Poincaré Series of Divisors on Finite Graphs

Graphs, Matrices and Applications, IIT Kharagpur

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- Based on the arxiv preprint " Poincaré Series of Divisors on Graphs and Chains of Loops" (arXiv:2011.11910), 25 November, 2020.
- We start with a gentle introduction to divisors and the chip firing game.

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- Example: $2(v_1) - 3(v_2) - (v_3) + (v_4),\ (v_1) - (v_3)$.
- Degree of a divisor D: $\deg(\mathrm{D}) := \sum_{v \in V(G)} a_v$ (the total number of chips).

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- **Move:** An arbitrary vertex u fires exactly one chip along every edge incident on it.
- **Final Configuration:** Another divisor $\tilde{D} = \sum_{v \in V(G)} b_v(v)$ where $b_u = a_u - \mathrm{val}(u)$ and $b_w = a_w + m_{u,w}$ for all $w \neq u$ with $m_{u,w}$: the number of edges between u and w.

An Example

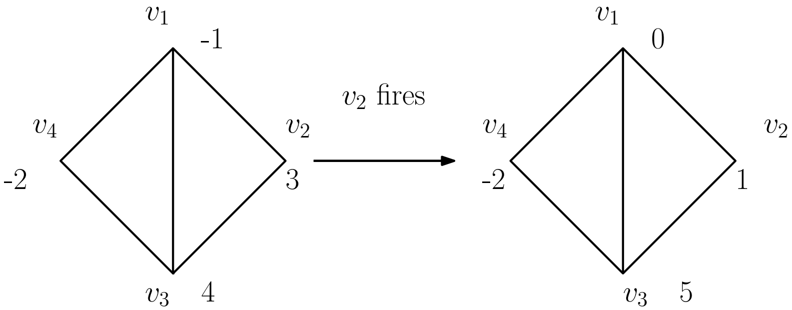

Figure: A Chip Firing Move: An Example

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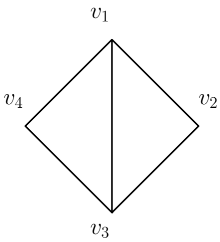
- Recall that the Laplacian matrix $Q(G)$ of G is defined as $D - A$ where D is a diagonal matrix $\mathrm{diag}(\mathrm{val}(v_1), \ldots, \mathrm{val}(v_n))$ where $\mathrm{val}(v_i)$ is the valence of the vertex v_i and A is the vertex to vertex adjacency matrix.

- Example:



$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

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- We have $\tilde{D} = D - Q(G) \cdot e_u$.
- For a sequence of chip firings, $\tilde{D} = D - Q(G) \cdot \mathbf{w}$ where the i-th entry of $\mathbf{w}$ is the number of times v_i fires.

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- For a sequence of chip firings, $\tilde{D} = D - Q(G) \cdot \mathbf{w}$ where the i-th entry of $\mathbf{w}$ is the number of times v_i fires.
- Two divisors are said to be *linearly equivalent* if they can be reached from one to another via a sequence of chip firings.

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Call a divisor *effective* if every vertex has a non-negative number of chips.

- Given a divisor D (with possibly negative coefficients), is there a sequence of chip firing moves that transforms it into an effective divisor?
- No, if its degree is negative. More subtle, if the degree is non-negative.

Rank of a divisor

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- The answer to our question is yes if and only $r_G(D) \geq 0$.
- **Riemann-Roch Problem:** Determine the rank of a divisor (in terms of the underlying graph and suitable quantities).

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- For now, we consider $P_{G,D_1,\ldots,D_k}(z_1, \ldots, z_k)$ as a formal power series.

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- We say that a power series in $z_1, \ldots, z_k$ is rational if there is a rational function f/g where $f,\ g \neq 0 \in \mathbb{C}[z_1, \ldots, z_k]$ such that the power series agrees with this rational function at every $(z_1, \ldots, z_k) \in \mathbb{C}^k$ where it is absolutely convergent.

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- Via the upper bound $r_G(D) \leq \deg(D)$ (if $\deg(D) \geq -1$): $P_{G,D_1,\ldots,D_k}(z_1, \ldots, z_k)$ is absolutely convergent in the set $\{(z_1, \ldots, z_k) |\ |z_i| < 1 \forall i\}$.

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- We discuss some key tools in the proof.

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Theorem (Baker and Norine 2007)

For every divisor D, the following formula holds:

$$r_G(D) = r_G(K_G - D) + \deg(D) - (g - 1)$$

where $g = m - n + 1$ is the first Betti number of G (G has m edges and n vertices).

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As a consequence:

- If $\deg(D) > 2g - 2$, then $r_G(D) = \deg(D) - g$.

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- They are of the form $Q(G) \cdot \mathbf{v}$ for $\mathbf{v} \in \mathbb{Z}^n$ and also form a group denoted by $\mathrm{Prin}(G)$ (group of principal divisors).
- Note that the degree of every principal divisor is zero.

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- Kirchoff's matrix-tree theorem: The Jacobian is a finite group of order equal to the number of spanning trees of G.
- A closely related group $\mathrm{Div}(G)/\mathrm{Prin}(G)$ and is isomorphic to $\mathrm{Jac}(G) \oplus \mathbb{Z}$.

