On the spectral radius of bi-block graphs with given independence number α

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Outline

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Definitions

- A graph G is an ordered pair, $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$ is the set of vertices and $E \subset V \times V$ is the set of edges in G.
- We write $i \sim j$ to indicate that the vertices $i, j \in V$ are adjacent in G and $i \not\sim j$ when they are not adjacent.

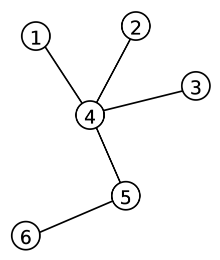

Figure: Graph on 6 vertices

- The degree of the vertex i, denoted by δ_i, equals the number of vertices in V that are adjacent to i.

Subgraph

A subgraph H of a graph G is a graph whose set of vertices and set of edges are all subsets of G.

A subgraph H of G is said to be an induced subgraph with vertex set S if H is a maximal subgraph of G with vertex set $V(H) = S$.

Connected Graph

In an undirected graph G, two vertices u and v are called connected if G contains a path from u to v. Otherwise, they are called disconnected.

A graph is said to be connected if every pair of vertices in the graph is connected.

Examples: Path, Cycles, Tree, Complete Graphs, Complete Bipartite Graphs.

K_n and $K_{m,n}$

A graph with n vertices is called complete, if each vertex of the graph is adjacent to every other vertex and is denoted by K_n.

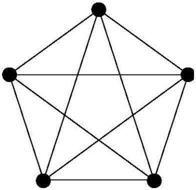

Figure: $G = K_5$

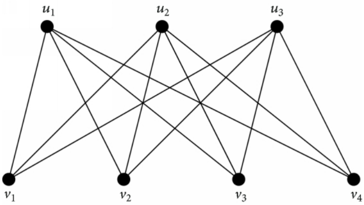

Figure: $G = K_{4,3}$

A graph $G = (V, E)$ said to be bipartite if V can be partitioned into two subsets V_1 and V_2 such that $E \subset V_1 \times V_2$. A bipartite graph $G = (V, E)$ with the partition V_1 and V_2 is said to be a complete bipartite graph, if every vertex in V_1 is adjacent to every vertex of V_2.

Independence Number

A set $\mathcal{I}$ of vertices in a graph G is an independent set if no pair of vertices of $\mathcal{I}$ are adjacent. The independence number of G is denoted by $\alpha(G)$, is the cardinality of the largest independent set in G.

An independent set of cardinality $\alpha(G)$ is called an $\alpha(G)$-set.

Adjacency Matrix

The adjacency matrix of G is the $n \times n$ matrix, denoted as $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j, i \sim j \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

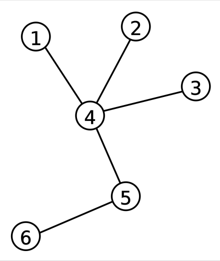

Figure: G

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Cut Vertex and Blocks

A vertex v of a connected graph G is a *cut vertex* of G if $G - v$ is disconnected. A block of the graph G is a maximal connected subgraph of G that has no cut-vertex.

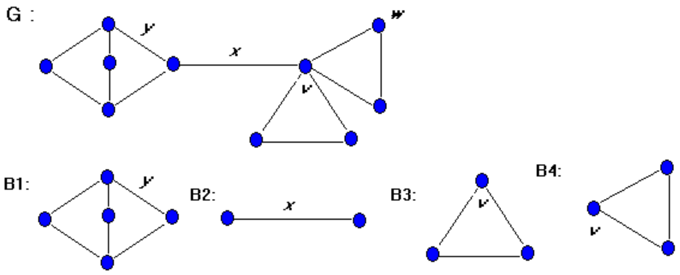

Figure: Graph with cut-vertex.

- A block is said to be a leaf block if its deletion does not disconnect the graph.
- Given two blocks F and H of graph G are said to be neighbours if they are connected via a cut-vertex. We write $F \circledcirc H$ to represent the induced subgraph on the vertex set of two neighbouring blocks F and H.

Block and Bi-block Graphs

A graph is said to be *block graph* if each of its blocks are complete graphs.

A graph is said to be *bi-block graph* if each of its blocks are complete bipartite graphs.

For $v \in V$, the block index of v is denoted by $bi_G(v)$, equals the number of blocks in G that contain the vertex v. Here we consider the star $K_{1,n}$ as a complete bipartite graph instead of a bi-block graph.

