Blowup-polynomials of graphs

Projesh Nath Choudhury
Indian Institute of Science

(Joint with Apoorva Khare)

E-seminar on Graphs and Matrices
IIT Kharagpur

Distance matrices of graphs

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- Between any two nodes v, w of G, there is a shortest path of integer length $d(v, w) \geqslant 0$ (i.e., $d(v, w)$ edges).
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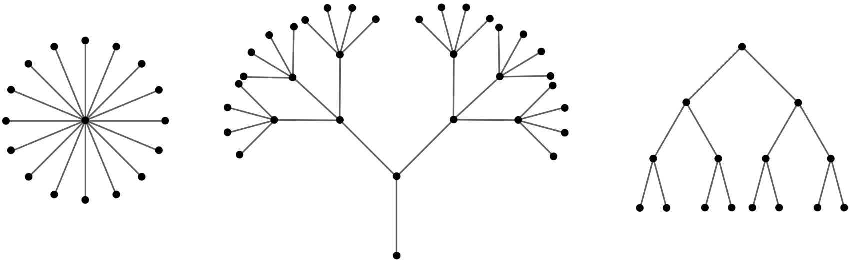
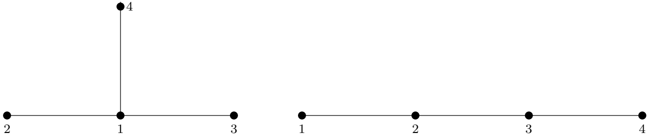
- The *distance matrix* D_G is a $V \times V$ matrix with entries $d(v, w)$.
- Extensively studied quantity: the determinant of D_G for G a tree.



Algebraic fact: The Graham–Pollak result

Examples of distance matrices (on 4 nodes):

T_1, T_2 are the star graph $K_{1,3}$ and the path graph P_4, respectively.



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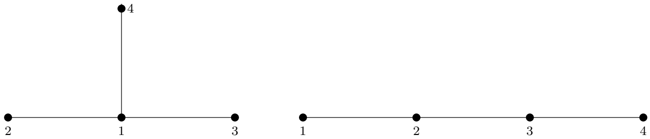

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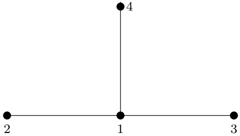

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Theorem (Graham–Pollak, *Bell Sys. Tech. J.*, 1971)

Given a tree T on k nodes, $\quad \det D_T = (-1)^{k-1} 2^{k-2} (k-1).$

Analysis fact: co-spectral matrices

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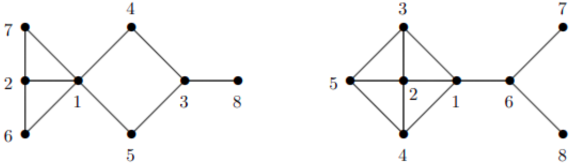

Thus, $\det(D_G - x\,\mathrm{Id}_V)$ does not detect the graph (up to isomorphism).

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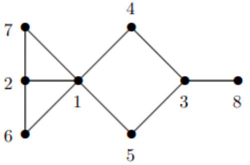

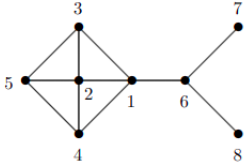

Thus, $\det(D_G - x\,\mathrm{Id}_V)$ does not detect the graph (up to isomorphism).

Inter-related Motivations/Goals:

1. Find a(nother) family $\{G_i : i \in I\}$ of graphs (e.g., trees on k vertices) such that $i \mapsto \det D_{G_i}$ is a "nice" function.
2. Find an invariant of the matrix D_G which detects G (and is related to the distance spectrum – eigenvalues of D_G).

Graph blowups

The key construction is of a *graph blowup* $G[\mathbf{n}]$, where $\mathbf{n} = (n_v)_{v \in V}$ is a V-tuple of positive integers. This is a finite simple connected graph $G[\mathbf{n}]$, with:

- n_v copies of the vertex $v \in V$, and
- a copy of vertex v and one of w are adjacent in $G[\mathbf{n}]$ if and only if $v \neq w$ and v, w are adjacent in G.

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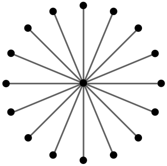
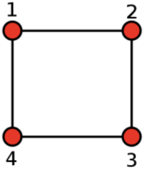
More examples:



Star graph: $K_{1,n} \cong K_2[(1,n)]$

4-cycle: $C_4 \cong K_2[(2,2)]$.



