Spectrum of Cayley sum graphs

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Preliminaries

Definition 1

A *graph* is a pair $\mathbb{G} = (V, E)$, where V is called the *vertex set*, E is called the *edge set*, and E is a subset of $\binom{V}{2}$.

Definition 2

A *multi-graph* is a pair $\mathbb{G} = (V, E)$, where V is called the *vertex set*, E is called the *edge multi-set*, and E is a multi-subset of $\binom{V}{2}$.

Definition 3

Let $\mathbb{G} = (V, E)$ be a multi-graph. The *neighbourhood* $N(V_1)$ of a subset V_1 of V is the set of vertices in V adjacent to some element of V_1. The *boundary* $\delta(V_1)$ of a subset V_1 of V is the set of vertices in V that lie outside V_1 and are adjacent to some element of V_1.

Let $\mathbb{G} = (V, E)$ be a graph. Let $\ell^2(V)$ denote the space of complex valued functions $f : V \to \mathbb{C}$, equipped with the inner product, defined by

$$\langle f, g \rangle = \sum_{v \in V} f(v)\overline{g(v)}.$$

Definition 4

The *adjacency operator* $A : \ell^2(V) \to \ell^2(V)$ is defined by

$$(Af)(v) = \sum_{w \in V, \{v,w\} \in E} f(w), \quad f \in \ell^2(V).$$

Definition 5

For a d-regular multi-graph (V, E), its *normalized Laplacian operator* is defined by

$$\Delta = \mathrm{id} - \frac{1}{d} A.$$

- Tao, *Expansion in finite simple groups of Lie type*.

Lemma 6

For a d-regular graph (V, E), the eigenvalues of its adjacency operator A lies in $[-d, d]$, and d is an eigenvalue of A.

Proof.

Note that d is an eigenvalue of A since $A1 = d1$, where 1 denotes the constant function sending $v \mapsto 1$. For any $f, g \in \ell^2(V)$ having norm one,

$$\begin{aligned}
\left|\langle Af, g\rangle_{\ell^2(V)}\right| &= \left|\sum_{v,w\in V,\{v,w\}\in E} f(w)\overline{g(v)}\right| \\
&\leq \frac{1}{2}\sum_{v,w\in V,\{v,w\}\in E} (|f(w)|^2 + |g(v)|^2) \\
&\leq \frac{d}{2}\sum_{w\in V}|f(w)|^2 + \frac{d}{2}\sum_{v\in V}|g(v)|^2 \\
&= d.
\end{aligned}$$

This shows that the eigenvalues of A lie in the interval $[-d, d]$.

For a d-regular graph (V, E), the eigenvalues of $\frac{1}{d}A, \Delta$ are denoted by

$$-1 \leqslant t_n \leqslant t_{n-1} \leqslant \cdots \leqslant t_2 \leqslant t_1 = 1,$$
$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n-1} \leqslant \lambda_n \leqslant 2$$

respectively, where $\lambda_i = 1 - t_i$.

Definition 7 (Vertex Cheeger constant)

The *vertex Cheeger constant* of the multi-graph $\mathbb{G} = (V, E)$, denoted by $h(\mathbb{G})$, is defined as

$$h(\mathbb{G}) := \inf \left\{ \frac{|\delta(V_1)|}{|V_1|} : \emptyset \neq V_1 \subseteq V, |V_1| \leqslant \frac{|V|}{2} \right\}.$$

Definition 8 ((n, d, ε)-expander, Alon (1986))

Let $\varepsilon > 0$. An (n, d, ε)-*expander* is a graph $\mathbb{G} = (V, E)$ on $|V| = n$ vertices, having maximal degree d, such that for every set $\emptyset \neq V_1 \subseteq V$ satisfying $|V_1| \leqslant \frac{n}{2}$, $|\delta(V_1)| \geqslant \varepsilon |V_1|$ holds (equivalently, $h(\mathbb{G}) \geqslant \varepsilon$).

Like vertex expansion, one also has a notion of edge expansion and a corresponding edge Cheeger constant.

Definition 9 (Edge expansion)

Let $\mathbb{G} = (V, E)$ be a d-regular multi-graph. For a subset $\emptyset \neq V_1 \subseteq V$, let $E(V_1, V \setminus V_1)$ be the *edge boundary* of V_1, defined as

$$E(V_1, V \setminus V_1) := \{(v_1, v_2) \in E : v_1 \in V, v_2 \in V \setminus V_1\}.$$

Then the *edge expansion ratio* $\phi(V_1)$ of V_1 is defined as

$$\phi(V_1) := \frac{|E(V_1, V \setminus V_1)|}{d|V_1|}.$$

Definition 10 (Edge-Cheeger constant)

The edge-Cheeger constant $\mathfrak{h}(\mathbb{G})$ of a multi-graph $\mathbb{G}$ is defined by

$$\mathfrak{h}(\mathbb{G}) := \inf_{\emptyset \neq V_1 \subseteq V, |V_1| \leqslant |V|/2} \phi(V_1).$$

