Distance Equienergetic graphs

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23rd October, 2020

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- 2008 Distance energy - - G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. 60 (2008), 461 - 472.

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- 2008 Distance energy - - G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. 60 (2008), 461 - 472.
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Basic Concepts

- **Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, \ldots, v_p\}$ and size q.**

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- **The distance matrix D of G is $D = [d_{ij}]$ where d_{ij} is the distance between v_i and v_j in G.**

- **Let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, \ldots, v_p\}$ and size q.**
- **The distance matrix D of G is $D = [d_{ij}]$ where d_{ij} is the distance between v_i and v_j in G.**
- **The eigenvalues of $D-$ are the $D-$ eigenvalues of G and form its $D-$ spectrum , $spec_D(G)$.**

Definitions

$D-$cospectral

Two graphs G and H with $spec_D(G) = spec_D(H)$ are $D-$cospectral.

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Definitions

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Two graphs G and H with $spec_D(G) = spec_D(H)$ are $D-$cospectral.

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The $D-$ energy $E_D(G)$ of G is defined as $E_D(G) = \sum_{i=1}^{p} |\mu_i|$.

$D-$ equi energetic

Two non $D-$ cospectral graphs with the same $D-$ energy are $D-$ equi energetic.

$D-$ spectrum of the join

Theorem

For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues of the adjacency matrix A_{G_i},
$\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. The distance spectrum of $G_1 \nabla G_2$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$ and two more eigenvalues of the form

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{\left(n_1 - n_2 - \frac{r_1 - r_2}{2}\right)^2 + n_1 n_2}. \quad (1)$$

D.Stevanovic, G.Indulal, *The distance spectrum and energy of the compositions of regular graphs*, Appl. Math lett., 2009, 136 - 140

Proof.

The distance matrix D of the join $G_1 \nabla G_2$ has the form

$$D = \begin{bmatrix} 2(J-I) - A_{G_1} & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & 2(J-I) - A_{G_2} \end{bmatrix}.$$

□

Equienergetic graphs from Join

- $D_E(K_{m,n}) = 4(m+n-2)\,,\, m, n \geq 2$

Equienergetic graphs from Join

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- $\left\{K_{2,n-2}, K_{3,n-3}, \ldots.., K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right\}, n \geq 4$

Equienergetic graphs from Join

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Theorem

Let n be a given positive integer ≥ 6. Let $\{p_1, p_2\}$ and $\{q_1, q_2\}$ be two partitions of n into two parts each of size atleast 2. Then K_{p_1,p_2} and K_{q_1,q_2} are $D-$ equienergetic.

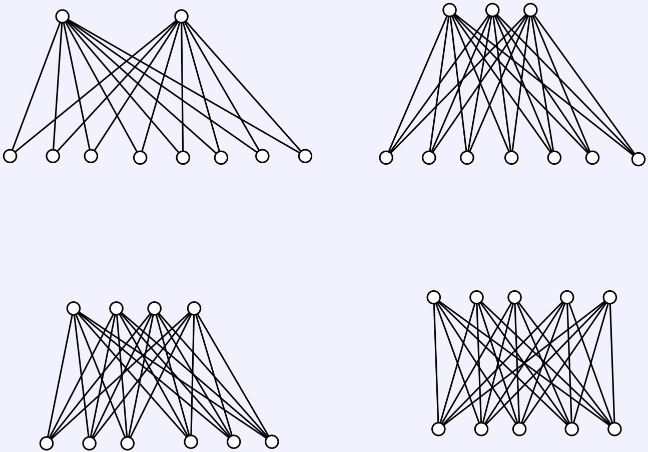


Figure: $D-$ equienergetic complete bipartite graphs on 10 vertices.

Equienergetic graphs from Join

- $D_E(K_{\frac{m}{2},\frac{m}{2}} \nabla K_{\frac{n}{2},\frac{n}{2}}) = 4(m+n-4)\,, m, n \geq 4$

Equienergetic graphs from Join

- $D_E(K_{\frac{m}{2},\frac{m}{2}} \nabla K_{\frac{n}{2},\frac{n}{2}}) = 4(m+n-4)$, $m, n \geq 4$
- $\left\{K_{2,2} \nabla K_{\frac{n-4}{2},\frac{n-4}{2}}, K_{3,3} \nabla K_{\frac{n-6}{2},\frac{n-6}{2}}, \ldots., K_{\frac{n}{4},\frac{n}{4}} \nabla K_{\frac{n}{4},\frac{n}{4}}\right\}$, $n = 4t \geq 12$

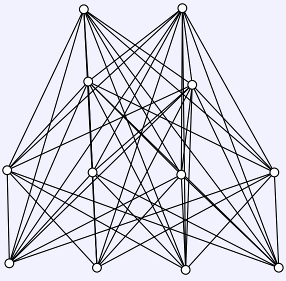
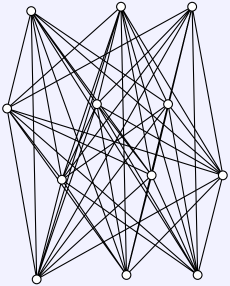



Figure: $D-$ equienergetic complete multi-partite graphs on 12 vertices.