Distance matrix of graph blowup, and its determinant

Suggestive example: Compute $\det D_{G[\mathbf{n}]}$ in all examples above:

$$\det D_{K_2[(r,s)]} = (-2)^{r+s-2}(3rs - 4r - 4s + 4).$$

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Moreover, p_G is multi-affine in $\mathbf{n}$, with constant term $(-2)^{|V|}$ and linear term $-(-2)^{|V|} \sum_{v \in V} n_v$. (In fact, have closed-form expression for every monomial.)

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Definition: Define $p_G(\cdot)$ to be the *blowup-polynomial* of G.

Proof: Zariski density

Define the *modified distance matrix* $M_G := D_G + 2\,\mathrm{Id}_V$, and $\Delta_{\mathbf{n}} := \mathrm{diag}(n_v)_{v \in V}$. The above proof reveals:

$$(-2)^{-\sum_v (n_v - 1)} \cdot \det D_{G[\mathbf{n}]} = p_G(\mathbf{n}) = \det(\Delta_{\mathbf{n}} M_G - 2\,\mathrm{Id}_V).$$

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4. Finally, specialize from $\mathbb{Z}[\{m_{vw}\}]$ to values in arbitrary commutative R
   – e.g., in $\mathbb{R}$.

Blowup-polynomials: further properties

1. $p_G(\mathbf{n}) = \det(\Delta_{\mathbf{n}} M_G - 2\,\mathrm{Id}_V)$ is a polynomial in the entries of M_G and in the sizes n_v. Thus: in the above proof, *we also let n_v be indeterminates*, and work over $\widetilde{R_0} := \mathbb{Q}(\{m_{vw}, n_v\})$ (and apply Zariski density).

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   $$(-2)^{|V \setminus I|} \det(M_G)_{I \times I},$$
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Univariate specialization of p_G

Definition: The *univariate blowup-polynomial* is $u_G(n) := p_G(n, n, \ldots, n)$.

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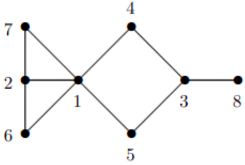
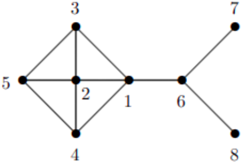
In particular, u_G also does not recover G:





What about p_G – does it recover G?

p_G is a graph invariant

Note: If G has an automorphism sending a vertex $v \in V$ to w, then the blowup-polynomial is "symmetric" under $n_v \longleftrightarrow n_w$.

- Thus, the self-isometries/automorphisms of G determine the *symmetries* of p_G. Does this process work in reverse?

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Theorem (C.–Khare, 2021)

The symmetries of p_G coincide with the self-isometries of G. More strongly, the "purely quadratic" part of p_G, i.e. its "Hessian" $\mathcal{H}(p_G)$, recovers G.

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Proof: For all graphs G,

$$\mathcal{H}(p_G) := ((\partial_{n_v}\partial_{n_w} p_G)(\mathbf{0}))_{v,w\in V} = (-2)^{|V|}\mathbf{1}_{V\times V} - (-2)^{|V|-2}(D_G + 2\,\mathrm{Id}_V)^{\circ 2},$$

where given a matrix $M = (m_{vw})$, $M^{\circ 2} := (m_{vw}^2)$ is its entrywise square. □

(Answers **Goal 2**.)

Real-rootedness of u_G

- The polynomial $u_{K_2}(n) = 3n^2 - 8n + 4 = (n-2)(3n-2)$, so it is real-rooted.
- One can compute: $u_{K_k}(n) = (n-2)^{k-1}(kn+n-2)$ – also real rooted.

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Answer: Yes. In fact, much more is true – and for p_G itself:

Theorem (C.–Khare, 2021)

For all graphs G, the polynomial $p_G(\mathbf{n})$ is real-stable.

Real-rootedness of u_G

- The polynomial $u_{K_2}(n) = 3n^2 - 8n + 4 = (n-2)(3n-2)$, so it is real-rooted.
- One can compute: $u_{K_k}(n) = (n-2)^{k-1}(kn + n - 2)$ – also real rooted.

Question: Is $u_G(n)$ real-rooted for all graphs G?

Answer: Yes. In fact, much more is true – and for p_G itself:

Theorem (C.–Khare, 2021)

For all graphs G, the polynomial $p_G(\mathbf{n})$ is real-stable.