Relation between the Cheeger constants

Lemma 11

Let $\mathbb{G} = (V, E)$ *be a* d*-regular multi-graph. Then*

$$\frac{h(\mathbb{G})}{d} \leqslant \mathfrak{h}(\mathbb{G}) \leqslant h(\mathbb{G}).$$

The Cheeger constants and the spectrum

Proposition 12 (Discrete Cheeger–Buser inequality)

Let $\mathbb{G} = (V, E)$ *be a finite d-regular multi-graph. Let* λ_2 *denote the second smallest eigenvalue of its normalised Laplacian operator and* $\mathfrak{h}(\mathbb{G})$ *be the (edge) Cheeger constant. Then*

$$\frac{\mathfrak{h}(\mathbb{G})^2}{2} \leqslant \lambda_2 \leqslant 2\mathfrak{h}(\mathbb{G}).$$

Proof.

Lubotzky, *Discrete groups, expanding graphs and invariant measures.* □

The lower spectrum

- Breuillard–Green–Guralnick–Tao argued qualitatively that in the case of non-bipartite Cayley graphs, the lower spectrum cannot be arbitrarily close to -1 (2013).
- Biswas established a quantitative version of this fact (2018).
- Moorman–Ralli–Tetali obtained an improvement to Biswas's bound (2020).

Definition 13

Let G be a group and S be a subset of a group G. The *Cayley sum graph* $C_\Sigma(G, S)$ is the graph having G as its set of vertices, and two vertices $g, h \in G$ are adjacent if $gh = s$ (or equivalently, $h = g^{-1}s$).

Definition 14

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Cayley graphs vs Cayley sum graphs

A Cayley graph 'looks' the same around every vertex (it is vertex transitive). The Cayley sum graphs do not have this property.

Example 15

The Cayley sum graph $C_\Sigma(\mathbb{Z}/(2k+1)\mathbb{Z}, \{\pm 1\})$ has loops at the vertices $k, k+1$, and does not admit loops at the remaining vertices.

Lemma 16

The Cayley sum graph $C_\Sigma(G, S)$ is undirected if and only if S is closed under conjugation by the elements of G.

Proof.

The graph $C_\Sigma(G, S)$ is undirected if and only if for any $g \in G, s \in S$, $g = (g^{-1}s)^{-1}t$ holds for some $t \in S$, i.e., S is closed under conjugation. □

Lemma 17

Suppose the Cayley sum graph $C_\Sigma(G,S)$ is connected. Then the graph $C_\Sigma(G,S)$ is bipartite if and only if there is an index two subgroup H of G that avoids S.

Proof.

($\Leftarrow$) Suppose G contains a subgroup H of index two which does not intersect S. Then the set H is an independent set of vertices. Otherwise, $h' = h^{-1}s$ for some $h, h' \in H, s \in S$, which implies $s = hh' \in H$. The set $G \setminus H$ is also an independent set of vertices since $h' = h^{-1}s$ for some $h, h' \in G \setminus H, s \in S$, which implies $s = hh' \in (G \setminus H)^2 = H$.

□

The lower spectra of Cayley sum graphs

Theorem 18 (Biswas-S, 2019)

Let $h_\Sigma(G)$ denote the vertex Cheeger constant of the Cayley sum graph $C_\Sigma(G,S)$. Then if $C_\Sigma(G,S)$ is non-bipartite, we have

$$\lambda_n < 2 - \frac{h_\Sigma(G)^4}{2^9 d^8} \;\; (\text{equivalently } -1 + \frac{h_\Sigma(G)^4}{2^9 d^8} < t_n),$$

where λ_n (respectively, t_n) is the largest (respectively, the smallest) eigenvalue of the normalised Laplacian operator (respectively, the normalised adjacency operator) of $C_\Sigma(G,S)$.

Key steps of the proof

Proposition 19

Let $C_\Sigma(G,S)$ be a non-bipartite (n,d,ε)-vertex expander for some $\varepsilon > 0$. Suppose the normalised adjacency operator of $C_\Sigma(G,S)$ has an eigenvalue in the interval $(-1,-1+\zeta]$ for some ζ satisfying $0 < \zeta \leqslant \frac{\varepsilon^2}{4d^4}$. Then for some subset A of G, the following conditions hold with $\beta = d^2\sqrt{2\zeta(2-\zeta)}$.

1. $\left(\frac{1}{2+\beta+\frac{d\beta}{\varepsilon}}\right)|G| \leqslant |A| \leqslant \frac{1}{2}|G|$.
2. $|Ag \cap (Ag)^{-1}S| \leqslant \frac{\beta}{\varepsilon}|A|$ *for all* $g \in G$.
3. $|(Ag)^{-1}s\Delta(Ag)^c| \leqslant \frac{\beta}{\varepsilon}(\varepsilon + d + 2)|A|$ *for all* $s \in S, g \in G$.
4. $|A^{-1}g \cap (A^{-1}g)^{-1}S| \leqslant \frac{\beta}{\varepsilon}|A|$ *for all* $g \in G$.
5. $|(A^{-1}g)^{-1}s\Delta(A^{-1}g)^c| \leqslant \frac{\beta}{\varepsilon}(\varepsilon + d + 2)|A|$ *for all* $s \in S, g \in G$.

