q-spectrum of $\mathcal{M}$-join of $\mathcal{H}_k$ can be obtained by taking $k_2 = 0$. Thus we can obtain the A-spectra, the L-spectra and the Q-spectra of the graphs defined in Definition 3.5.

Some new variants of the join of graphs using unary graphs

Definition 3.6.

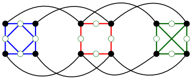
Let G be a graph with n vertices and H be a graph having k vertices with $A(H) = [h_{ij}]$. Let $H_i = U_i(G)$, where $U_i \in \mathcal{U}$ for $i = 1, 2, \ldots, k$. Let M be one of the matrix as given in Table 9. Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ and $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k_2}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, where $M_{ij} = \begin{bmatrix} h_{ij}M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ for $i, j = 1, 2, \ldots, k$; $i < j$. Then we call the $\mathcal{M}$-join of $\mathcal{H}_k$ as in Table 9.

S. No	M	Name of the graph operation
1.	I_n	The vertex-identity join of $\mathcal{H}_k$ with respect to H
2.	$A(G)$	The vertex-G-neighbourhood join of $\mathcal{H}_k$ with respect to H
3.	$A(G) + I_n$	The vertex-G-closed neighbourhood join of $\mathcal{H}_k$ with respect to H
4.	$J_n - I_n$	The vertex-complemented join of $\mathcal{H}_k$ with respect to H
5.	$J_n - A(G)$	The vertex-G-neighbourhood complemented join of $\mathcal{H}_k$ with respect to H
6.	$J_n - I_n - A(G)$	The vertex-G-closed neighbourhood complemented join of $\mathcal{H}_k$ with respect to H

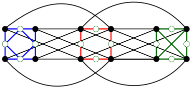
Table 9: Some new variants of join of graphs constructed as particular cases of $\bigvee_{\mathcal{M}} \mathcal{H}_k$ using unary graph operations

Example 4.4

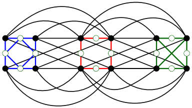
Let $H_1 = \mathcal{Q}(C_4)$, $H_2 = S(C_4)$, $H_3 = Ct(C_4)$. Then the graphs constructed by using these graphs and the graph operations mentioned in Table 9 are as shown in Figure 6.



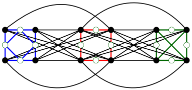
The vertex-identity join of $\mathcal{H}_3$ with respect to H



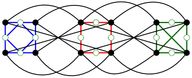
The vertex-G-neighbourhood join of $\mathcal{H}_3$ with respect to H



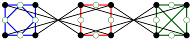
The vertex-G-closed neighbourhood join of $\mathcal{H}_3$ with respect to H



The vertex-complemented join of $\mathcal{H}_3$ with respect to H



The vertex-G-neighbourhood complemented join of $\mathcal{H}_3$ with respect to H



The vertex-G-closed neighbourhood complemented join of $\mathcal{H}_3$ with respect to H

Figure 6: Examples for new variants of join of graphs defined in Table 9

Theorem 3.4.

Let G be an r-regular graph with n vertices and m $(= \frac{1}{2}nr)$ edges. Let $M \in \{I_n, A(G), A(G)+I_n, A(\overline{G}), A(\overline{G})+I_n, J_n - I_n, J_n, \mathbf{0}\}$. Let H be an r_1-regular graph having k vertices with $A(H) = [h_{ij}]$, $i, j = 1, 2, \ldots, k$ and commutes with M. Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ and $\mathcal{M}=(M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, $M_{ij} = \begin{bmatrix} h_{ij}M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ for $i, j = 1, 2, \ldots, k$; $i < j$. Let s be the sum of the entries in a row of M. Then we have the following.

(1) *If $U \in \mathcal{U}_1$ and $H_i = U(G)$ for $i = 1, 2, \ldots, k$, then the A-spectrum, the L-spectrum and the Q-spectrum of $\bigvee_{\mathcal{M}} \mathcal{H}_k$ are*

(i) $\frac{1}{2}\left(\alpha_t^{(i)} + \beta_t \pm \sqrt{(\alpha_t^{(i)} - \beta_t)^2 - 4\nu_t(G)}\right)$,
where $\alpha_t^{(i)} = b + \rho(sr_1 + r) + \theta_t nb' + b''\nu_t(G) + \alpha\lambda_i(H)\lambda_t(M)$,
$\beta_t = c + 2\rho + \theta_t mc' + c''\nu_t(G)$ *for* $i = 1, 2, \ldots, k$; $t = 1, 2, \ldots, n$,

(ii) $c + 2\rho$ *with multiplicity* $k(m-n)$.