Recall: $p(\mathbf{z})$ is real-stable if $p(z_1, \dots, z_k) \neq 0$ whenever $\Im(z_j) > 0 \ \forall j$.
(Henceforth, $|V| = k$.)

Real-stability – recent applications

Borcea and Brändén [*Duke* 2008, *Ann. of Math.* 2009, *Invent. Math.* 2009...]

- Provided far-reaching generalizations of the Laguerre–Pólya–Schur program on entire functions / multipliers / root-location / ...
- Developed a multi-variable Szász principle and multi-dimensional Jensen multipliers.
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Taken forward by Marcus–Spielman–Srivastava:

- Proved the Kadison–Singer conjecture. [*Ann. of Math.* 2015]
- Existence of bipartite Ramanujan graphs of all degrees and orders – proved conjectures of Bilu–Linial and Lubotzky. [*Ann. of Math.* 2015]

Real-stability of p_G

Theorem (C.–Khare, 2021)

For all graphs G, the polynomial $\mathbf{z} \mapsto p_G(\mathbf{z})$ is real-stable.

Provides novel families of examples of real-stable examples, from graphs and distance matrices.

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The proof uses two ingredients:

1. A result of Brändén [*Adv. in Math.* 2007]: if $A_1, \ldots, A_k$ are positive semidefinite matrices, and B is real symmetric, then the map

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is real-stable.

2. *"Inversion" preserves real-stability:* If $g(z_1, \dots, z_k)$ is a polynomial with z_j-degree $d_j \geq 1$ that is real-stable, then so is $z_1^{d_1} g(-z_1^{-1}, z_2, \dots, z_k)$. (This is because the map $z \mapsto -1/z$ preserves the upper half-plane.)

Beyond real-stability: Lorenztian / strongly Rayleigh

Recall from above (with $|V| = k$) that $p_G(\mathbf{z})$ has constant term $(-2)^k$ and linear term $-(-2)^k \sum_{j=1}^k z_j$.

Thus, the real-stable polynomial p_G does not satisfy two further properties:

1. The coefficients are not all of the same sign. [Can consider $p_G(-\mathbf{z})$.]
2. p_G is not homogeneous. [Can consider $z_0^k p_G(z_0^{-1}\mathbf{z})$.]

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Stable polynomials with these properties were studied (in broader settings) by:

1. Borcea–Brändén–Liggett [*J. Amer. Math. Soc.* 2009] – *strongly Rayleigh distributions/polynomials*;
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Our next result characterizes the graphs for which this holds.
Remarkably – if and only if all coefficients have same sign (strongly Rayleigh)!

Strongly Rayleigh graphs are complete multi-partite

Theorem (C.–Khare, 2021)

Given a graph G as above, define its homogenized blowup-polynomial

$$\widetilde{p}_G(z_0, z_1, \dots, z_k) := (-z_0)^k p_G\left(\frac{z_1}{-z_0}, \dots, \frac{z_k}{-z_0}\right) \in \mathbb{R}[z_0, z_1, \dots, z_k].$$

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3. *$(-1)^k p_G(-1, \dots, -1) > 0$, and the normalized "reflected" polynomial*

$$q_G : (z_1, \dots, z_k) \quad \mapsto \quad \frac{p_G(-z_1, \dots, -z_k)}{p_G(-1, \dots, -1)}$$

is strongly Rayleigh, *i.e., q_G is real-stable, has non-negative coefficients (of all monomials $\prod_{j \in J} z_j$), and these sum up to 1.*

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4. *The graph G is a blowup of a complete graph – that is, G is a complete multipartite graph.*

Lorentzian graphs are also complete multi-partite!

This provides a novel characterization of complete multi-partite graphs, in terms of real-stability – of the homogenized polynomial

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Further equivalent conditions:

Theorem (C.–Khare, 2021)

A graph G is complete multi-partite if and only if any of the following holds:

5. *The matrix $M_G = D_G + 2\,\mathrm{Id}_k$ is positive semidefinite.*
6. *The polynomial $\widetilde{p}_G(z_0, z_1, \ldots, z_k)$ is Lorentzian. (Brändén–Huh, 2020)*
7. *The polynomial $\widetilde{p}_G(z_0, z_1, \ldots, z_k)$ is strongly log-concave. (Gurvits, 2009)*
8. *The polynomial $\widetilde{p}_G(z_0, z_1, \ldots, z_k)$ is completely log-concave. (Anari–Oveis Gharan–Vinzant, 2018)*

Blowup-polynomials of metric spaces

- The graph blowup and blowup-polynomial (defined above for finite connected graphs) – can be defined in greater generality: *for all finite metric spaces*.