Equienergetic graphs from Join

- $D_E(C_m \nabla \overline{K_n}) = 4(m+n-3)\,, m, n \geq 3$
- $\left\{ C_3 \nabla \overline{\mathrm{K}_{n-3}}, C_4 \nabla \overline{\mathrm{K}_{n-4}}, ..., C_{n-3} \nabla \overline{\mathrm{K}_3} \right\}, n \geq 7$

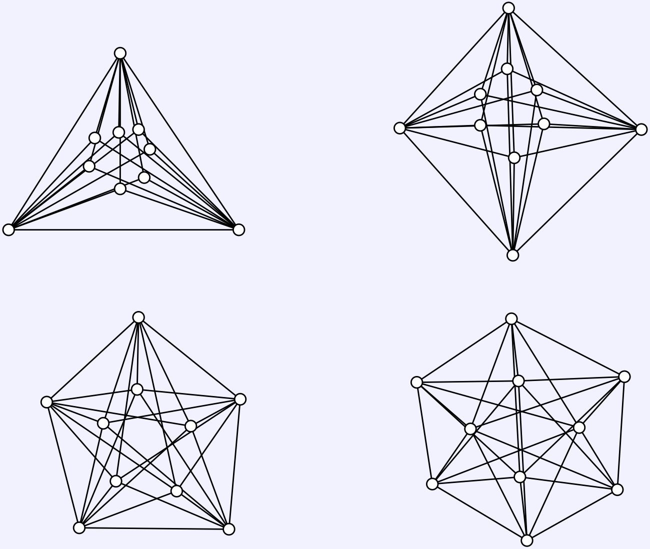


Figure: $D-$ equienergetic on 10 vertices.

Equienergetic graphs from Join

- $D_E(C_m \nabla K_{\frac{n}{2},\frac{n}{2}}) = 4(m+n-4)$, $m, n \geq 4$
-
$$\{C_3 \nabla K_{t-1,t-1}, C_5 \nabla K_{t-2,t-2}, ..., C_{2t-3} \nabla K_{2,2}\}, n = 2t+1 \geq 9$$
$$\{C_4 \nabla K_{t-2,t-2}, C_6 \nabla K_{t-3,t-3}, ..., C_{2t-4} \nabla K_{2,2}\}, n = 2t \geq 10$$

Equienergetic graphs from Join

- $D_E(C_m \nabla C_n) = 4(m+n-4)\,, m, n \geq 4$
- $\{C_3 \nabla C_{n-3}, C_4 \nabla C_{n-4}, ..., C_k \nabla C_{n-k}\}\,, n \geq 8, k = 1, 2, 3.. \left\lfloor \frac{n}{2} \right\rfloor$

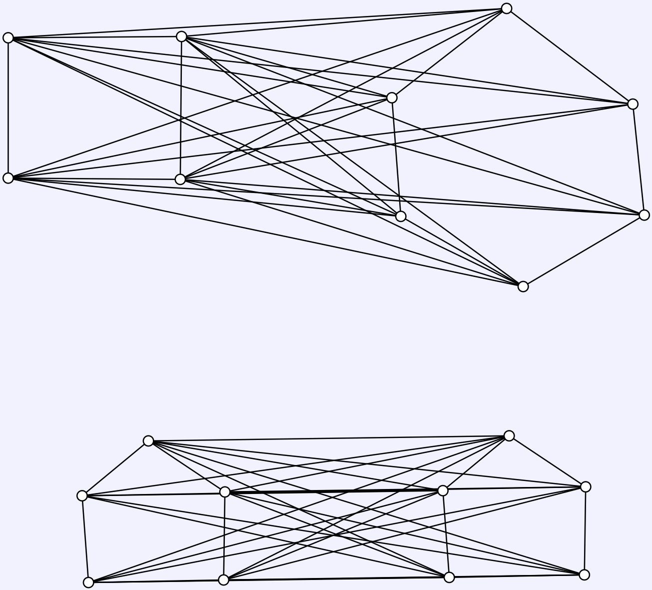


Figure: $D-$ equienergetic on 10 vertices.

Equienergetic graphs from Join

Theorem

For every $p \equiv 0(mod6) \geq 18$, *there exists a pair of equi* $D-$ *energetic regular graphs.*

G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **60** (2008),461-472

Proof.

- Let $p = 6t, \ t \geq 3$. Let G_1 and G_2 be the non cospectral cubic graphs on $2t$ vertices .
- Then their line graphs $L(G_1)$ and $L(G_2)$ are $4-$ regular on $3t$ vertices
- The only positive $D-$ eigenvalues of $L(G_1)\nabla L(G_1)$ are $9t-6$ and $3t-6$. Similarly for $L(G_2)\nabla L(G_2)$ also.
- Thus $E_D(L(G_1)\nabla L(G_1)) = E_D(L(G_2)\nabla L(G_2)) = 24(t-1)$. Also both are on $6t$ vertices. Hence the theorem.

□

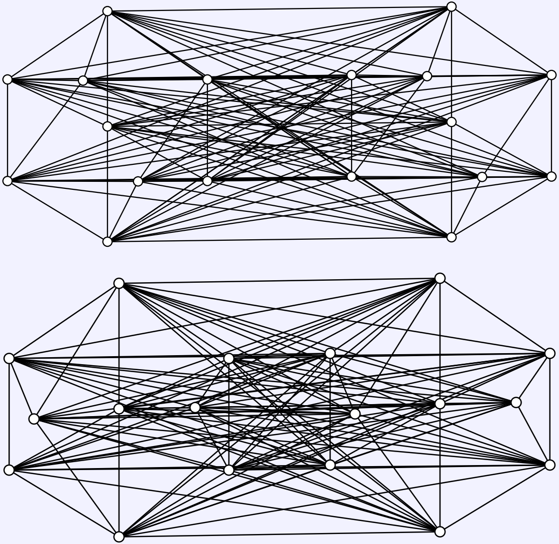


Figure: Equi $D-$ energetic graphs on 18 vertices with energy 48.