(b, b', b'', c, c', c'') is the sequence of scalars corresponding to $A(U(G))$, $L(U(G))$ and $Q(U(G))$ in $\mathcal{U}_1$ for the A-spectrum, the L-spectrum and the Q-spectrum of $\bigvee_{\mathcal{M}} \mathcal{H}_k$, respectively.

(2) If $U \in \mathcal{U}_2$ and $H_i = U(G)$ for $i = 1, 2, \ldots, k$, then the A-spectrum, the L-spectrum and the Q-spectrum of $\bigvee_{\mathcal{M}} \mathcal{H}_k$ are

$$\frac{1}{2}\left(\alpha_t^{(i)} + \beta_t \pm \sqrt{(\alpha_t^{(i)} - \beta_t)^2 - 4\gamma_t^2}\right),$$

where $\alpha_t^{(i)} = b_1 + \rho(sr_1 + b_3 + nb_3' + 2rb_3'') + \theta_t nb_1' + b_1''\nu_t(G) + \alpha\lambda_i(H)\lambda_t(M)$, $\beta_t = b_2 + \rho(b_3 + nb_3 + 2rb_3'') + \theta_t nb_2' + b_2''\nu_t(G)$, $\gamma_t = b_3 + \theta_t nb_3' + b_3''\nu_t(G)$ for $i = 1, 2, \ldots, k$; $t = 1, 2, \ldots, n$,
$(b_1, b_1', b_1'', b_2, b_2', b_2'', b_3, b_3', b_3'')$ is the sequence of scalars corresponding to $A(U(G))$, $L(U(G))$ and $Q(U(G))$ in $\mathcal{U}_2$ for the A-spectrum, the L-spectrum and the Q-spectrum of $\bigvee_{\mathcal{M}} \mathcal{H}_k$, respectively.

(3) If $U \in \mathcal{U}_3$ and $H_i = U(G)$ for $i = 1, 2, \ldots, k$, then the A-spectrum, the L-spectrum and the Q-spectrum of $\bigvee_{\mathcal{M}} \mathcal{H}_k$ are
   (i) $\frac{1}{2}\left(\alpha_t^{(i)} + \beta_t \pm \sqrt{(\alpha_t^{(i)} - \beta_t)^2 - 4\gamma_t}\right)$,
   where $\alpha_t^{(i)} = b + \rho(sr_1 + m - r) + \theta_t nb' + b''\nu_t(G) + \alpha\lambda_i(H)\lambda_t(M)$, $\beta_t = c + (n-2)\rho + \theta_t mc' + c''\nu_t(G)$, $\gamma_t = \theta_t(mn - nr - 2m) + \nu_t(G)$ for $i = 1, 2, \ldots, k$; $t = 1, 2, \ldots, n$;
   (ii) $c + \rho(n-2)$ with multiplicity $k(m-n)$,

(b, b', b'', c, c', c'') is the sequence of scalars corresponding to $A(U(G))$, $L(U(G))$ and $Q(U(G))$ in $\mathcal{U}_3$ for the A-spectrum, the L-spectrum and the Q-spectrum of $\bigvee_{\mathcal{M}} \mathcal{H}_k$, respectively.

Corollary 3.4.

Let G and G' be regular cospectral graphs and let H be an regular graph having k vertices with $A(H) = [h_{ij}]$ for $i, j = 1, 2, \ldots, k$ which commutes with $A(G)$ and $A(G')$. Let $U \in \mathcal{U}$, $H_i = U(G)$, $H'_i = U(G')$ $i = 1, 2, \ldots, k$ and let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ and $\mathcal{H}'_k = (H'_1, H'_2, \ldots, H'_k)$. Then the vertex G-identity join of $\mathcal{H}_k$ with respect to H and the vertex G-identity join of $\mathcal{H}'_k$ with respect to H are simultaneously A-cospectral, L-cospectral and Q-cospectral.

Remark 3.2.

For each graph operation defined in Table 9, we can construct cospectral graphs by using a similar procedure described in Corollary 3.4.

$(H, H_1, H_2, \ldots, H_k)$-merged F-subdivision-edge complemented graph of a graph with respect to T_1 and T_2

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let F be a graph with $V(F) = \{u_1, u_2, \ldots, u_k\}$, and let T_1, $T_2 \subseteq V(F)$. Let H be a graph with $V(H) = \{w_1, w_2, \ldots, w_n\}$ and let H_i be a graph with $V(H_i) = \{u_{i1}, u_{i2}, \ldots, u_{im}\}$ for $i = 1, 2, \ldots, k$.