Lattice Point Enumeration of Rational Polyhedra

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- A central result: The lattice point enumeration function is rational with both the numerator and denominator having integer coefficients.
- See Barvinok-Pommershein'98 for more details.

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- Upcoming Area: "Tropical algebraic geometry": Several applications to algebraic geometry and combinatorics. See Maclagan-Sturmfels 2015.
- Poincaré series of algebraic curves has been studied by Cutkosky 2003, goes back to Cutkosky-Srinivas 1993.

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1. If $\deg(D) < 0$, then $P_{G,D} = 0$ (since $r(nD) = -1$ for every $n \in \mathbb{N}$).
2. If $\deg(D) > 0$, then by Riemann-Roch $r(nD) = \deg(nD) - g = n\deg(D) - g$ for $n >> 0$. Hence, $P_{G,D}(z) = p(z) + m/(1 - z)^2 - gz^m/(1 - z)$ for some positive integer m.

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- Note that $\ker(\phi_{G,D})$ is a subgroup of $\mathbb{Z}$ and is hence, of the form $m \cdot \mathbb{Z}$ where m is the order of $[D]$ in the Jacobian.

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Hence, the Poincaré series is rational.

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$$\begin{cases} \sum_{(n_1,\ldots,n_k)\in Q^{(l)}} (r(n_1 D_1 + \cdots + n_k D_k) + 1) z_1^{n_1} \cdots z_k^{n_k}, \text{if } Q^{(l)} \neq \emptyset \\ 0, \text{otherwise.} \end{cases}$$

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- Note $P_{G,D_1,\ldots,D_k} = \sum_{l\in\mathbb{Z}} P^{(l)}_{G,D_1,\ldots,D_k}$.

- Decompose $P_{G,D_1,\ldots,D_k}$ into three pieces:

$$\underbrace{\sum_{l<0} P^{(l)}_{G,D_1,\ldots,D_k}}_{\text{small}} + \underbrace{\sum_{0\le l\le 2g-2} P^{(l)}_{G,D_1,\ldots,D_k}}_{\text{intermediate}} + \underbrace{\sum_{l\ge 2g-1} P^{(l)}_{G,D_1,\ldots,D_k}}_{\text{large}}$$

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- Since the rank of a divisor of negative degree is -1, the small piece is zero.
- Key Idea 2: Show that the intermediate and large piece are both rational by interpreting them in term lattice point enumerating functions of rational polyhedra (Barvinok-Pommershein'98).

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- If $l > 2g-2$, then by Riemann-Roch $P^{(l)}_{G,D_1,\ldots,D_k} = \sum_{(n_1,\ldots,n_k)\in Q^{(l)}} (\sum_{i=1}^{k} n_i d_i - g) z_1^{n_1} \cdots z_k^{n_k}$.

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- Hence, $\sum_{l=2g-1}^{\infty} P^{(l)}_{G,D_1,\dots,D_k}$ can be expressed in terms of the lattice point enumerating function E_C of the rational polyhedron:

$$C = \{(n_1,\dots,n_k) \in \mathbb{R}^k |\ \sum_{i=1}^{k} n_i d_i \geq 2g-2, n_i \geq 0\ \forall i\}$$

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- In other words, $E_C(z_1,\dots,z_k) = \sum_{(n_1,\dots,n_k)\in C\cap\mathbb{Z}^k} z_1^{n_1}\cdots z_k^{n_k}$

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- E_C is a rational function and hence, so is the large piece.

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- Key Idea 3: Further refine $P^{(I)}_{G,D_1,\ldots,D_k}$ based on each divisor class $[D]$.
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- Let $P^{[D]}_{G,D_1,\ldots,D_k} =$

$$\begin{cases} \sum_{(n_1,\ldots,n_k)\in Q^{(D]}} (r(\sum_{i=1}^{k} n_i D_i) + 1) z_1^{n_1} \cdots z_k^{n_k}, \text{if } Q^{([D])} \neq \emptyset \\ 0, \text{otherwise.} \end{cases}$$

- Note that $P^{(l)}_{G,D_1,\ldots,D_k} = \sum_{[D]\in \mathrm{Jac}^l(G)} P^{[D]}_{G,D_1,\ldots,D_k}$.

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- Here $\mathrm{Jac}^{(l)}(G)$ is the set of divisor classes of degree l and is finite.

- Hence, it suffices to show that each $P^{[D]}_{G,D_1,\ldots,D_k}$ is rational.

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- We show: $\sum_{(n_1,\ldots,n_k)\in Q^{[D]}} z_1^{n_1}\cdots z_k^{n_k}$ is rational by interpreting $Q^{[D]}$ in terms of the "set of lattice points in a rational polyhedron".

- Key Idea 4: Consider the homomorphism
$$\phi_{G,D_1,\ldots,D_k} : \mathbb{Z}^k \to \mathrm{Div}(G)/\mathrm{Prin}(G)$$
taking $(n_1, \ldots, n_k)$ to $[\sum_{i=1}^k n_i D_i]$.

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- Study the fiber of $\phi_{G,D_1,\ldots,D_k}$ over $[D]$.

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- The notion of Poincaré series can be defined for metric graphs.
- See the arxiv preprint for rationality of a family called chains of loops.

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Thank You