Equienergetic graphs from Join

Theorem

For every $p \equiv 1 (mod 3) \geq 10$*, there exists a pair of equi* $D-$ *energetic graphs.*

G. Indulal, I. Gutman, A. Vijayakumar, *On distance energy of graphs*, MATCH Commun. Math. Comput. Chem. **60** (2008),461-472

$D-$ spectrum of the join over union

Theorem

For $i = 0, 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$ of the adjacency matrix A_{G_i}. If $r_1 \neq r_2$, then the distance spectrum of $G_0 \nabla (G_1 \cup G_2)$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 0, 1, 2$ and $j = 2, 3, \ldots, n_i$ and three more eigenvalues which are solutions of the cubic equation in ν:

$$(2n_0 - r_0 - 2 - \nu)(\nu + r_1 + 2)(\nu + r_2 + 2)$$
$$+[2(\nu + r_0 + 2) - 3n_0][n_1(\nu + r_2 + 2) + n_2(\nu + r_1 + 2)] = 0.$$

D.Stevanovic, G.Indulal, *The distance spectrum and energy of the compositions of regular graphs*, Appl. Math lett., 2009, 136 - 140

$D-$ Equienergetic graphs from the join over union

Theorem

Let G be any regular graph with least eigenvalue atleast -2. For a fixed n let $\mathscr{P}_n$ be the set of all integer partitions on n into parts, each of size atleast 3. For $p = \{p_1, p_2, ..., p_k\} \in \mathscr{P}_n$ let $\mathbf{C}_p$ be the union of cycles with vertices $p_1, p_2, ..., p_k$. Then the graphs $K_1 \nabla (\mathbf{C}_p \cup G)$, $p \in \mathscr{P}_n$ forms a set of $D-$ equienergetic graphs

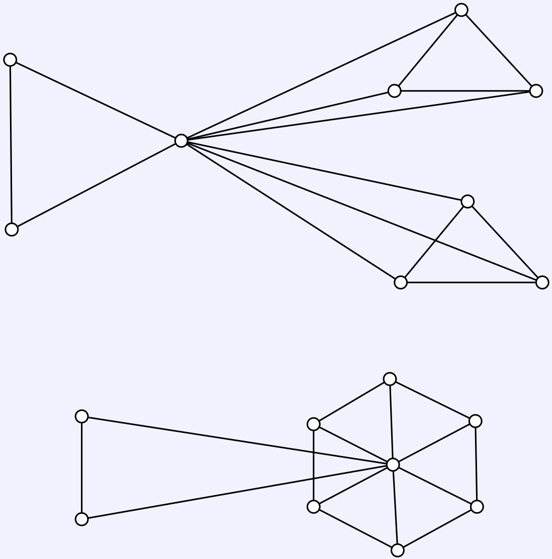


Figure: $D-$ equienergetic on 9 vertices. n=6

Joined Union

Definition

Let G be a graph on n vertices $\{v_1, v_2, ..., v_n\}$. Now corresponding to each vertex v_i introduce a graph G_i and make every vertex of G_i adjacent to every vertex of G_j whenever v_i and v_j are adjacent in G. The resulting graph denoted by $G[G_1, G_2, ..., G_n]$ is the ***joined union of constituent graphs****. The ordinary join of two graphs G and H is $K_2[G, H]$*

$D-$ spectrum of the joined union

Theorem

The distance spectrum of the joined union $G[G_1, G_2, ..., G_n]$ consists of $-\lambda_{i,j} - 2$ for $i = 1, 2, ..., n$ and $j = 2, 3, ..., m_i$ and the eigenvalues of

$$\begin{bmatrix} 2m_1 - r_1 - 2 & d_G(v_1, v_2)m_2 & d_G(v_1, v_3)m_3 & \dots & d_G(v_1, v_n)m_n \\ d_G(v_2, v_1)m_1 & 2m_2 - r_2 - 2 & d_G(v_2, v_3)m_3 & \dots & d_G(v_2, v_n)m_n \\ . & . & . & . & . \\ . & . & . & . & . \\ d_G(v_n, v_1)m_1 & . & . & . & 2m_n - r_n - 2 \end{bmatrix}$$

D.Stevanovic,Large sets of long distance equienergetic graphs,ARS MATHEMATICA CONTEMPORANEA 2 (2009) 35–40

$D-$ spectrum of the joined union

Long equidistance energetic graphs

- $P_n\left[C_6, C_6, C_6, ..., C_6\right]$
- $P_n\left[C_6, C_6, C_6, ..., C_3 \cup C_3\right]$
- $P_n\left[C_6, C_6, C_6, .., C_3 \cup C_3, C_3 \cup C_3\right]$
- $P_n\left[C_3 \cup C_3, C_3 \cup C_3, ..., C_3 \cup C_3, C_3 \cup C_3\right]$

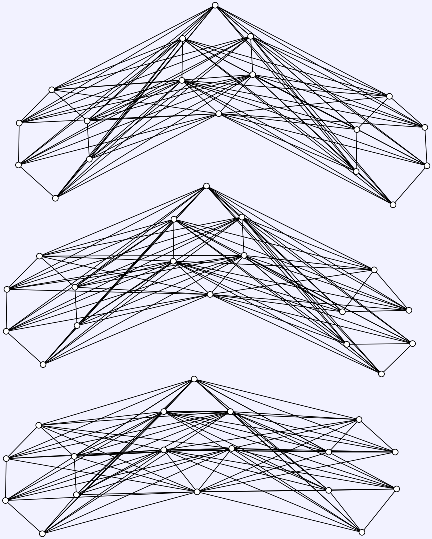
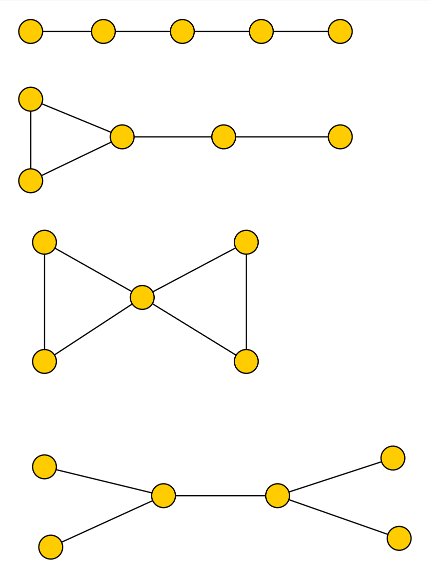