Definition 3.7.

The ***F*-subdivision-edge complement graph of *G* with respect to *T_1* and *T_2*** is the graph obtained by taking one copy of G and *a* copy F corresponding to each edge of G, and

(i) joining each vertex in T_1 to the end vertices of the corresponding edge;

(ii) joining each vertex in T_2 to the vertices of G other than the end vertices of the corresponding edge;

(iii) deleting all the edges of G.

Notice that, if $T_1 = V(F)$ and $T_2 = \phi$, then the F-subdivision-edge complement graph of G with respect to T_1 and T_2 is the graph $S(G, F)$ defined in [51], which we call it as *F-subdivision graph of G*.

Definition 3.8.

The **$(H, H_1, H_2, \ldots, H_k)$-merged F-subdivision-edge complement graph of G with respect to T_1 and T_2** is the graph obtained by taking one copy of F-subdivision-edge complement graph of G with respect to T_1 and T_2, and

(i) joining the vertices v_r and v_s if and only if the vertices w_r and w_s are adjacent in H for $r, s = 1, 2, \ldots, n$;

(ii) joining the vertices u_r in the i-th and j-th copy of F if and only if the vertices u_{ri} and u_{rj} are adjacent in H_r for $r = 1, 2, \ldots, k$; $i, j = 1, 2, \ldots, m$.

Notice that if F has a single vertex, $T_1 = V(F)$ and $T_2 = \phi$, then the $(H, H_1, H_2, \ldots, H_k)$-merged F-subdivision graph-edge complement of G with respect to T_1 and T_2 is same as the (H, H_1)-merged subdivision graph of G.

Example 4.5

Consider the graphs G, F, H, H_1, H_2, H_3 as in Figure 7. Let $T_1 = \{u_1, u_3\}$, $T_2 = \{u_3\}$. Then F-subdivision-edge complement graph of G with respect to T_1 and T_2 and (H, H_1, H_2, H_3)-merged subdivision-edge complement graph of G with respect to T_1, T_2 are as shown in Figure 7.

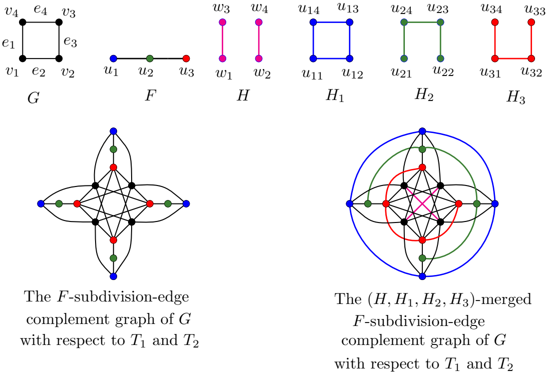


Figure 7: Examples of the $(H, H_1, H_2, \ldots, H_k)$-merged F-subdivision-edge complement graph of a graph G with respect to T_1 and T_2

Theorem 3.5.

Let Γ *be the* $(H, H_1, H_2, \ldots, H_k)$*-merged* F*-subdivision-edge complement graph of* G *with respect to* T_1 *and* T_2*. Also assume the following.*

(1) F *is a graph with* $V(F) = \{u_1, u_2, \ldots, u_k\}$; $A(F) = [f_{ij}]$, $i, j = 1, 2, \ldots, k$ *and* d_h *is the degree of* u_h *in* F *for* $h = 1, 2, \ldots, k$.

(2) $T_1, T_2 \subseteq V(F)$; $t_1 = |T_1 \setminus T_2|$; $t_2 = |T_2 \setminus T_1|$; $t_3 = |T_1 \cap T_2|$.

(3) $$d_i' = \begin{cases} 2 & \text{if } u_i \in T_1 \setminus T_2; \\ m-2 & \text{if } u_i \in T_2 \setminus T_1; \\ m & \text{if } u_i \in T_1 \cap T_2; \\ 0 & \text{otherwise}, \end{cases}$$

$$p_i^{(t)} = \begin{cases} r + \lambda_t(G) & \text{if } u_i \in T_1 \setminus T_2; \\ m - r - \lambda_t(G) & \text{if } u_i \in T_2 \setminus T_1; \\ mn & \text{if } u_i \in T_1 \cap T_2; \\ 0 & \text{otherwise}. \end{cases}$$

$$\beta_i = \begin{cases} 1 & \text{if } u_i \in T_1 \setminus T_2; \\ -1 & \text{if } u_i \in T_2 \setminus T_1; \\ 0 & \text{otherwise}. \end{cases}$$

