Characteristic center of a tree

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- $L(G)$ is symmetric and positive semi-definite.

- $S(G) = (\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G))$ is the **Laplacian spectrum**, where $0 \leq \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ are the eigenvalues of $L(G)$.

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- **Corollary** Let G be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. The number of spanning trees of G equals $\frac{\lambda_2 \lambda_3 \cdots \lambda_n}{n}$.

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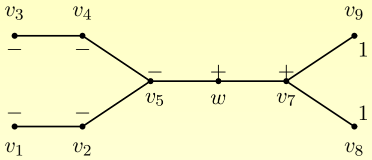
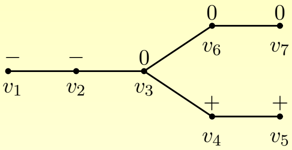
- Let Y be a Fiedler vector. By $Y(v)$, we mean the co-ordinate of Y corresponding to the vertex v.





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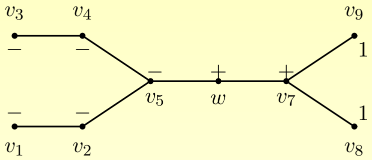

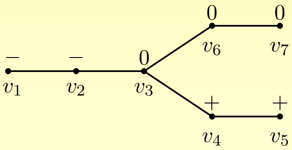

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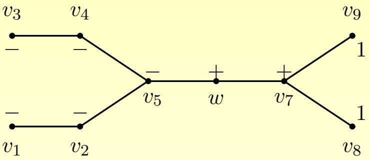

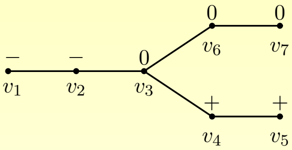

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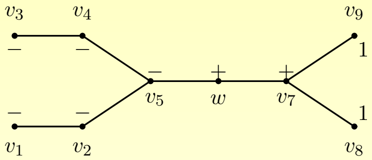

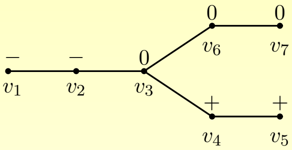

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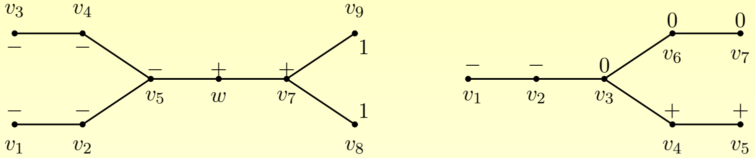

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- **Characteristic set:** Collection of characteristic edges and characteristic vertices and is denoted by $\mathcal{C}(T, Y)$.

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2. **Some entries of Y are zero.** The subgraph of T induced by the set of vertices corresponding to zero's in Y is connected. Moreover, there is a unique vertex u such that $Y(u) = 0$ and u is adjacent to a vertex v with $Y(v) \neq 0$. The entries of Y are either increasing, decreasing or identically zero along any path in T which starts at u.

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A square matrix A of order $n \geq 2$, is called **reducible** if there is a permutation matrix P such that $P^t A P = \begin{pmatrix} B & C \\ \mathbf{0} & D \end{pmatrix}$, where B and D are square submatrices. Otherwise A is called **irreducible**.

Perron-Frobenius Theorem: An irreducible non-negative matrix A has a real positive simple eigenvalue r such that $r \geq |\lambda|$ for any eigenvalue λ of A. Furthermore, there is a positive eigenvector corresponding to r. Also if $\mathbf{u}$ is an eigenvector of A with positive entries then $\mathbf{u}$ is the eigenvector corresponding to the eigenvalue r mentioned above.

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Corollary: Let G be a connected graph. Then the smallest eigenvalue of $L(G)$ is simple and there is a positive eigenvector associated with it. Furthermore, if M is a principal submatrix of $L(G)$ then the smallest eigenvalue of $L(G)$ is strictly smaller than the smallest eigenvalue of M.

For non-negative square matrix A and B(not necessarily same order), the notation $A \ll B$ is used to mean that there exist a permutation matrix P such that P^tAP is entry wise dominated by a pricipal submatrix of B, with strict inequality in atleast one position in case A and B have the same order.

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Corollary: Let A and B be two non-negative square matrices. If B is irreducible and $A \ll B$ then the largest eiginvalue of A is strictly less that the largest eigenvalue of B.

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M-matrices are closed under the extraction of principal submatrices and the inverse of an irreducible nonsingular M-matrix has positive entries.