- For any column vector X of order $|V|$, if x_u represent the entry of X corresponding to the vertex $u \in V$, then $X^t A(G) X = 2\sum_{u\sim w} x_u x_w$.
- For a connected graph G on k ≥ 2 vertices, by Perron-Frobenius theorem, the spectral radius $\rho(G)$ of $A(G)$ is a simple positive eigenvalue and the associated eigenvector is entry-wise positive. We will refer to such an eigenvector as the Perron vector of G.
- By Min-max theorem, we have

$$\rho(G) = \max_{X \neq 0} \frac{X^t A(G) X}{X^t X} = \max_{X \neq 0} \frac{2\sum_{u\sim w} x_u x_w}{\sum_{u \in V} x_u^2}.$$

- For a graph G if $\Delta(G)$ and $\delta(G)$ denote the maximum and the minimum of the vertex degrees of G, respectively, then

$$\delta(G) \leq \rho(G) \leq \Delta(G).$$

If G is a connected graph such that for $x, y \in V(G)$, $xy \notin E(G)$, then

$$\rho(G) < \rho(G + xy).$$

Motivation

- If G is a bipartite graph with vertex partition M and N, then
$$\alpha(G) = \max\{|M|, |N|\}.$$
- Since every bi-block graph is a bipartite graph, so given a bi-block graph G on k vertices, the independence number $\alpha(G)$, satisfies
$$\left\lceil \frac{\mathrm{k}}{2} \right\rceil \leq \alpha(G) \leq \mathrm{k} - 1.$$
- Let G be a bi-block graph. Let H be any leaf block connected to the graph G at a cut-vertex $v \in V(G)$ and $G - H$ be the graph obtained from G by removing $H - v$. Given an $\alpha(G)$-set $\mathcal{I}$, we denote
$$\mathcal{I}\big|_{G-H} = \{u \in \mathcal{I} \mid u \in V(G - H)\}.$$
- We will denote the class of bi-block graphs on k vertices with a given independence number α by $\mathcal{B}(\mathrm{k}, \alpha)$.

Observations

Let $G = (V, E)$ be a bi-block graph consisting of two blocks F and H connected by cut-vertex v, *i.e.*, $G = F \circledcirc H$. Let $F = K(P, Q)$ with $|P| = p$, $|Q| = q$ and $H = K(M, N)$ with $|M| = m$, $|N| = n$ such that $Q \cap M = \{v\}$.
Let A be the adjacency matrix of G and (ρ, X) be the eigen-pair corresponding to the spectral radius of A. Let x_u denote the entry of X corresponding to the vertex $u \in V$. Let $q, m \geq 2$. Using $AX = \rho X$, we have $\rho x_u = \sum_{w \sim u} x_w = \sum_{w \in M} x_w$ for all $u \in N$. Thus x_u is a constant, whenever $u \in N$ and we denote it by a_n. Using similar arguments, let us denote

$$x_u = \begin{cases} a_n & \text{if } u \in N, \\ a_m & \text{if } u \in M, u \neq v, \\ a_p & \text{if } u \in P, \\ a_q & \text{if } u \in Q, u \neq v. \end{cases} \tag{1}$$

Adjacency Relations

Now using $AX = \rho X$, we have the following identities:

(I1) $(q-1)a_q + x_v = \rho a_p$.

(I2) $pa_p = \rho a_q$.

(I3) $pa_p + na_n = \rho x_v$.

(I4) $na_n = \rho a_m$.

(I5) $x_v + (m-1)a_m = \rho a_n$.

Using the identities (I2),(I3) and (I4), we have $x_v = a_q + a_m$. Substituting $x_v = a_q + a_m$ in (I1) and (I5), we have

(I1*) $qa_q + a_m = \rho a_p$,

(I5*) $a_q + ma_m = \rho a_n$.

Without loss of generality if we assume that $a_p = 1$, then

(I6) $a_q = \dfrac{p}{\rho}$, $a_m = \dfrac{\rho^2 - pq}{\rho}$ and $a_n = \dfrac{\rho^2 - pq}{n}$.

Similarly, if we assume that $a_n = 1$, then

(I7) $a_m = \dfrac{n}{\rho}$, $a_q = \dfrac{\rho^2 - mn}{\rho}$ and $a_p = \dfrac{\rho^2 - mn}{p}$.