(4) $H \in \{G, \overline{G}, K_n, \overline{K}_n\}$, $H_i \in \{\mathcal{L}(G), \overline{\mathcal{L}(G)}, K_m, \overline{K}_m\}$ and let b_i, b_i' and b_i'' be such that

$$b_i I_m + b_i' J_m + b_i'' B(G)^T B(G) = \begin{cases} A(H_i) & \text{for the } A\text{-spectrum of } \Gamma; \\ L(H_i) & \text{for the } L\text{-spectrum of } \Gamma; \\ Q(H_i) & \text{for the } Q\text{-spectrum of } \Gamma \end{cases}$$

for $i = 1, 2, \ldots, k$.

(5)

$$E_t = \begin{bmatrix} \lambda^{(t)} & \alpha p_1^{(t)} & \alpha p_2^{(t)} & \ldots & \alpha p_k^{(t)} \\ \alpha\beta_1 & \lambda_1^{(t)} & \alpha f_{12} & \ldots & \alpha f_{1k} \\ \alpha\beta_2 & \alpha f_{21} & \lambda_2^{(t)} & \ldots & \alpha f_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha\beta_k & \alpha f_{k1} & \alpha f_{k2} & \ldots & \lambda_k^{(t)} \end{bmatrix},$$

where

$$\lambda^{(t)} = \begin{cases} \lambda_t(H) & \text{for the } A\text{-spectrum of } \Gamma; \\ \mu_t(H) + rt_1 + t_2(n-r) + nt_3 & \text{for the } L\text{-spectrum of } \Gamma; \\ \nu_t(H) + rt_1 + t_2(n-r) + nt_3 & \text{for the } Q\text{-spectrum of } \Gamma, \end{cases}$$

with $\lambda_i^{(t)} = b_i + \theta_t m + b_i'' \nu_t(G) + \rho(d_i + d_i')$ for $t = 1, 2, \ldots, n$; $i = 1, 2, \ldots, k$.

(6) $\hat{B} = \alpha A(F) + diag(b_1 + \rho(d_1 + d_1'), b_2 + \rho(d_2 + d_2'), \ldots, b_k + \rho(d_k + d_k'))$.

Then the A-spectrum, the L-spectrum and the Q-spectrum of $(H, H_1, H_2, \ldots, H_k)$-merged F-subdivision-edge complement graph of G can be obtained from

$$\sum_{t=1}^{n} \sigma(E_t) + (m-n)\sigma(\hat{B}).$$

Quadruple join of graphs

Definition 3.9.

Let H_i be a graph and let $T_i \subseteq V(H_i)$ for $i = 1, 2, \ldots, k$. Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ and $\mathcal{T} = (T_1, T_2, \ldots, T_k)$. Let F_i be a graph with $V(F_i) = \{u_{i1}, u_{i2}, \ldots, u_{ik}\}$ for $i = 1, 2, 3, 4$. Then the **quadruple join of $\mathcal{H}_k$ with respect to (F_1, F_2, F_3, F_4) constrained by $\mathcal{T}$** is the graph obtained by taking one copy of the graphs H_1, H_2, ..., H_k, and

(i) joining each vertex of T_i to all the vertices of T_j if and only if u_{1i} and u_{1j} are adjacent in F_1 for $i, j = 1, 2, \ldots, k$;

(ii) joining each vertex of T_i to all the vertices of T_j^c if and only if u_{2i} and u_{2j} are adjacent in F_2 for $i, j = 1, 2, \ldots, k$; $i < j$;

(iii) joining each vertex of T_i^c to all the vertices of T_j if and only if u_{3i} and u_{3j} are adjacent in F_3 for $i, j = 1, 2, \ldots, k$; $i < j$;

(iv) joining each vertex of T_i^c to all the vertices of T_j^c if and only if u_{4i} and u_{4j} are adjacent in F_4 for $i, j = 1, 2, \ldots, k$.

Example 4.6

Consider the graphs H_1, H_2, H_3, F_1, F_2, F_3 and F_4 as shown in Figure 8. The quadruple join of the graphs (H_1, H_2, H_3) with respect to the graphs (F_1, F_2, F_3, F_4) constrained by $\mathcal{T} = (T_1, T_2, T_3)$ is as shown in Figure 8, where T_i is a vertex subset of H_i whose vertices are colored with yellow for $i = 1, 2, 3$ and F_1, F_2, F_3, F_4 are properly arrange in this figure.