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  This fact generalizes to:

Proposition (C.–Khare, 2021)

For a finite metric space X, with distance matrix D_X, the blowup-polynomial $p_X(\mathbf{n})$ is symmetric in the variables $\{n_x : x \in X\}$, if and only if (X, d) is discrete up to scaling. That is, there exists $c > 0$ such that $d(x, y) = c$ if $x \neq y \in X$, and 0 otherwise.

Matroids

A *matroid* is a notion common to linear algebra and graph theory (among other areas):

1. A finite set E (called the *ground set*);
2. A nonempty family of subsets $\mathcal{F} \subset 2^E$ called the *independent* sets – closed under taking subsets + under "exchange axiom".

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4. *Graphic matroid* $\mathcal{F}_G$*:* Let $G = (V, E)$ be a graph. Now $\mathcal{F}_G$ includes those ("independent") sets $F \subset E$ which do not contain a cycle.

Delta-matroids

A related well-studied notion is that of a *delta-matroid*.

Example 1: Restrict to the *bases* of $\mathrm{Col}(A)$, not all linearly independent subsets. These satisfy the "Symmetric Exchange Axiom":

$$A, B \in \mathcal{F},\ x \in A \Delta B \quad \Longrightarrow \quad \text{there exists } y \in A\Delta B \ \text{ s.t. } A\Delta\{x, y\} \in \mathcal{F}.$$

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Example 2: *Linear delta-matroid* – given a symmetric or skew-symmetric matrix $A_{n\times n}$ over a field, let $E := \{1, \dots, n\}$.

A subset $F \subset E$ is *feasible* $\iff$ $\det A_{F\times F} \neq 0$.
The set of feasible subsets is the linear delta-matroid, denoted by $\mathcal{M}_A$.

From blowup-polynomials to blowup delta-matroids

Brändén (*Adv. Math.* 2007) showed: if $p(z_1, \dots, z_k)$ is a real-stable multi-affine polynomial, then the set of monomials in p forms a delta-matroid with ground set $E = \{1, \dots, k\}$.

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In fact, this delta-matroid is linear: $\mathcal{M}_{M_G}$.

Example: For $G = P_3$ (path graph), with $E = \{1, 2, 3\}$,

$$\mathcal{M}_{M_{P_3}} = 2^E \setminus \{\{1, 3\}, \{1, 2, 3\}\}.$$

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Questions:

1. Does this hold for all k?
2. Regardless of (1), is the right-hand side a delta-matroid for all k?

Another delta-matroid for trees

Proposition (C.–Khare, 2021)

The right-hand side is a delta-matroid for every k, and it equals $\mathcal{M}_{P_k}$ if and only if $k \leq 8$.

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In particular, for $k \geq 9$, the right-hand side yields a different novel delta-matroid for P_k. How to generalize this phenomenon?

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Note that the induced subgraph in P_k on $I := \{i, i+1, i+2\}$ is a tree which is a blowup-graph: $P_3 = K_2[(2,1)]$, and $i, i+2$ are copies of a vertex in K_2. Hence $(M_{P_3})_{I \times I}$ has two identical rows and columns, so $\det(M_{P_3})_{I \times I} = 0$.

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Another delta-matroid for trees (cont.)

Thus, if e.g. G is a tree, and two vertices are leaves in any sub-tree of G on vertices $I \subset V(G)$ with the same parent, then $\det(M_G)_{I \times I} = 0$.
Is the converse true – i.e., does setting all such I as the *infeasible* subsets yield a delta-matroid? (Notice, this recovers the "right-hand" delta-matroid for P_k for all k.)

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Answer: *Yes:*

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Suppose T is a tree. Define a subset of vertices I to be infeasible if its Steiner tree $T(I)$ has two leaves, which are in I and have the same parent. Then the remaining, "feasible" subsets form a delta-matroid $\mathcal{M}'(T)$.

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- Note that $\mathcal{M}'(P_k) \supsetneq \mathcal{M}_{M_{P_k}}$ for $k \geq 9$.
- We also show that the construction of $\mathcal{M}'(T)$ does *not* extend to arbitrary graphs.

References

[1] P.N. Choudhury and A. Khare.
The blowup-polynomial of a metric space: connections to stable polynomials, graphs and their distance spectra.
Preprint, arXiv:2105.12111, 2021.

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