Moreover, since the ratio $\dfrac{a_p}{a_n}$ is constant for the Perron vector X, so using (I6) and (I7), we have

(I8) $pn = (\rho^2 - pq)(\rho^2 - mn)$.

If $m = 1$ and $q > 1$, then by choosing $a_m = x_v - a_q$, all the above identities are true. Similarly, for $q = 1$ and $m > 1$, we choose $a_q = x_v - a_m$.

Bi-block with 2 blocks

Let $G \in \mathcal{B}(\mathsf{k}, \alpha)$. If G consists of two blocks, then $\rho(G) < \rho(K_{\alpha,\mathsf{k}-\alpha})$.

Proof: Let G be a bi-block graph consists of two blocks F and H connected by the cut-vertex v. Let $F = K(P, Q)$, where $|P| = p$, $|Q| = q$ and $H = K(M, N)$, where $|M| = m$, $|N| = n$ such that $Q \cap M = \{v\}$. Then $\mathsf{k} = p + q + m + n - 1$.

If $m = 1$ and $q = 1$, then $\mathsf{k} = p + n + 1$ and $G = K_{1,p+n}$ with independence number $\alpha(G) = p + n$. Thus, for $\alpha = p + n$ the class $\mathcal{B}(\mathsf{k}, \alpha)$ consists of only the star $G = K_{1,p+n}$ and hence result is vacuously true. We complete the proof by considering the following cases.

<u>**Case 1:**</u> If $p \geq q$ and $n \geq m$, then $\mathcal{I} = P \cup N$ is the $\alpha(G)$-set. We consider the complete bipartite graph $G^* = K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup N$ and $\widetilde{Q} = Q \cup M$. Thus $\alpha(G) = \alpha(G^*) = p + n$. Since G^* is obtained from G by adding extra edges, so we have $\rho(G) < \rho(G^*)$.

<u>**Case 2:**</u> If $q > p$ and $m \geq n$, then $\mathcal{I} = Q \cup M$ is an $\alpha(G)$-set. We consider the complete bipartite graph $G^* = K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup N$ and $\widetilde{Q} = Q \cup M$. Thus $\alpha(G) = \alpha(G^*) = q + m - 1$. Since G^* is obtained from G by adding extra edges, so we have $\rho(G) < \rho(G^*)$.

Case 3: If $q > p$ and $n > m$, then $\mathcal{I} = (Q \setminus \{v\}) \cup N$ is an $\alpha(G)$-set and hence $\alpha(G) = q + n - 1$. Now we subdivide this case as follows:

Subcase 3.1: Let $p = q - 1$. Then $\mathcal{L} = P \cup N$ is an independent set in G and $|\mathcal{L}| = q + n - 1$. This implies that $\mathcal{L}$ is also an $\alpha(G)$-set. We consider the complete bipartite graph $G^* = K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup N$ and $\widetilde{Q} = Q \cup M$. Thus, $\alpha(G) = \alpha(G^*) = q + n - 1$. Since G^* is obtained from G by adding extra edges, so we have $\rho(G) < \rho(G^*)$.

Subcase 3.2: Let $p < q-1$. In view of the $\alpha(G)$-set $\mathcal{I} = (Q \setminus \{v\}) \cup N$, we consider the complete bipartite graph $G^* = K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup M$ and $\widetilde{Q} = (Q \setminus \{v\}) \cup N$. So $\alpha(G) = \alpha(G^*) = q+n-1$. Observe that, we can obtain the graph G^* from G using the following operations:

1. Delete the edges between vertex v and the vertices of P.
2. Add edges between vertices of M and $Q \setminus \{v\}$.
3. Add edges between vertices of P and N.