Figure: $D-$ equienergetic $P_3[C_6, C_6, C_6], P_3[C_6, C_6, C_3 \cup C_3]$ and $P_3[C_6, C_3 \cup C_3, C_3 \cup C_3]$



$D-$ Equienergetic line graphs

Theorem

Let G be any $r-$ regular graph on n vertices of diameter ≤ 2 in which none of the above graphs is an induced subgraph. Then none of these graphs can be an induced subgraph of $L(G)$, diameter of $L(G) \leq 2$ and $E_D(L(G)) = 2nr - 4r$

H. S. Ramane, D. S. Revankar, I. Gutman, and H. B. Walikar,DISTANCE SPECTRA AND DISTANCE ENERGIES OF ITERATED LINE GRAPHS OF REGULAR GRAPHS,PUBLICATIONS DE L'INSTITUT MATHEMATIQUE Nouvelle serie, tome 85(99) (2009), 39–46

$D-$ Equienergetic line graphs

Theorem

Let G_1 and G_2 be any two non co spectral $r-$ regular graph on n vertices of diameter ≤ 2 in which none of the above graphs is an induced subgraph. Then the iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ are $D-$ equienergetic with energy

$$4n\prod_{i=0}^{k-1}\left(2^{i-1}r-2^{i}+1\right)-2\left(2^{k}r-2^{k+1}+4\right)$$

Theorem

Let G be an $r-$ regular graph on p vertices. If $r \leq \frac{p-1}{2}$, then $d(\overline{G}) = 2$.

Proof.

Let u and v be two vertices of G. If u not adjacent to v in G, then $d(u, v) = 1$ in $\overline{G}$.
Now let u adjacent to v in G. Then u and v are collectively adjacent to atmost $2r - 2$ vertices other than themselves. Hence they are collectively adjacent to atmost $p - 3$ vertices. Therefore always there exists a vertex w in G not adjacent to u and v. Thus in $\overline{G}$, w is adjacent to both u and v and hence $d(u, v) = 2$. □

Theorem

Let G be an $r-$ *regular graph on* $p \geq 8$ *vertices with* $r \leq \dfrac{p-1}{2}$.
Then $d(\overline{L^k(G)}) = 2$ *for all* $k \geq 1$.

Theorem

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Theorem

Let G be an $r \geq 4$ *regular graph on* $p \geq 8$ *vertices with with* $r \leq \dfrac{p-1}{2}$. *Then* $E_D(\overline{L^2(G)}) = 3pr(r-2)$.

Theorem

Let G be an $r-$ *regular graph on p vertices with with* $r \leq \dfrac{p-1}{2}$. *Let* p_k *and* r_k *denote the order and degree of its* k^{th} *iterated line graph* $L^k(G)$. *Then*

$$E_D(\overline{L^k(G)}) = 3p(r-2)(2^{k-2}r - 2^{k-1} + 2)\prod_{i=0}^{k-3}(2^i r - 2^{i+1} + 2)$$

Indulal G, D-SPECTRUM AND D-ENERGY OF COMPLEMENTS OF ITERATED LINE GRAPHS OF REGULAR GRAPHS,Algebraic Structures and Their Applications Vol. 4 No. 1 (2017) pp 53-58.

Theorem

Let G_1 and G_2 be two non-cospectral regular graphs of the same order $p \geq 9$ and of the same degree $r \geq 4$ with $2r \leq p-1$. Then for any $k \geq 2$, $r \geq 4$, the graphs $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ form a pair of non $D-$cospectral, $D-$ equienergetic graphs of equal order and of equal number of edges with energy

$$3p(r-2)(2^{k-2}r - 2^{k-1} + 2) \prod_{i=0}^{k-3} \left(2^i r - 2^{i+1} + 2\right) .$$

Theorem

Let G and H be two distance regular graphs on p and n vertices with distance regularity k and t respectively. Let $spec_D(G) = \{k, \mu_2, \mu_3, ..., \mu_p\}$ *and* $spec_D(H) = \{t, \eta_2, \eta_3, ..., \eta_n\}$*. Then distance spectrum of their cartesian product*

$$spec_D(G + H) = \{nk + pt, n\mu_i, p\eta_j, 0\}$$

$i = 2, ..., p$, $j = 2, ..., n$ *and* 0 *is with multiplicity* $(p-1)(n-1)$.

Indulal G, Distance spectrum of graph compositions,ARS MATHEMATICA CONTEMPORANEA 2 (2009) 93Ű100

Theorem

$$E_D(G + H) = nE_D(G) + pE_D(H).$$

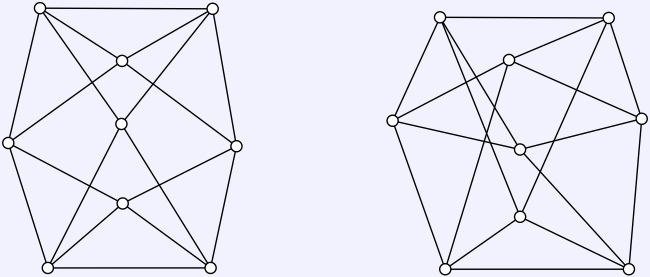

Figure: Pair of $D-$ equienergetic graphs on 9 vertices.

Theorem

Let G_1 and G_2 be two distance regular $D-$ equienergetic graphs. Then $G_1 + H$ and $G_2 + H$ are $D-$ equienergetic.

Theorem

Let G be a distance regular graph of order p , diameter k and $G_n = G_{n-1} + K_2$ with $G_0 = G$, $n = 1, 2, ..., ...$ Then $d(G_n) = k + n$ and $E_D(G_n) = 2^n \left(E_D(G) + np\right)$

Proof.

Let G be a distance regular graph of order p and diameter d. Let $G_n = G_{n-1} + K_2$ with $G_0 = G$. Then $d(G_n) = d + n$ by its repeated application.

Proof.

