Quantum State Transfer on Non-Complete Extended P-Sum of the Path on Three Vertices

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Introduction

- Continuous-time quantum walk (CTQW) in Quantum Algorithmic Problems was first used by Farhi and Gutmann.

  E. Farhi, S. Gutmann, *Quantum computation and decision trees*, Phys. Rev. A **58**:915-928 (1998).

- CTQW plays an important role in studying several Quantum transportation phenomena.

- Quantum State Transfer is one such phenomena where the characteristic vector of a initial vertex is transited to the characteristic vector of an another vertex.

We discuss two types of state transfer:

- Perfect state transfer (PST) $\longrightarrow$ introduced by Bose

  S. Bose, *Quantum communication through an unmodulated spin chain*, Physical Review Letters, **91**(20):207901 (2003).

- Pretty good state transfer (PGST) $\longrightarrow$ introduced by Chris Godsil

  C. Godsil, *State transfer on graphs*, Discrete Math., **312**(1): 129–147 (2012).

State transfer has significant applications (see [1, 7]) in

- Quantum Information Processing
- Cryptography

Definition

The transition matrix [9] of a graph G with adjacency matrix A is

$$H(t) := \exp\left(-itA\right) = \sum_{n=1}^{\infty} \frac{(-it)^n}{n!} A^n, \; t \in \mathbb{R}.$$

Let u and v be two vertices in G.

- PST occurs at τ if
$$\left| \mathbf{e_u}^T H(\tau) \mathbf{e_v} \right| = 1.$$
- PGST occurs w.r.t. a sequence t_k if
$$\lim_{k \to \infty} H(t_k) \mathbf{e_u} = \gamma \mathbf{e_v}, \; |\gamma| = 1.$$

Remark: Continuous-time quantum walk can also be defined with respect to the Laplacian matrix as well.

An Example

- The adjacency matrix of P_2 is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

.

- Note that

$$A^n = \begin{cases} I & \text{if } n \text{ is even,} \\ A & \text{if } n \text{ is odd.} \end{cases}$$

- Transition matrix is

$$\begin{aligned} H(t) = \sum_{n=1}^{\infty} \frac{(-it)^n}{n!} A^n &= \cos(t) I - i \sin(t) A \\ &= \begin{pmatrix} \cos(t) & -i\sin(t) \\ -i\sin(t) & \cos(t) \end{pmatrix} \end{aligned}$$

- Hence P_2 admits PST at $\frac{\pi}{2}$.

Lemma 1 ([9])

If a graph G admits perfect state transfer from u to v, then

$$Aut(G)_u = Aut(G)_v.$$

Proof.

Let P be a permutation matrix associated to an automorphism of G. If PST occurs between u and v then

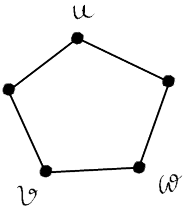
$$H(\tau)\mathbf{e_u} = \gamma\mathbf{e_v},\ \gamma \in \mathbb{C}, |\gamma| = 1.$$

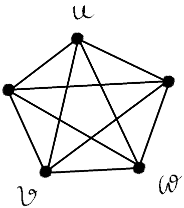
Note that the transition matrix $H(t)$ is a polynomial in A. Since P commutes with A, P commutes with $H(t)$ as well. This gives

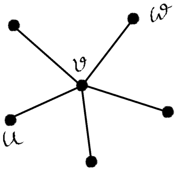
$$H(\tau)\,(P\mathbf{e_u}) = P\,(H(\tau)\mathbf{e_u}) = \gamma P\mathbf{e_v}.$$

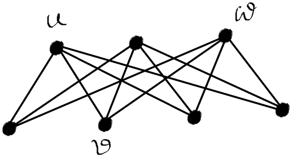
$\square$

Graphs without PST









More Examples

Example 1 (Graphs with/without PST)

- The path P_2 and P_3 at $\frac{\pi}{2}$ and $\frac{\pi}{\sqrt{2}}$, respectively. See [4, 5].
- Cartesian powers of P_2 and P_3 at $\frac{\pi}{2}$ and $\frac{\pi}{\sqrt{2}}$, respectively. See [4, 5].
- Cubelike graphs (or NEPS of P_2) at $\frac{\pi}{2}$ and $\frac{\pi}{4}$. See [2, 3].
- No PST: Paths with more than three vertices. See [4, 9].