Let A be the adjacency matrix of G and (ρ, X) be the eigen-pair corresponding to the spectral radius of A. Let A^* be the adjacency matrix of G^*.

$$\begin{aligned}
\frac{1}{2}X^t(A^*-A)X &= -x_v\sum_{w\in P}x_w + \sum_{\substack{u\sim w\\ u\in M, w\in Q\setminus\{v\}}} x_ux_w + \sum_{\substack{u\sim w\\ u\in P, w\in N}} x_ux_w\\
&= -pa_p(a_q+a_m) + (q-1)a_q(a_q+ma_m) + pna_pa_n\\
&= -pa_p(a_q+a_m) + (q-1)\rho a_qa_n + pna_pa_n\\
&= -pa_p(a_q+a_m) + (q-1)pa_pa_n + pna_pa_n\\
&= p\left[(q-1)a_n + \rho a_m - (a_q+a_m)\right]\\
&= \frac{p}{\rho n}\left[\rho(q-1)(\rho^2-pq) + \rho n(\rho^2-pq) - n(p+\rho^2-pq)\right]\\
&= \frac{p}{\rho n}\left[\rho(q-1)(\rho^2-pq) + \rho n(\rho^2-pq) - n(\rho^2-pq) - (\rho^2-pq)(\rho^2-mn)\right]\\
&= \frac{p(\rho^2-pq)}{\rho n}\left[\rho(q-1) + \rho n - n - (\rho^2-mn)\right]\\
&= \frac{p(\rho^2-pq)}{\rho n}\left[\rho(q+n-1) - \rho^2 + n(m-1)\right].
\end{aligned}$$

We have $\rho \leq \max\{p+n, q\}$. And using the assumption $p < q-1$, we always have $q+n-1 > \rho$.

<u>Case 4:</u> If $p > q$ and $m > n$, then $\mathcal{I} = P \cup (M \setminus \{v\})$ is an $\alpha(G)$-set and $\alpha(G) = p + m - 1$. This case is analogous to Case 3 and hence proceeding similarly, we have $\rho(G) < \rho(G^*)$.

Bi-block with $bi_G(u) = 2$ and b blocks

Let $G \in \mathcal{B}(\mathrm{k}, \alpha)$. If $bi_G(u) = 2$ for all cut-vertex u in G, then $\rho(G) \leq \rho(K_{\alpha,\mathrm{k}-\alpha})$ and equality holds if and only if $G = K_{\alpha,\mathrm{k}-\alpha}$.

Proof:(Induction) Let $H = K(M, N)$ with $|M| = m$ and $|N| = n$ be a leaf block connected to the graph G at a cut-vertex v. Since $bi_G(v) = 2$, so there exists a unique block $F = K(P, Q)$ with $|P| = p$ and $|Q| = q$ which is a neighbour of H connected via the cut-vertex v. Without loss of generality, we assume that $M \cap Q = \{v\}$. Let $\mathcal{I}$ be an $\alpha(G)$-set of G, *i.e.*, $|\mathcal{I}| = \alpha$.

Case 1: $\mathcal{I} \cap P = \emptyset$ and $\mathcal{I} \cap Q = \emptyset$. In this case, either $M \setminus \{v\} \subset \mathcal{I}$ or $N \subset \mathcal{I}$. We consider the complete bipartite graph $K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup N$ and $\widetilde{Q} = Q \cup M$. Let G^* be the graph obtained from G by replacing the induced subgraph $F \circledcirc H$ with $K(\widetilde{P}, \widetilde{Q})$. Then, the resulting graph G^* consists of $b-1$ blocks and $\mathcal{I}$ is an $\alpha(G^*)$-set, *i.e.,* $G^* \in \mathcal{B}(\mathrm{k}, \alpha)$.

Case 2: $\mathcal{I} \cap P = \emptyset$ and $\mathcal{I} \cap Q \neq \emptyset$. For $m \geq n$, we can assume $M \subset \mathcal{I}$. We consider graph G^* which is obtained from G by replacing the induced subgraph $F \circledcirc H$ with $K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup N$ and $\widetilde{Q} = Q \cup M$, which implies that $\mathcal{I}$ is an $\alpha(G^*)$-set.

Case 3: $\mathcal{I} \cap P = \emptyset$ and $\mathcal{I} \cap Q \neq \emptyset$. For $n > m$, if $v \in \mathcal{I}$, then $\mathcal{L} = (\mathcal{I}\big|_{G-H} \setminus \{v\}) \cup N$ is an independent set of G and $|\mathcal{L}| > |\mathcal{I}|$, which leads to a contradiction. Thus $v \notin \mathcal{I}$ and we have the following:

$$\begin{cases} v \notin \mathcal{I} \text{ and } \mathcal{I} = \mathcal{I}\big|_{G-H} \cup N, \\ \alpha(G) = \left|\mathcal{I}\big|_{G-H}\right| + n. \end{cases} \tag{2}$$