Let v be a vertex of a tree T. Let $T_1, T_2, \cdots, T_k$ be the connected components of $T - v$. For each such component, let $\hat{L}(T_i), i = 1, 2, \cdots, k$ denote the principal submatrix of the Laplacian matrix L corresponding to the vertices of T_i. Then $\hat{L}(T_i)$ is invertible and $\hat{L}(T_i)^{-1}$ is a positive matrix which is called the bottleneck matrix for T_i.

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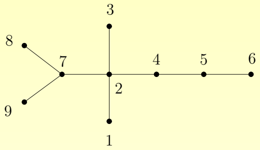
Proposition[Kirkland, Neumann and Shader(1996)]: Let T be a tree and v be any vertex of T. Let T_1 be a connected component of $T - v$. Then $\hat{L}(T_1)^{-1} = [m_{ij}]$, where m_{ij} is the number of edges in common between the path P_{iv} joining the vertex i and v and the path P_{jv} joining the vertex j and v.



Let C_1 be the connected component of $T-2$ containing the vertex 6. Then

$$\widehat{L}(C_1) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \widehat{L}(C_1)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Theorem[Kirkland, Neumann and Shader(1996)]: Let T be a tree on n vertices. Then the edge $\{i, j\}$ is the characteristic edge of T if and only if the component T_i at vertex j containing the vertex i is the unique Perron component at j while the component T_j at vertex i containing the vertex j is the unique Perron component at i. Moreover in this case there exists a $\gamma \in (0, 1)$ such that

$$\frac{1}{\mu(T)} = \rho(\widehat{L}(C_i)^{-1} - \gamma J) = \rho(\widehat{L}(C_j)^{-1} - (1-\gamma)J).$$

Furthermore, any eigenvector Y of $L(T)$ corresponding to $\mu(T)$ acn be permuted and partitioned into block form $Y^t = [Y_1^t | - Y_2^t]$, where Y_1 is a Perron vector for $\rho(\widehat{L}(C_i)^{-1} - \gamma J)$ and Y_2 is a Perron vector for $\rho(\widehat{L}(C_j)^{-1} - (1-\gamma)J)$. Here J is the all one matrix and ρ stands for spectral radius.

Theorem[Kirkland, Neumann and Shader(1996)]: Let T be a tree on n vertices. Then the vertex v is the characteristic vertex of T if and only if there are two or more Perron components of T at v. Moreover in this case,

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Theorem[Kirkland, Neumann and Shader(1996)]: Let T be a tree. Then for any vertex v that is neither a characteristic vertex nor an end vertex of the characteristic edge, the unique Perron component at v contains the characteristic set of T.

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  For any tree T, $C(T)$ is same as the center of any $u - v$ path in T of length $diam(T)$.

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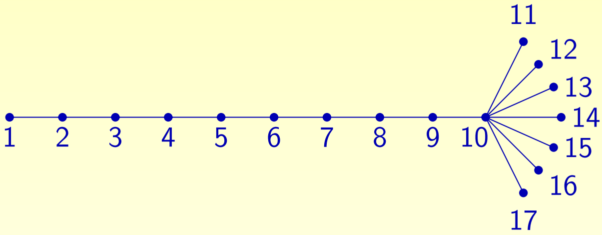
- If $n \geq 3$, then neither the center nor the centroid of T contain pendant vertices.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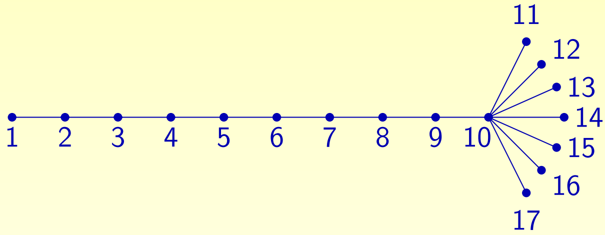

For the above tree T, the vertex 6 is the center as its eccentricity is 5, less than any other vertex. The vertex 9 is the centroid as it has weight 8, less than any other vertex.

Also $\mu(T) = .0483$ and $Y =$
$(-0.4116, -0.3917, -0.3528, -0.2970, -0.2267, -0.1455, -0.0573, 0.0337, 0.1231, 0.2065, 0.2170, 0.2170, 0.2170, 0.2170, 0.2170, 0.2170, 0.2170)^t$ is a Fiedler vector. So $\chi(T) = \{7, 8\}$, which is disjoint from each of the center, centroid and subtree core.