Proposition 20

Under the notations and assumptions as above, and the additional hypothesis

$$\beta < \frac{\varepsilon^2}{4d(d+1)},$$

it follows that for a given element $g \in G$,

1. *exactly one of the inequalities*

   $$|A \cap Ag| \leqslant \frac{d\beta}{\varepsilon^2}(\varepsilon + d + 2)|A|, \quad |A \cap Ag| \geqslant \left(1 - \frac{d\beta}{\varepsilon^2}(\varepsilon + d + 2)\right)|A|$$

   holds, and

2. *exactly one of the inequalities*

   $$|A \cap A^{-1}g| \leqslant \frac{d\beta}{\varepsilon^2}(\varepsilon + d + 2)|A|, \quad |A \cap A^{-1}g| \geqslant \left(1 - \frac{d\beta}{\varepsilon^2}(\varepsilon + d + 2)\right)|A|$$

   holds.

Theorem 21

Suppose $C_\Sigma(G,S)$ be a non-bipartite (n,d,ε)-vertex expander for some $\varepsilon > 0$. Then the eigenvalues of the normalised adjacency operator of this graph are greater than $-1+\ell_{\varepsilon,d}$ with

$$\ell_{\varepsilon,d} = \frac{\varepsilon^4}{2^9 d^8}.$$

Proof.

1. To define two subsets H_+, H_- of G.
2. H_+ is a subgroup of G of index two.
3. A dichotomy result for H_-.
4. To use the dichotomy to conclude the proof.

□

The subsets H_+, H_-

Let us assume that an eigenvalue of the normalised adjacency operator of the graph $C_\Sigma(G, S)$ lies in the interval $[-1, -1 + \ell_{\varepsilon,d}]$. Note that -1 is not an eigenvalue of its normalised adjacency operator. Hence an eigenvalue of the normalised adjacency operator of the graph $C_\Sigma(G, S)$ lies in the interval $(-1, -1 + \ell_{\varepsilon,d}]$. Set

$$\tau = d^2 \sqrt{2\ell_{\varepsilon,d}(2 - \ell_{\varepsilon,d})},$$
$$r = 1 - \frac{d\tau}{\varepsilon^2}(\varepsilon + d + 2).$$

Define the subsets H_+, H_- of G by

$$H_+ := \{g \in G \ : \ |A \cap Ag| \geqslant r|A|\},$$
$$H_- := \{g \in G \ : \ |A \cap A^{-1}g| \geqslant r|A|\}.$$

H_+ is a subgroup of G of index two

We use an argument due to Freĭman.

- H_+ contains the identity element of G.
- By the triangle inequality, it follows that H_+ is a subgroup of G.
- By counting arguments, it follows that $H_+ \neq G$.
- Furthermore,

$$|A| \leqslant |H_+| + \frac{d\tau}{\varepsilon^2}(\varepsilon + d + 2)(|G| - |H_+|).$$

Using Proposition 19(1), we obtain

$$\left(\frac{1}{2 + \tau + \frac{d\tau}{\varepsilon}}\right)|G| - \frac{d\tau}{\varepsilon^2}(\varepsilon + d + 2)|G| \leqslant \left(1 - \frac{d\tau}{\varepsilon^2}(\varepsilon + d + 2)\right)|H_+|.$$

If $|G| \geq 3|H_+|$, then one obtains a contradiction. This implies that H_+ is a subgroup of G of index two.

A dichotomy result for H_-

We use the strategy of Freĭman once again.

- By Proposition 19(2), H_- does not intersect the set S (in particular $H_- \neq G$).
- Using counting arguments, it follows that
$$|H_-| > \frac{|G|}{3},$$
and consequently, H_- is nonempty.
- Using the triangle inequality once again, it follows that H_- contains $h_- h_+$ for $h_- \in H_-, h_+ \in H_+$, i.e.,
$$H_- H_+ \subseteq H_-.$$
- (Dichotomy) Since H_- is a nonempty proper subset of G, it follows that
$$H_- = H_+, \quad \text{or } H_- = G \setminus H_+.$$

Concluding the proof

- If $H_{-} = H_{+}$, then H_{+} is an index two subgroup of G avoiding S, which implies that $C_{\Sigma}(G, S)$ is bipartite.
- Suppose H_{-} is not equal to H_{+}. Then H_{+} of G contains S. Since the graph $C_{\Sigma}(G, S)$ is connected, every element of G is connected to the identity element. So, any of element of G can be expressed as a product of elements of the set $S \cup S^{-1}$. This shows that G is contained in H_{+}, which is impossible.

This completes the proof.

Thank you