<u>Subcase 1:</u> Suppose that all the vertices of Q are cut-vertices. Let $u \in Q \setminus \{v\}$ be a cut-vertex and $u \in \mathcal{I}$. Since $bi_G(u) = 2$, so let $B = K(R, S)$ be the neighbour of the block F via the cut-vertex u, where $R \cap Q = \{u\}$. Thus, $u \in \mathcal{I}$ and $u \in R$ implies that $\mathcal{I} \cap S = \emptyset$. Consider the bi-block graph G^* obtained from G by replacing the induced subgraph $F \circledcirc B$ with the complete bipartite graph $K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup S$ and $\widetilde{Q} = Q \cup R$. It is easy to see that $\mathcal{I}$ is an $\alpha(G^*)$-set and $G^* \in \mathcal{B}(\mathrm{k}, \alpha)$ consists of $b - 1$ blocks.

Subcase 2: Let $c \in Q$ and c is not a cut-vertex. Since $\mathcal{I} \cap P = \emptyset$, so $c \in \mathcal{I}$. Let A be the adjacency matrix of G and (ρ, X) be the eigen-pair corresponding to the spectral radius of A. Let x_u denote the entry of X corresponding to the vertex $u \in V$. Using $AX = \rho X$ we find a few identities as follows. For $m \geq 2$, let us denote

$$x_u = \begin{cases} b_n & \text{if } u \in N, \\ b_m & \text{if } u \in M, u \neq v. \end{cases} \tag{3}$$

Using $c \in Q$, c is not a cut-vertex and $AX = \rho X$, we have the following identities:

(J1) $\rho x_c = \sum_{w \in P} x_w$.

(J2) $\rho x_v = \sum_{w \in P} x_w + n b_n$.

(J3) $\rho b_n = (m-1) b_m + x_v$.

(J4) $\rho b_m = n b_n$.

Using indentities (J1), (J2) and (J4), we have $x_v = x_c + b_m$. Thus the identity (J3) reduces to:

(J3*) $\rho b_n = m b_m + x_c$.

Next, if $m = 1$, then by choosing $b_m = x_v - x_c$, all the above identities are true.

Subcase 2.1: Whenever $b_m \geq b_n$.
Let G^* be a bi-block graph obtained from G by replacing the induced subgraph $F \circledcirc H$ with the complete bipartite graph $K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup M$ and $\widetilde{Q} = (Q \setminus \{v\}) \cup N$. Thus, $\mathcal{I}$ is an $\alpha(G^*)$-set and $G^* \in \mathcal{B}(\mathrm{k}, \alpha)$ consists of $b-1$ blocks. Note that, we can obtain the graph G^* from G using the following operations:

1. Delete the edges between vertex v and the vertices of P.
2. Add edges between vertices of M and $Q \setminus \{v\}$.
3. Add edges between vertices of P and N.

Let A^* be the adjacency matrix of G^*. Using the above identities, we have

$$
\begin{aligned}
\frac{1}{2}X^t(A^*-A)X &= -x_v\sum_{w\in P}x_w + \sum_{\substack{u\sim w\\ u\in M, w\in Q\setminus\{v\}}} x_ux_w + \sum_{\substack{u\sim w\\ u\in P, w\in N}} x_ux_w \\
&= -(x_c+b_m)\sum_{w\in P}x_w + (mb_m+x_c)\sum_{w\in Q\setminus\{v\}}x_w + nb_n\sum_{w\in P}x_w && \text{[By Eq.(3)]} \\
&= -(x_c+b_m)\rho x_c + (mb_m+x_c)\sum_{w\in Q\setminus\{v\}}x_w + nb_n\rho x_c && \text{[Using (J1)]} \\
&= -(x_c+b_m)\rho x_c + (mb_m+x_c)\sum_{w\in Q\setminus\{v\}}x_w + \rho^2 b_m x_c && \text{[Using (J4)]} \\
&\geq -(x_c+mb_m)\rho x_c + (mb_m+x_c)\sum_{w\in Q\setminus\{v\}}x_w + \rho^2 b_m x_c \\
&= -\rho^2 b_n x_c + (mb_m+x_c)\sum_{w\in Q\setminus\{v\}}x_w + \rho^2 b_m x_c && \text{[Using (J3}^*\text{)]} \\
&= \rho^2(b_m-b_n)x_c + (mb_m+x_c)\sum_{w\in Q\setminus\{v\}}x_w.
\end{aligned}
$$

Since $b_m \geq b_n$, and X is the Perron vector of G, so $X^t(A^*-A)X \geq 0$. Thus, by Min-max theorem, we have $\rho(G) \leq \rho(G^*)$ and hence the induction hypothesis yields the result.