For a given tree T, we denote by $d_T(C, C_d) = \min\{d(u, v) | u \in C \text{ and } v \in C_d\}$ the distance between the center and the centroid of T.

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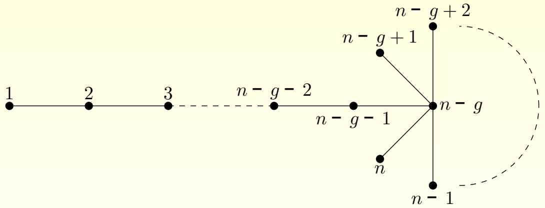
Let $P_{n-g,g}$, $n \geq 5$, $2 \leq g \leq n-3$, denote the tree on n vertices which is obtained from the path P_{n-g} by adding g pendant vertices to the vertex $n-g$. Such a tree $P_{n-g,g}$ is called a **path-star tree**.



Theorem[-, 2007]: Among all trees on $n \geq 5$ vertices, the distance between the center and the characteristic center is maximized by a path-star tree $P_{n-g,g}$, for some positive integer g.
Proof:
Case 1: Characteristic center lies in one of the longest path

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Proof:

Case 1: Characteristic center lies in one of the longest path

Case 2: Characteristic center does not lie in any of the longest path

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Theorem[-, 2007]: Let $P_{n-g,g}$ be a path-star tree. Then the characteristic center of $P_{n-g,g}$ lies in the path from $C(P_{n-g,g})$ to $C_d(P_{n-g,g})$.

The position of the center of $P_{n-g,g}$ can also be expressed in terms of $n - g$. The following result is straight-forward.

The position of the center of $P_{n-g,g}$ can also be expressed in terms of $n-g$. The following result is straight-forward.

Lemma: The center of the path-star tree $P_{n-g,g}$ is given by

$$C(P_{n-g,g}) = \begin{cases} \left\{\frac{n-g+2}{2}\right\}, & \text{if } n-g \text{ is even,} \\ \left\{\frac{n-g+1}{2}, \frac{n-g+3}{2}\right\}, & \text{if } n-g \text{ is odd.} \end{cases}$$

The position of the centroid of a path-star tree $P_{n-g,g}$ can be expressed in terms of g. The following result is straight-forward.

The position of the centroid of a path-star tree $P_{n-g,g}$ can be expressed in terms of g. The following result is straight-forward.

Lemma: The centroid of the path-star tree $P_{n-g,g}$ is given by

$$C_d(P_{n-g,g}) = \begin{cases} \begin{cases} \{\frac{n+1}{2}\}, & \text{if } g \leq \frac{n-1}{2} \\ \{n-g\}, & \text{if } g > \frac{n-1}{2} \end{cases}, & \text{if } n \text{ is odd,} \\ \begin{cases} \{\frac{n}{2}, \frac{n}{2}+1\}, & \text{if } g \leq \frac{n}{2}-1 \\ \{n-g\}, & \text{if } g > \frac{n}{2}-1 \end{cases}, & \text{if } n \text{ is even.} \end{cases}$$

Theorem[-, 2007]: Among all trees on $n \geq 5$ vertices, the distance between the center and the centroid is maximized by $P_{n-\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$. Futhermore,

$$d_{P_{n-\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}}(C, C_d) = \left\lfloor \frac{n-3}{4} \right\rfloor.$$

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Corollary: $\lim_{n\to\infty} \frac{\delta_n(C,C_d)}{n} = \frac{1}{4}$.

Theorem[Kirkland et al.,2017]: Let z be the unique root of the equation $\tan(z) + z = 0$ that lies in the interval $(\frac{\pi}{2}, \pi]$. Then

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Theorem[Kirkland et al.,2017]:

$$\lim_{n\to\infty} \frac{\delta_n(C, \chi)}{n} = \frac{c_0\pi}{4}\left(c_0\pi - \sqrt{c_0^2\pi^2 - 4(1 - c_0)}\right) - \frac{1 - c_0}{2},$$

where $c_0 \in \left(\frac{2\sqrt{\pi^2+1}}{\pi^2} - 1, 1\right)$ is the unique solution of $w(c) = \frac{\pi}{2(1-r)}$, r is a function of c, $r \in (c, \frac{c+1}{2})$, $c \in (0, 1)$.

Conjecture: For $n \geq 5$ and $2 \leq g \leq n-3$, the path star-tree is a Type-II tree.

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THANK YOU