Example 2 (Graphs with/without PGST)

- P_n where n is either 2^k; p; $2p$ ($p \rightarrow$ odd prime). See [10].
- Double Star. See [8].
- No PGST: Complete graphs with more than two vertices. See [9]
- No PGST: Vertex transitive graphs of odd order.See [9]

Non-complete Extended P-Sum (NEPS)

- The NEPS [6] of n graphs $G_1, \ldots, G_n$ with $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$ is denoted by

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- If all the factor graphs are G then we simply write $NEPS_n\left(G, \Omega\right)$.

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A. Bernasconi, C. Godsil and S. Severini, *Quantum networks on cubelike graphs*, Physical Review A, **78**:052320 (2008).

W. Cheung and C. Godsil, *Perfect state transfer in cubelike graphs*, Linear Algebra and Its Applications, **435**(10):2468-2474 (2011).

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- We investigated quantum state transfer on NEPS of P_3 in [11, 12] and found few partial characterizations on these graphs.

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If $\sum_{\beta \in \Omega^*} \beta \neq \mathbf{0}$ *in* $\mathbb{Z}_2^n$ *then* $NEPS_n(P_3, \Omega)$ *allows PST at time* $\frac{\pi}{(\sqrt{2})^k}$.

Example 4

The $NEPS_n(P_3, \Omega)$ with Ω as follows exhibits PST:

- $\Omega = \{(1,0,0),(0,1,0),(0,0,1)\}$.
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Further we find that

Corollary 2 ([11])

- *For any* $n \in \mathbb{N} \setminus \{1\}$ *and an odd positive integer* $k < n$
- *There exists* $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$

so that $NEPS_n(P_3, \Omega)$ *is connected and exhibits PST at* $\frac{\pi}{(\sqrt{2})^k}$.

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Then PST occurs in $NEPS_n(P_3, \Omega) \times G$ at time $\frac{\tau_k}{r}$.

The graph $NEPS_n(P_3, J - I) \times K_m$, m even, allows PST at $\frac{\pi}{(\sqrt{2})^{n-1}}$.

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Theorem 5 ([12])

- *Let $\Omega \subset \mathbb{Z}_2^n \setminus \{\mathbf{0}\}$, and*
- *both Ω_e and Ω_o are non-empty.*

Then $NEPS_n(P_3, \Omega)$ does not exhibit PST.

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Corollary 5 ([12])

- *Let a graph G be integral.*
- *Let $NEPS_n(P_3, \Omega)$ satisfies all the conditions of Theorem 3 with k odd.*

Then the Cartesian product $G \square NEPS_n(P_3, \Omega)$ admits PGST.

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Example: $K_m \square NEPS_n(P_3, \Omega)$ exhibits PGST with appropriate Ω.

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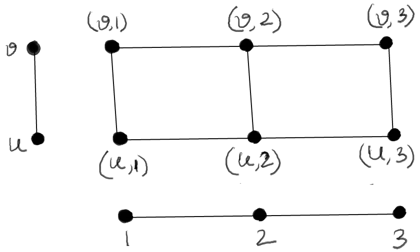

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1. $\sum_{\beta \in \Omega^*} \beta \neq \mathbf{0}$ *in* $\mathbb{Z}_2^m$,
2. $\sum_{\beta \in \Omega'} \beta \neq \mathbf{0}$ *in* $\mathbb{Z}_2^n$.

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Thank You