Subcase 2.2: Whenever $b_m < b_n$.
For this case we partition the set $N \subset \mathcal{I}$ as $N = N_1 \cup N_2$ and $N_1 \cap N_2 = \emptyset$ such that $|N_1| = m$ and $|N_2| = n - m$. We consider the complete bipartite graph $K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup N_1$ and $\widetilde{Q} = Q \cup M \cup N_2$. Let G^* be a bi-block graph obtained from G by replacing the induced subgraph $F \circledcirc H$ with $K(\widetilde{P}, \widetilde{Q})$. Thus, by Eq. (2), we obtain that $\mathcal{I}^* = \mathcal{I}\big|_{G-H} \cup M \cup N_2$ is an $\alpha(G^*)$-set and $\alpha(G^*) = \alpha(G) = \left|\mathcal{I}\big|_{G-H}\right| + n$, which implies that $G^* \in \mathcal{B}(\mathrm{k}, \alpha)$ consists of $b - 1$ blocks. Note that, we can obtain the graph G^* from G using the following operations:

1. Delete the edges between vertices of M and N_2.
2. Add edges between vertices of N_1 and $Q \setminus \{v\}$.
3. Add edges between vertices of P and N_2.
4. Add edges between vertices of N_1 and N_2.
5. Add edges between vertices of $M \setminus \{v\}$ and P.

Let A^* be the adjacency matrix of G^*. Then,

$$\frac{1}{2}X^t(A^*-A)X = -\sum_{\substack{u\sim w\\ u\in M, w\in N_2}} x_u x_w + \sum_{\substack{u\sim w\\ u\in N_1, w\in Q\setminus\{v\}}} x_u x_w + \sum_{\substack{u\sim w\\ u\in N_2, w\in P}} x_u x_w$$

$$+ \sum_{\substack{u\sim w\\ u\in N_1, w\in N_2}} x_u x_w + \sum_{\substack{u\sim w\\ u\in M\setminus\{v\}, w\in P}} x_u x_w$$

$$= -(n-m)(mb_m + x_c)b_n + mb_n \sum_{w\in Q\setminus\{v\}} x_w + (n-m)b_n \sum_{w\in P} x_w$$

$$+ (n-m)mb_n^2 + b_m(m-1)\sum_{w\in P} x_w \qquad \text{[By Eq.(3)]}$$

$$= -(n-m)mb_m b_n - (n-m)x_c b_n + mb_n \sum_{w\in Q\setminus\{v\}} x_w + \rho(n-m)b_n x_c$$

$$+ (n-m)mb_n^2 + \rho(m-1)b_m x_c \qquad \text{[Using (J1)]}$$

$$= (n-m)\left[mb_n(b_n-b_m) + (\rho-1)x_c b_n\right] + mb_n \sum_{w\in Q\setminus\{v\}} x_w + \rho(m-1)b_m x_c.$$

Since $b_m < b_n$ and $\rho \geq 1$ we are done.

Case 4: $\mathcal{I} \cap P \neq \emptyset$ and $\mathcal{I} \cap Q = \emptyset$. For $n \geq m$ or $m = n + 1$, we have $N \subset \mathcal{I}$. We consider graph G^* obtained from G by replacing the induced subgraph $F \circledcirc H$ with $K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup N$ and $\widetilde{Q} = Q \cup M$, which implies that $\mathcal{I}$ is an $\alpha(G^*)$-set. Thus, arguments similar to the Case 1 yields the result.