Let G be a distance regular graph of order p and diameter d. Let $G_n = G_{n-1} + K_2$ with $G_0 = G$. Then $d(G_n) = d + n$ by its repeated application.

Now as G is a graph of order p, we have G_n is a graph of order $2^n p$. Now using $E_D(K_2) = 2$ it follows that
$E_D(G_n) = 2E_D(G_{n-1}) + 2^n p$.

Proof.

Let G be a distance regular graph of order p and diameter d. Let $G_n = G_{n-1} + K_2$ with $G_0 = G$. Then $d(G_n) = d + n$ by its repeated application.
Now as G is a graph of order p , we have G_n is a graph of order $2^n p$. Now using $E_D(K_2) = 2$ it follows that
$E_D(G_n) = 2E_D(G_{n-1}) + 2^n p$.
Thus we get the recurrence relation

$$a_n = 2a_{n-1} + 2^n p \tag{2}$$

where $a_n = E_D(G_n)$.

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where $a_n = E_D(G_n)$.Solving the general solution to (**??**) is

$a_n = a_n^h + a_n^p = A2^n + pn2^n$.

Proof.

Let G be a distance regular graph of order p and diameter d. Let $G_n = G_{n-1} + K_2$ with $G_0 = G$. Then $d(G_n) = d + n$ by its repeated application.
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Thus we get the recurrence relation

$$a_n = 2a_{n-1} + 2^n p \qquad (2)$$

where $a_n = E_D(G_n)$.Solving the general solution to (**??**) is $a_n = a_n^h + a_n^p = A2^n + pn2^n$.Now using the initial condition that $a_1 = E_D(G_1) = E_D(G + K_2) = 2E_D(G) + 2p$, we get $2E_D(G) + 2p = 2A + 2p$ and hence $A = E_D(G)$.
Thus $E_D(G_n) = a_n = E_D(G)2^n + pn2^n = 2^n(E_D(G) + np)$.

□

Pair of $D-$ equienergetic graphs of diameter k for any $k \geq 2$

Theorem

Let k be any positive integer greater than or equal to 2. Then there exists a pair of $D-$ equienergetic graphs of diameter k.

Lexicographic product of G with H

Let G be a graph with $D-$ matrix D_G and H , an $r-$ regular graph with an adjacency matrix A. Let $spec_D(G) = \{\mu_1, \mu_2, ..., \mu_p\}$ and the ordinary spectrum of H be $\{r, \lambda_2, \lambda_3, ..., \lambda_n\}$. Then

$$spec_D G[H] = \begin{pmatrix} n\mu_i + 2n - r - 2 & -(\lambda_j + 2) \\ 1 & p \end{pmatrix}, \ i = 1 \text{ to } p \text{ and } j = 2 \text{ to } n$$

Indulal G, Distance spectrum of graph compositions,ARS MATHEMATICA CONTEMPORANEA 2 (2009) 93–100

$D-$ Equienergetic graphs from the lexicographic product of graphs

Theorem

Let G be any graph . For a fixed n let $\mathscr{P}_n$ be the set of all integer partitions on n into parts, each of size atleast 3. For $p = \{p_1, p_2, ..., p_k\} \in \mathscr{P}_n$ let $\mathbf{C}_p$ be the union of cycles with vertices $p_1, p_2, ..., p_k$. Then the graphs $G[\mathbf{C}_p]$, $p \in \mathscr{P}_n$, the lexicographic product of G with $\mathbf{C_p}$ forms a set of $D-$ equienergetic graphs

Definition

Let G be a graph on the vertex set $\{v_1, v_2, \ldots, v_p\}$. Define a bipartite graph H with $V(H) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_p\}$ in which v_i is adjacent to u_i for each $i = 1, 2, \ldots, p$ and v_i is adjacent to u_j if v_i is adjacent to v_j in G. The graph H is known as the extended double cover graph (EDC-graph) of G.

Theorem

Let G be an r-regular graph of diameter 2 on p vertices with (ordinary) spectrum $\{r, \lambda_2, \ldots, \lambda_p\}$. Then the D-spectrum of the EDC-graph of G consists of the numbers $5p - 2r - 4$, $2r - p$, $-2(\lambda_i + 2)$, $i = 2, 3, \ldots, p$, and $2\lambda_i$, $i = 2, 3, \ldots, p$.

Theorem

There exists a pair of regular non-D-cospectral D-equienergetic bipartite graphs on $24t$ vertices, for each $t \geq 3$.

Indulal G , I.Gutman,On the distance spectra of some graphs,Mathematical Communications 13(2008), 123-131

Proof.

Let G be a cubic graph on $2t$ vertices, $t \geq 3$. Consider $L^2(G)$, Then for $F = L^2(G) \nabla L^2(G)$, the D-spectrum of $EDC(F)$ is

$$\begin{pmatrix} 16\,(3t-1) & 12 & 0 & 2\,(\lambda_i+3) & 12\,t-16 & -4 & -12\,(t-1) & -2\,(\lambda_i+5) \\ 1 & 1 & 8t & 2 & 1 & 8t & 1 & 2 \end{pmatrix},\ i = 2, 3, \ldots, 2t\,.$$

Thus $E_D(EDC(F)) = 8\,(21\,t - 11)$.

Now let G_1 and G_2 be the two non-cospectral cubic graphs on $2t$ vertices . Further, let H_1 and H_2 be the EDC-graphs of $L^2(G_1) \nabla L^2(G_1)$ and $L^2(G_2) \nabla L^2(G_2)$, respectively. Then H_1 and H_2 are bipartite and $E_D(H_1) = E_D(H_2) = 8\,(21\,t - 11)$, proving the theorem. $\square$

Let G be a graph. Then the following construction [A. Farrugia] results in a self-complementary graph $\mathcal{H}$.