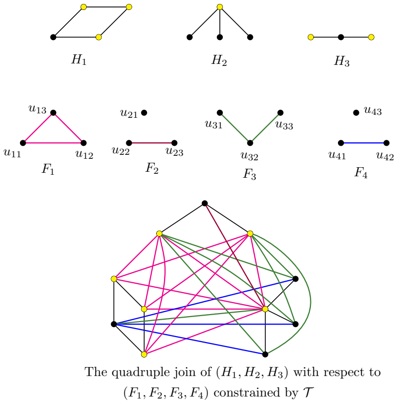


Figure 8: Example for the quadruple join of graphs (H_1, H_2, H_3) with respect to (F_1, F_2, F_3, F_4) constrained by $\mathcal{T}$

Remark 3.3.

Some special cases for k and the graphs F_1, F_2, F_3, F_4, H_i and T_i for $i = 1, 2, \dots, k$ in Definition 3.9, gives some existing variants of join of graphs: Let G_1 and G_2 be any two graphs.

(1) Taking F_2, F_3, F_4 as $\overline{K}_k$ and $F_1 = H$, where H is a graph with k vertices in Definition 3.9, we can obtain the H-generalized join of $\mathcal{H}_k$ constrained by vertex subsets $\mathcal{T}$.

(2) Taking $k = 2$, $H_1 = S(G_1)$, $H_2 = G_2$, $T_1 = V(G_1)$, $T_2 = V(G_2)$, $F_1 = F_2 = K_2$, $F_3 = F_4 = \overline{K}_2$ (resp. $F_1 = F_2 = \overline{K}_2$, $F_3 = F_4 = K_2$) in Definition 3.9, we can obtain the S-vertex join (resp. S-edge join) of G_1 and G_2.

(3) Taking $k = 2$, $H_1 = R(G_1)$, $H_2 = G_2$, $T_1 = V(G_1)$, $T_2 = V(G_2)$, $F_1 = F_2 = K_2$, $F_3 = F_4 = \overline{K}_2$ (resp. $F_1 = F_2 = \overline{K}_2$, $F_3 = F_4 = K_2$) in Definition 3.9, we can obtain the R-vertex join (resp. R-edge join) of G_1 and G_2.

(4) Taking $k = 2$, $H_1 = DG(G_1)$, $H_2 = G_2$, $T_1 = V(G_1)$, $T_2 = V(G_2)$, $F_1 = F_2 = K_2$, $F_3 = F_4 = \overline{K}_2$ (resp. $F_1 = F_2 = \overline{K}_2$, $F_3 = F_4 = K_2$) in Definition 3.9, we can obtain the DG-vertex join (resp. DG-add vertex join) of G_1 and G_2.

(5) Taking $k=2$, $H_1 = S(G_1)$, $H_2 = S(G_2)$, $T_1 = V(G_1)$, $T_2 = V(G_2)$, $F_1 = K_2$, $F_2 = F_3 = F_4 = \overline{K}_2$ (resp. $F_2 = K_2$, $F_1 = F_3 = F_4 = \overline{K}_2$) in Definition 3.9, we can obtain the subdivision vertex-vertex join (resp. subdivision vertex-edge join) of G_1 and G_2.

(6) Let H be a graph with $V(H) = \{u_1, u_2, u_3\}$ and $E(H) = \{\{u_1, u_2\}\}$, and let H' be a graph with $V(H') = \{v_1, v_2, v_3\}$ and $E(H') = \{\{v_1, v_3\}\}$. Then the subdivision double join (resp. R-graph double join, $\mathcal{Q}$-graph double join, total graph double join) of G_1, G_2 and G_3 can be obtained by taking $k=3$, $H_1 = S(G_1)$, (resp. $H_1 = R(G_1)$, $H_1 = \mathcal{Q}(G_1)$, $H_1 = T(G_1)$) $H_2 = G_2$, $H_3 = G_3$, $T_1 = V(G_1)$, $T_2 = V(G_2)$, $T_3 = V(G_3)$, $F_1 = H$, $F_3 = H'$ and $F_2 = F_4 = \overline{K}_3$ in Definition 3.9.

Theorem 3.6.