<u>Case 5:</u> $\mathcal{I} \cap P \neq \emptyset$ and $\mathcal{I} \cap Q = \emptyset$. For $m > n+1$, we have $(M \setminus \{v\}) \subset \mathcal{I}$. We consider all neighbouring blocks of $F = K(P, Q)$, say $B_i = K(R_i, S_i)$ for $1 \leq i \leq j$, connected via cut-vertices to the vertex partition P. Without loss of generality, we assume $S_i \cap P \neq \emptyset$. For any one of the such neighbour, if $\mathcal{I} \cap R_i = \emptyset$, then we consider the graph G^* which is obtained from G by replacing the induced subgraph $F \circledcirc B_i$ with $K(\widetilde{P}, \widetilde{Q})$, where $\widetilde{P} = P \cup S_i$ and $\widetilde{Q} = Q \cup R_i$. Since $\mathcal{I} \cap P \neq \emptyset$, so $\mathcal{I}$ is an $\alpha(G^*)$-set and argument similar to the Case 1 leads to the desired result. If no such neighbours exists, then proceeding inductively we need to look for B_i's neighbours with similar properties. Since G is a finite graph, either we will reach a neighbour with suitable properties or reach a leaf block does not satisfies requisite properties.

For the later case, we find a finite chain of blocks $C_i = K(M_i, N_i)$ for $1 \leq i \leq t$ satisfying the following:

1. $C_1 = H$ and C_t are leaf blocks.
2. For $i = 1, 2, \cdots, t-1$, the blocks C_i and C_{i+1} are neighbours such that $M_i \cap N_{i+1} \neq \emptyset$.
3. $\mathcal{I} \cap N_i = \emptyset$ for all $i = 1, 2, \ldots, t$.

Since C_t is a leaf block and is connected to C_{t-1} via a cut-vertex $u(say)$ with $bi_G(u) = 2$, so it can be seen $\mathcal{I} \cap N_{t-1} = \emptyset$ and $\mathcal{I} \cap N_t = \emptyset$ implies that $|M_t| > |N_t|$. Now, if we begin with the leaf block C_t, then this case is analogous to the Case 3. Hence the desired result follows.

Lemma

If $G \in \mathcal{B}(\mathrm{k}, \alpha)$, then there exists a bi-block graph $G^* \in \mathcal{B}(\mathrm{k}, \alpha)$ with $bi_{G^*}(u) = 2$ for all cut-vertex u in G^* such that $\rho(G) \leq \rho(G^*)$.

Proof of the Lemma

Proof: Let v be a cut-vertex of G with $bi_G(v) = t$, where $t \geq 3$. Let $B_i = K(M_i, N_i)$; $i = 1, 2, 3$ be any three neighbours connected via the cut-vertex v such that $v \in N_1 \cap N_2 \cap N_3$. Let $\mathcal{I}$ be an $\alpha(G)$-set. If $V(B_i) \cap \mathcal{I} \neq \emptyset$ for all $i = 1, 2, 3$, then either $M_i \cap \mathcal{I} \neq \emptyset$ or $N_i \cap \mathcal{I} \neq \emptyset$. Thus by pigeonhole principle, there exist $i, j \in \{1, 2, 3\}$ such that either $\mathcal{I} \cap N_i = \emptyset$ and $\mathcal{I} \cap N_j = \emptyset$ or $\mathcal{I} \cap M_i = \emptyset$ and $\mathcal{I} \cap M_j = \emptyset$. Let us consider a bi-block graph G^* obtained from G by replacing the induced subgraph $B_i \circledcirc B_j$ with $K(\widetilde{M}, \widetilde{N})$, where $\widetilde{M} = M_i \cup M_j$ and $\widetilde{N} = N_i \cup N_j$. It is easy to see that, $\mathcal{I}$ is an $\alpha(G^*)$-set and $bi_{G^*}(v) = t - 1$ and we have $\rho(G) \leq \rho(G^*)$. Hence proceeding inductively the result follows. If $V(B_{i_0}) \cap \mathcal{I} = \emptyset$ (*i.e.* $M_{i_0} \cap \mathcal{I} = \emptyset$ and $N_{i_0} \cap \mathcal{I} = \emptyset$) for some $i_0 \in \{1, 2, 3\}$, then for $j \neq i_0$ and choosing $K(\widetilde{M}, \widetilde{N})$, where $\widetilde{M} = M_{i_0} \cup M_j$ and $\widetilde{N} = N_{i_0} \cup N_j$, similar argument yields the desired result.

Main Result

Theorem

If $G \in \mathcal{B}(\mathsf{k}, \alpha)$, then $\rho(G) \leq \rho(K_{\alpha,\mathsf{k}-\alpha})$ and equality holds if and only if $G = K_{\alpha,\mathsf{k}-\alpha}$.

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The End

Thank you.