Construction:

$\mathcal{H}$: Replace each of the end vertices of P_4, the path on 4 vertices by a copy of G and each of the internal vertices by a copy of $\overline{G}$. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of P_4 are adjacent. Simply

$\mathcal{H} = P_4[G, \overline{G}, \overline{G}, G]$

Theorem

Let G be a connected $k-$ *regular graph on n vertices, with an adjacency matrix A and spectrum* $\{k, \lambda_2, \ldots, \ldots, \lambda_n\}$*. Then the distance spectrum of* $\mathcal{H}$ *consists of* $-(\lambda_i + 2)$ *and* $\lambda_i - 1$ *,* $i = 2, 3, ..., ..., n$*, each with multiplicity 2 together with the numbers*

$$\frac{1}{2}\left[7n - 3 \pm \sqrt{(2k+1)^2 + 45n^2 - 12nk - 6n}\right]$$

and

$$-\frac{1}{2}\left[n + 3 \pm \sqrt{(2k+1)^2 + 5n^2 + 4nk + 2n}\right]$$

Indulal G, I.Gutman , D-EQUIENERGETIC SELF-COMPLEMENTARY GRAPHS,Kragujevac J. Math. 32 (2009) 123-131.

Theorem

For every $n \geq 8$ *, there exists a pair of* $4-$ *regular non-cospectral graphs on* n *vertices.*

Case 1: $n = 2t\,, t \geq 4$

In this case form the two $t-$ cycles $u_1 u_2 \ldots\ldots u_t$ and $v_1 v_2 \ldots\ldots\ldots v_t$ and join u_i to v_i for each i. Let $\mathcal{A}$ be the resultant graph. Let $\mathcal{B}_1$ be the graph obtained from $\mathcal{A}$ by making u_i adjacent with v_{i+1} for each i and $\mathcal{B}_2$ be that obtained by making u_i adjacent with v_{i+2} for each i where suffix addition is modulo t. Then both $\mathcal{B}_1$ and $\mathcal{B}_2$ are $4-$ regular and the number of triangles in $\mathcal{B}_1$ is $2t$ and that in $\mathcal{B}_2$ is zero. Thus $\mathcal{B}_1$ and $\mathcal{B}_2$ are non- cospectral.

Illustration with $t = 4$

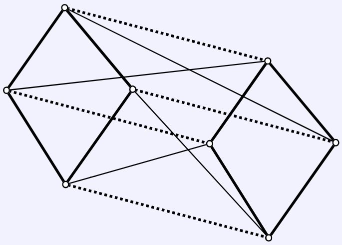


Figure: The graphs $\mathcal{B}_1$ and $\mathcal{B}_2$

Case 2: $n = 2t + 1\,, t \geq 4$

In this case form the $t+1$ cycle $v_1 v_2 ...... v_t v_{t+1}$ and the t cycle $u_1 u_2 .... u_t$. Now make v_{t-1} adjacent with v_1 and v_i to u_i, $i = 1$ to t. Then join v_j to u_{j+2}, $j = 2$ to $t-2$, v_t to u_2 and then v_{t+1} to u_1 and u_3. Let $\mathcal{F}_1$ be the resultant graph. Then $\mathcal{F}_1$ is $4-$ regular and contain two triangles $v_1 v_2 v_3$ and $v_5 u_1 v_1$ for $t = 4$ and only one triangle $v_{t+1} u_1 v_1$ for $t \geq 5$.

To get the other $4-$ regular graph, form the $2t+1$ cycle $v_1 v_2 ...... v_t v_{t+1} ..... v_{2t+1}$. Join v_i to $v_{i+2}\,, i = 1, 3, 5, ..., 2t+1, 2, 4, 6, ..., 2t$. Let $\mathcal{F}_2$ be the resultant graph. Then it is $4-$ regular and contain $2t+1$ triangles. Thus $\mathcal{F}_1$ and $\mathcal{F}_2$ are non cospectral.

Illustration with $t = 4$

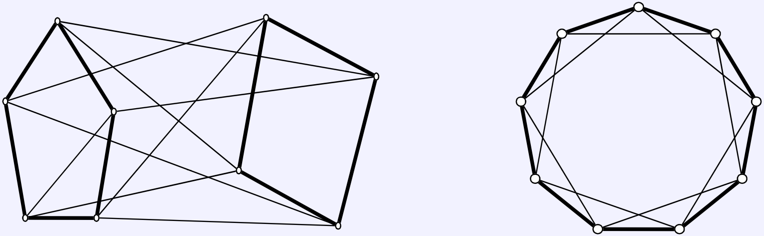


Figure: The graphs $\mathcal{F}_1$ and $\mathcal{F}_2$

Theorem

Let G be a connected $4-$ regular graph on n vertices, with an adjacency matrix A and spectrum $\{4, \lambda_2, \ldots, \ldots, \lambda_n\}$.
Let $H = L^2(G)$ and $\mathcal{H}$ be the P_4 self-complementary graph obtained from H.Then $E_D(\mathcal{H}) = 3[8(3n-1) + \sqrt{20n^2 + 28n + 49}]$

Theorem

For every $p = 48t$ *or* $24(2t+1)$, $t \geq 4$, *there exists a pair of* $D-$ *equienergetic self-complementary graphs.*

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two non-cospectral $4-$ regular graphs on $2t$ vertices . Let $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively denote their second iterated line graphs. Then both are on $12t$ vertices and are $6-$ regular. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be the respective self-complementary graphs on $48t$ vertices. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are $D-$ equienergetic

Let G_1 and G_2 be two graphs with $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$. Now add the edges $v_i v_i^{'}$ in the two disjoint copies of $G_1 \bigtriangledown G_2$ where primes denote vertices in the second copy of G_2 for $i = 1, 2, \ldots, n_2$. We denote this graph by $G_1 \bigtriangledown G_2 \equiv G_2 \bigtriangledown G_1$.