Let G_i be an r_i-regular graphs with n_i vertices and m_i edges for $i = 1, 2, \ldots, k_1$. Let $U \in \mathcal{U}$, $H_i = U(G_i)$, $T_i = V(G_i)$, and let H_j be a graph, $T_j = V(H_j)$ for $i = 1, 2, \ldots, k_1$; $j = k_1+1, k_1+2, \ldots, k_2$, and let F_1, F_2, F_3, F_4 be graphs with k_2 vertices and $A(F_s) = [h_{ij}^{(s)}]$ for $i, j = 1, 2, \ldots, k_2$; $s = 1, 2, 3, 4$. Let $\mathcal{H}_{k_2} = (H_1, H_2, \ldots, H_{k_2})$ and $\mathcal{T} = (T_1, T_2, \ldots, T_{k_2})$. Then the A-spectrum, the L-spectrum and the Q-spectrum of the quadruple join of $\mathcal{H}_{k_2}$ with respect to (F_1, F_2, F_3, F_4) constrained by $\mathcal{T}$ is

$$\sigma(E) + \sum_{i=1}^{k_2} \sigma(M_i) \setminus \{\sigma(\delta_{M_i})\},$$

where

$$M_i = \begin{cases} A(H_i) & \text{for the } A\text{-spectrum of } \Gamma; \\ L(H_i) & \text{for the } L\text{-spectrum of } \Gamma; \\ Q(H_i) & \text{for the } Q\text{-spectrum of } \Gamma; \end{cases}$$

for $i = 1, 2, \ldots, k_2$,

$$E = \begin{bmatrix} E_{11} & E_{12} & \ldots & E_{1k_2} \\ E_{21} & E_{22} & \ldots & E_{2k_2} \\ \vdots & \vdots & \ddots & \vdots \\ E_{k_21} & E_{k_22} & \ldots & E_{k_2k_2} \end{bmatrix},$$

with $E_{ii} = \left[\begin{smallmatrix} c_{1i} & c_{2i} \\ c_{3i} & c_{4i} \end{smallmatrix}\right]$ for $i, j = 1, 2, \ldots, k_1$;

$E_{ij} = \begin{bmatrix} h_{ij}^{(1)} n_j & h_{ij}^{(2)} t_j \\ h_{ij}^{(3)} n_j & h_{ij}^{(4)} t_j \end{bmatrix}$ for $i, j = 1, 2, \ldots, k_1$; $i \neq j$,

$c_{1i} = b_{1i} + n_i b'_{1i} + 2r_i b''_{1i} + \rho d_{1i}$;

$$c_{2i} = \begin{cases} r_i & \text{if } U \in \mathcal{U}_1; \\ b_{3i} + n_i b'_{3i} + 2r_i b''_{3i} & \text{if } U \in \mathcal{U}_2; , \\ m_i - r_i & \text{if } U \in \mathcal{U}_3; \end{cases}$$

$$c_{3i} = \begin{cases} 2 & \text{if } U \in \mathcal{U}_1; \\ c_{2i} & \text{if } U \in \mathcal{U}_2; \\ n_i - 2 & \text{if } U \in \mathcal{U}_3; \end{cases}$$

$$c_{4i} = b_{2i} + t_i b'_{2i} + 2r_i b''_{2i} + \rho d_{2i};$$

$$t_i = \begin{cases} m_i & \text{for } U \in \mathcal{U}_1 \cup \mathcal{U}_3; \\ n_i & \text{for } U \in \mathcal{U}_2; \end{cases}$$

for $i = 1, 2, \ldots, k_1$;

$$E_{ij} = \begin{bmatrix} h_{ij}^{(1)} n_j \\ h_{ij}^{(3)} n_j \end{bmatrix} = E_{ji}^T \quad \text{for } i = 1, 2, \ldots, k_1,\ j = k_1 + 1, k_1 + 2, \ldots, k_2;$$

$$E_{ii} = \begin{bmatrix} r_i + \rho d_{3i} \end{bmatrix} \quad \text{for } i = k_1 + 1, k_1 + 2, \ldots, k_2;$$

$$E_{ij} = \begin{bmatrix} h_{ij}^{(1)} n_j \end{bmatrix} \text{ for } i, j = k_1 + 1, k_1 + 2, \ldots, k_2;\ i \neq j;$$

$$
\begin{aligned}
d_{1i} &= \sum_{\substack{j=1\\ j\neq i}}^{k_1} h_{ij}^{(1)} n_j + \sum_{\substack{j=1\\ j\neq i}}^{k_2} h_{ij}^{(2)} t_j \text{ for } i = 1, 2, \ldots, k_1;\\
d_{2i} &= \sum_{\substack{j=1\\ j\neq i}}^{k_1} h_{ij}^{(3)} n_j + \sum_{\substack{j=1\\ j\neq i}}^{k_2} h_{ij}^{(4)} t_j \text{ for } i = 1, 2, \ldots, k_1;\\
d_{3i} &= \sum_{\substack{j=1\\ j\neq i}}^{k_1} h_{ij}^{(1)} n_j + \sum_{\substack{j=1\\ j\neq i}}^{k_2} h_{ij}^{(3)} t_j \text{ for } i = k_1+1, k_1+2, \ldots, k_2;
\end{aligned}
$$

$(b_{1i}, b'_{1i}, b''_{1i}, b_{2i}, b'_{2i}, b''_{2i})$ is the sequence of scalars corresponding to $A(U(G_i))$ for $U \in \mathcal{U}_1 \cup \mathcal{U}_3$ and $(b_{1i}, b'_{1i}, b''_{1i}, b_{2i}, b'_{2i}, b''_{2i}, b_{3i}, b'_{3i}, b''_{3i})$ is the sequence of scalars corresponding to $A(U(G_i))$ for $U \in \mathcal{U}_2$ for $i = 1, 2, \ldots, k_2$.

Corollary 3.5.

Let G_i and G_i' be regular cospectral graphs, $U \in \mathcal{U}$, $H_i = U(G_i)$, $H_i' = U(G_i')$, $T_i = V(G_i)$, $T_i' = V(G_i')$ for $i = 1, 2, \ldots, k_1$. Let H_j and H_j' be regular cospectral graphs, $T_j = V(H_j)$, $T_j' = V(H_j')$ for $j = k_1 + 1, k_1 + 2, \ldots, k_2$. Let F_1, F_2, F_3, F_4 be graphs with k_2 vertices. Let $\mathcal{H}_{k_2} = (H_1, H_2, \ldots, H_{k_2})$, $\mathcal{H}'_{k_2} = (H_1', H_2', \ldots, H_{k_2}')$, $\mathcal{T} = (T_1, T_2, \ldots, T_{k_2})$, $\mathcal{T}' = (T_1', T_2', \ldots, T_{k_2}')$. Then the quadruple join of $\mathcal{H}_{k_2}$ with respect to (F_1, F_2, F_3, F_4) constrained by $\mathcal{T}$ and the quadruple join of $\mathcal{H}'_{k_2}$ with respect to (F_1, F_2, F_3, F_4) constrained by $\mathcal{T}'$ are simultaneously A-cospectral, L-cospectral and Q-cospectral.

Spectra of the existing variants of join of graphs

Corollary 3.6.

Let G_i be an r_i-regular graphs with n_i vertices and m_i edges for $i = 1, 2, \ldots, k_1$. Let $U \in \mathcal{U}$, $H_i = U(G_i)$, $T_i = V(G_i)$ and let H_j be a graph, $T_j = V(H_j)$ for $i = 1, 2, \ldots, k_1$; $j = k_1 + 1, k_1 + 2, \ldots, k_2$, and let H be a graph with k_2 vertices and $A(H) = [h_{ij}]$ for $i, j = 1, 2, \ldots, k_2$. Let $\mathcal{H}_{k_2} = (H_1, H_2, \ldots, H_{k_2})$ and $\mathcal{T} = (T_1, T_2, \ldots, T_{k_2})$. Then the A-spectrum, the L-spectrum and the Q-spectrum of H-generalized join of $\mathcal{H}_{k_2}$ constrained by $\mathcal{T}$ can be obtained by taking $h^{(2)}_{ij} = h^{(3)}_{ij} = h^{(4)}_{ij} = 0$ and $h^{(1)}_{ij} = h_{ij}$ for $i, j = 1, 2, \ldots, k_2$ in Theorem 3.6.

Remark 3.4.

Let G_i be an r_i-regular graph with n_i vertices and m_i edges for $i = 1, 2, 3$.

(1) The A-spectrum, the L-spectrum and the Q-spectrum of S-vertex (resp. S-edge) join of G_1 and G_2 can be obtained by taking $k_1 = 1$, $k_2 = 2$, $c_{11} = 0$, $c_{21} = r_1$, $c_{31} = 2$, $c_{41} = 0$, $h^{(1)}_{ij} = h^{(2)}_{ij} = 1$ and $h^{(3)}_{ij} = h^{(4)}_{ij} = 0$ (resp. $h^{(1)}_{ij} = h^{(2)}_{ij} = 0$ and $h^{(3)}_{ij} = h^{(4)}_{ij} = 1$) in Theorem 3.6. (cf. Theorem 1.1 and 1.2 in [30]).