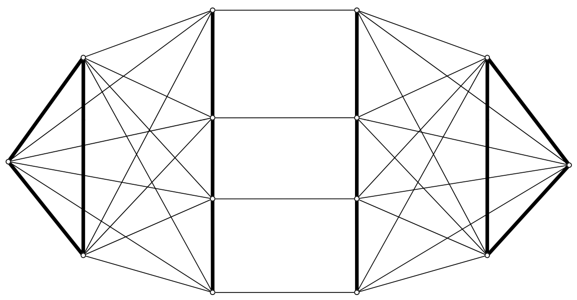


Figure: The graph $K_3 \blacktriangledown P_4$

Theorem

For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues of the adjacency matrix A_{G_i},
$\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,3} \geq \cdots \geq \lambda_{i,n_i}$. *Then the $D-$ spectrum of $G_1 \bigtriangledown G_2 \equiv G_2 \bigtriangledown G_1$ consists of the eigenvalues $-(\lambda_{1,j} + 2)$ for $j = 2, 3, \ldots, n_1$, each with multiplicity 2, the eigenvalues $-2(\lambda_{2,j} + 2)$ for $j = 2, 3, \ldots, n_2$ and 0 with multiplicity $2(n_2 - 1)$ together with four more eigenvalues which are roots of the biquadratic equation:*

$$\left[v^2 + (6 - 5(n_1 + n_2) + r_1 + 2r_2)\, v + (2r_1 r_2 + 4(r_1 + r_2) - 10n_1 r_2 - 5n_2 r_1 - 10(2n_1 + n_2) + 16n_1 n_2 + 8)\right]$$
$$\times \quad \left[v^2 + (2 + n_1 + n_2 + r_1)\, v + n_2(r_1 + 2)\right] = 0$$

Theorem

There exists a pair of distance equienergetic graphs of diameter 3 *on* p *vertices for every* $p = 18 + 2k$, $k \geq 1$.

G.Indulal, R.Balakrishnan,Distance spectrum of Indu–Bala product of graphs,AKCE International Journal of Graphs and Combinatorics 13 (2016) 230–234

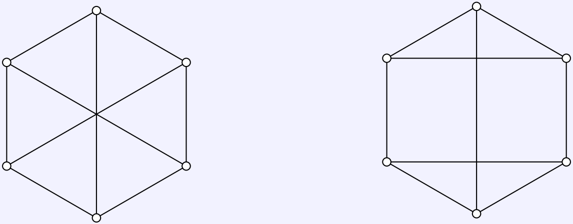


Figure: The graphs H_1 and H_2

Let G_1 and G_2 denote respectively their line graphs. Let $F_1^k = \overline{K_k} \blacktriangledown G_1$ and $F_2^k = \overline{K_k} \blacktriangledown G_2$. Then each of the graphs F_1^k and F_2^k has $p = 18 + 2k$ vertices. By previous Theorem $E_D(F_1^k) = E_D(F_2^k) = 2(31 + 5k)$. Moreover, as H_1 and H_2 are non-isomorphic [?], so are the pairs $G_1, \ G_2$ and $F_1^k, \ F_2^k$.

Theorem

Let G be a distance regular graph on p vertices $\{v_1, v_2, ..., v_p\}$ with distance regularity k, a distance matrix D and $spec_D = \{k = \mu_1, \mu_2, ..., \mu_p\}$. Let H be an $r-$ regular graph on n vertices with an adjacency matrix A and $spec_A = \{r = \lambda_1, \lambda_2, ..., \lambda_n\}$. Then the distance spectrum of $G \circ H$ consists of the following numbers:

(a) $\dfrac{n(2p+k)+k-r-2 \pm \sqrt{(n(2p+k)+k-r-2)^2 + 4(np^2 + k(r+2))}}{2}$ *each with multiplicity* 1

(b) $\dfrac{\mu_i(n+1)-r-2 \pm \sqrt{(\mu_i(n+1)-r-2)^2 + 4\mu_i(r+2)}}{2}$ *for each* $\mu_i \in spec_D$, $i = 2, 3, ..., p$

(c) $-\lambda_i - 2$ *with multiplicity* p *for each* $\lambda_i \in spec_A(H)$, $i = 2, 3, ..., n$.

Indulal G,Dragan Stevanovic,The distance spectrum of corona and cluster of two graphs,AKCE International Journal of Graphs and Combinatorics 12 (2015) 186Ű192

Theorem

Let G be a distance regular graph. For a fixed n let $\mathscr{P}_n$ be the set of all integer partitions on n into parts, each of size atleast 3. For $p = \{p_1, p_2, ..., p_k\} \in \mathscr{P}_n$ let $\mathbf{C}_p$ be the union of cycles with vertices $p_1, p_2, ..., p_k$. Then the graphs $G \circ \mathbf{C}_p$, $p \in \mathscr{P}_n$ forms a set of $D-$ equienergetic graphs

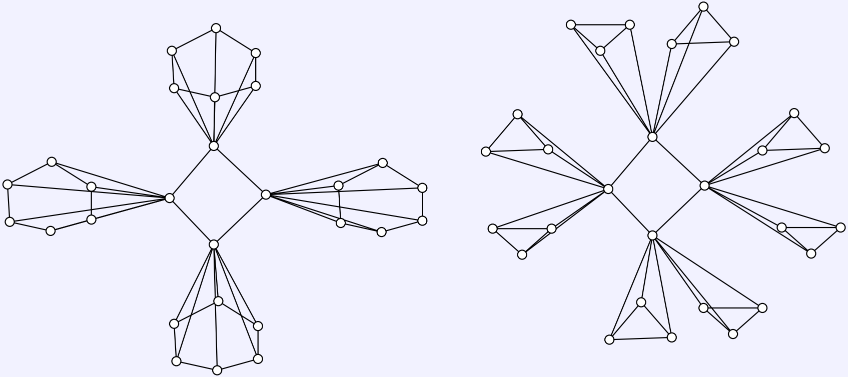


Figure: $D-$ equienergetic graphs on 28 vertices.

Definition

The subdivision-vertex join of two vertex disjoint graphs G_1 *and* G_2*, denoted by* $G_1 \dot{\vee} G_2$*, is the graph obtained from* $S(G_1)$ *and* G_2 *by joining each vertex of* $V(G_1)$ *with every vertex of* $V(G_2)$*.*

Theorem

Let G_i be an r_i regular graph on p_i vertices and q_i edges with an adjacency matrix A_i and adjacency spectrum $\{r_i, \lambda_{i2}, \lambda_{i3}, ..., \lambda_{ip_i}\}, i = 1, 2$. Then the distance spectrum of $G_1 \dot{\vee} G_2$ consists of the following numbers: $-2(\lambda_{1j} + r_1 + 1), j = 2, 3, ..., p_1$; 0 of multiplicity $q_1 - 1$; $-(\lambda_{2j} + 2), j = 2, 3, ..., p_2$ alongwith the eigenvalues of the matrix

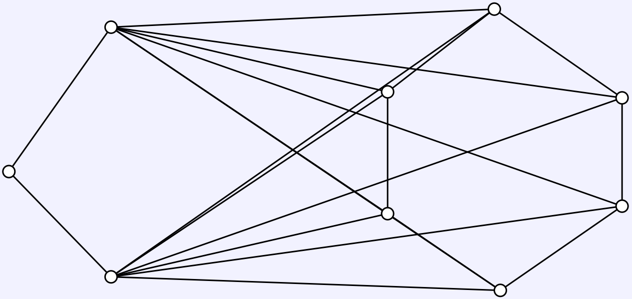
$$\begin{bmatrix} 2(p_1 - 1) & 3q_1 - 2r_1 & p_2 \\ 3p_1 - 4 & 4(q_1 - r_1) & 2p_2 \\ p_1 & 2q_1 & 2p_2 - 2 - r_2 \end{bmatrix}$$

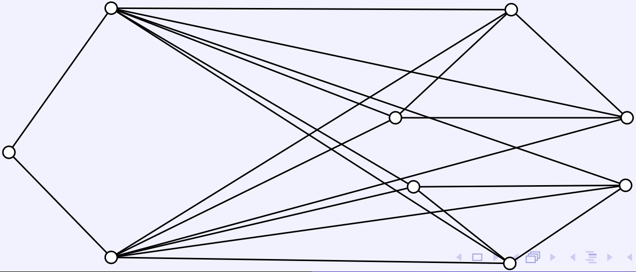
Indulal G, Deena C.S, Xiaogang Liu, The distance spectrum of the subdivision vertex join and subdivision edge join of two regular graphs, Discrete Math. Lett. 1 (2019) 36–41

$D-$ Equienergetic graphs from the subdivision vertex join

Theorem

Let G be any regular graph . For a fixed n let $\mathscr{P}_n$ be the set of all integer partitions on n into parts, each of size atleast 3. For $p = \{p_1, p_2, ..., p_k\} \in \mathscr{P}_n$ let $\mathbf{C}_p$ be the union of cycles with vertices $p_1, p_2, ..., p_k$. Then the graphs $G\dot{\vee}\mathbf{C}_p$, $p \in \mathscr{P}_n$ forms a set of $D-$ equienergetic graphs. Also for every $p \geq 9$ there exists pair of equienergetic graphs from this class





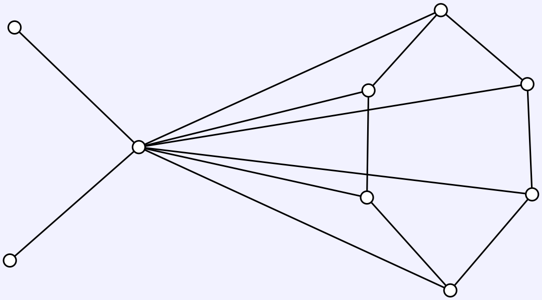
Definition

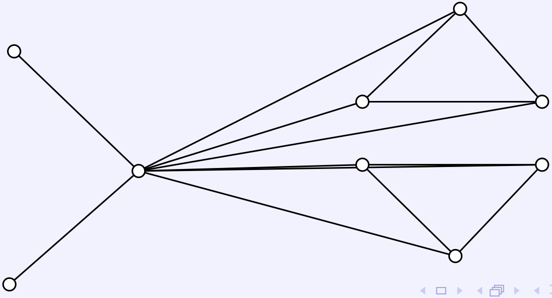
The subdivision-edge join of two vertex disjoint graphs G_1 *and* G_2*, denoted by* $G_1 \veebar G_2$*, is the graph obtained from* $S(G_1)$ *and* G_2 *by joining each vertex of* $I(G_1)$ *with every vertex of* $V(G_2)$*.*

$D-$ Equienergetic graphs from the subdivision edge join

Theorem

Let G be any regular graph . For a fixed n let $\mathscr{P}_n$ be the set of all integer partitions on n into parts, each of size atleast 3. For $p = \{p_1, p_2, ..., p_k\} \in \mathscr{P}_n$ let $\mathbf{C}_p$ be the union of cycles with vertices $p_1, p_2, ..., p_k$. Then the graphs $G \underline{\vee} \mathbf{C}_p$, $p \in \mathscr{P}_n$ forms a set of $D-$ equienergetic graphs





A Conjecture

There exists no pair of noncospectral D-equienergetic trees up to 20 vertices.

D.Stevanovic, G.Indulal, *The distance spectrum and energy of the compositions of regular graphs*, Appl. Math lett., 2009, 136 - 140

Surprise

Equienergetic with respect to adjacency and distance matrices

An open problem

- Characterise those graphs which are equienergetic with respect to both the adjacency and distance matrices

An open problem

- Characterise those graphs which are equienergetic with respect to both the adjacency and distance matrices
- A weaker Problem: Construct families of graphs which are equienergetic with respect to both the adjacency and distance matrices

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Thank you