(2) The A-spectrum, the L-spectrum and the Q-spectrum of R-vertex (resp. R-edge join) join of G_1 and G_2 can be obtained by taking $k_1 = 1$, $k_2 = 2$, $c_{11} = r_1$, $c_{21} = r_1$, $c_{31} = 2$, $c_{41} = 0$, $h_{ij}^{(1)} = h_{ij}^{(2)} = 1$ and $h_{ij}^{(3)} = h_{ij}^{(4)} = 0$ (resp. $h_{ij}^{(1)} = h_{ij}^{(2)} = 0$ and $h_{ij}^{(3)} = h_{ij}^{(4)} = 1$) in Theorem 3.6. (cf. Theorems 4, 7, 10, 13, 16, 19 in [19]).

(3) The A-spectrum, the L-spectrum and the Q-spectrum of subdivision vertex-vertex join (resp. subdivision vertex-edge join) of G_1 and G_2 can be obtained by taking $k_1 = 2$, $k_2 = 2$, $c_{1i} = 2$, $c_{2i} = r_i$, $c_{3i} = r_i$, $c_{4i} = 0$ for $i = 1, 2$, $h_{ij}^{(1)} = 1$ and $h_{ij}^{(2)} = h_{ij}^{(3)} = h_{ij}^{(4)} = 0$ (resp. $h_{ij}^{(2)} = 0$ and $h_{ij}^{(1)} = h_{ij}^{(3)} = h_{ij}^{(4)} = 1$) in Theorem 3.6. (cf. Theorems 5 and 7 in [47]).

(4) The A-spectrum, the L-spectrum and the Q-spectrum of subdivision double join of G_1, G_2 and G_3 can be obtained by taking $k_1 = 1$, $k_2 = 3$, $c_{11} = 0$, $c_{21} = r_1$, $c_{31} = r_1$, $c_{41} = 0$, $h_{ij}^{(1)} = 1$ and $h_{ij}^{(2)} = h_{ij}^{(3)} = h_{ij}^{(4)} = 0$ in Theorem 3.6. (cf. Theorem 4 in [57]).

(5) The A-spectrum, the L-spectrum and the Q-spectrum of $\mathcal{Q}$-graph double join G_1 with G_2 and G_3 can be obtained by taking $k_1 = 1$, $k_2 = 3$, $c_{11} = 0$, $c_{21} = r_1$, $c_{31} = r_1$, $c_{41} = 2r_1 - 2$, $h_{ij}^{(1)} = 1$ and $h_{ij}^{(2)} = h_{ij}^{(3)} = h_{ij}^{(4)} = 0$ in Theorem 3.6. (cf. Theorem 6 in [57]).

(6) The A-spectrum, the L-spectrum and the Q-spectrum of R-graph double join of G_1 with G_2 and G_3 can be obtained by taking $k_1 = 1$, $k_2 = 3$, $c_{11} = r_1$, $c_{21} = r_1$, $c_{31} = 2$, $c_{41} = 0$, $h_{ij}^{(1)} = 1$ and $h_{ij}^{(2)} = h_{ij}^{(3)} = h_{ij}^{(4)} = 0$ in Theorem 3.6. (cf. Theorem 7 in [57]).

(7) The A-spectrum, the L-spectrum and the Q-spectrum of total graph double join of G_1 with G_2 and G_3 can be obtained by taking $k_1 = 1$, $k_2 = 3$, $c_{11} = r_1$, $c_{21} = r_1$, $c_{31} = r_1$, $c_{41} = 2r_1 - 2$, $h_{ij}^{(1)} = 1$ and $h_{ij}^{(2)} = h_{ij}^{(3)} = h_{ij}^{(4)} = 0$ in Theorem 3.6. (cf. Theorem 8 in [57]).

Publications

1. R. Rajkumar and M. Gayathri, Spectra of (H_1, H_2)-merged subdivision graph of a graph, **Indagationes Mathematicae** 30 (2019), 1061–1076.
2. M. Gayathri and R. Rajkumar, Adjacency and Laplacian spectra of variants of neighbourhood corona of graphs constrained by vertex subsets, **Discrete Mathematics, Algorithms and Applications**, 11(6) (2019) Article No. 1950073 (19 pages).
3. Rajkumar and M. Gayathri, Spectra of generalized corona of graphs constrained by vertex subsets, **Le Matematiche**, 1 (2021) 175–205.
4. M. Gayathri and R. Rajkumar, Spectra of partitioned matrices and the $\mathcal{M}$-join of graphs, **Ricerche di Matematica**, (2021). (Published online)
5. R. Rajkumar and M. Gayathri, M-generalized corona of graphs constrained by vertex subsets: Generalized characteristic polynomial, Adjacency spectrum and Laplacian spectrum. (Communicated).
6. M. Gayathri and R. Rajkumar, Spectra of $(M, \mathcal{M})$-corona-join of graphs. (Communicated).

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