Complex Adjacency Spectra of (Multi)Digraphs

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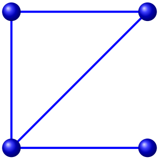
Graphs, Digraphs and Multidigraphs

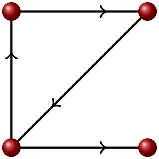
$$G = (V, E)$$

V: set of vertices, E: set of edges



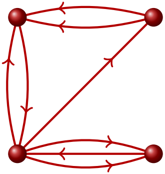
(Undirected) Graph

E : distinct pairs of
vertices called edges



Digraph

E : distinct ordered pairs
of vertices called directed edges



Multidigraph

E : ordered pairs of vertices
not necessarily distinct

Spectral Graph Theory

Given the eigenvalues of a matrix associated with a graph, what can be said about the structure of the graph?

The goal of spectral graph theory is to see how the eigenvalues and eigenvectors of a matrix representation of a graph are related to the graph structure.

Finding inter-relationship between graph structure and spectrum of its associated matrix.

Adjacency matrix of a graph

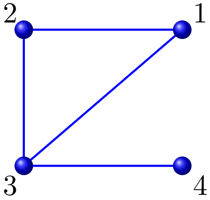
$$G = (V, E)$$
$$V = \{1, 2, \ldots, n\}$$

$A(G) = [a_{ij}]$: called the adjacency matrix of G, $n \times n$ matrix whose rows and columns are indexed by V

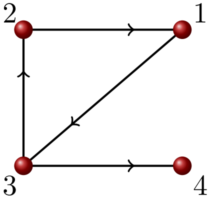
$a_{ij} :=$ the number of edges, or arcs, originating from the vertex i and terminating at the vertex j

$\sigma_A(G)$: called the adjacency spectrum of G, is the collection of all the eigenvalues of $A(G)$

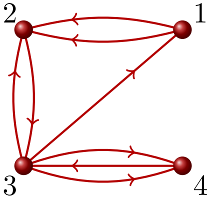
Example



G_1



G_2



G_3

$$A(G_1) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A(G_2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A(G_3) = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Adjacency spectrum of an undirected simple graph

Let G be an undirected graph on n vertices.

Adjacency spectrum :

- Eigenvalues are all real.

$$\sigma_A(G) = \left(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\right)$$

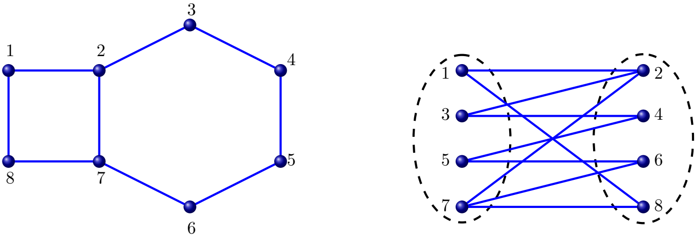
where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

- The adjacency matrix of G has a complete set of orthonormal eigenvectors.

Spectral properties of undirected graphs

- G is bipartite if and only if $\sigma_A(G)$ is symmetric about origin.

G:



$$\sigma_A(G) = (-2.53, \ -1.47, \ -1.09, \ -0.26, 0.26, \ 1.09, \ 1.47, \ 2.35)$$

More spectral properties of undirected graphs

Let G be an undirected graph on n vertices.

- If $\lambda_{n-1} = -1$, then G is the complete graph.
- If $\lambda_{n-1} = 0$, then G is complete multipartite.
- If $\lambda_{n-2} < -1$, then G is isomorphic to P_3.
- If $\lambda_{n-2} = -1$, then G^c is isomorphic to the union of a complete bipartite graph and some isolated vertices.

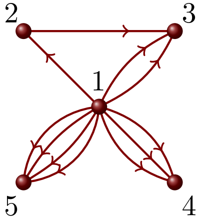
- If G is a connected (non-complete) graph with $n \geq 3$, then $\lambda_1 \leq \lambda_1(K_{n-1}^1)$ with equality is true if and only if $G \equiv K_{n-1}^1$, where K_{n-1}^1 is the graph obtained by the coalescence of K_{n-1} with P_2.
- G has multiple eigenvalues equal to -1 if the third least eigenvalue of its complement is zero.
- If $n \geq 7$ and $\lambda_{n-3} < \frac{1-\sqrt{5}}{2}$, then the chromatic number of G is 3.
- Let $n \geq 7$. Then $\lambda_3 = 0$ implies $\lambda_n \leq -2\lambda_1$.

Spectral Properties of Multidigraphs

Adjacency spectrum of a multidigraph

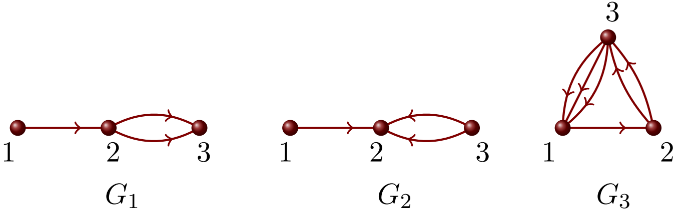
$$A(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$G:$



$$\sigma_A(G) = (0, 0, 0, 0, 0)$$

$$\mathcal{B}_A(G) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$



$\sigma_A(G_1) = (0,0,0,0)$; $\sigma_A(G_2) = (0,0,0,0)$; $\sigma_A(G_3) = (\sqrt[3]{6}, \sqrt[3]{6}\omega, \sqrt[3]{6}\omega^2)$

where $\omega = \frac{-1+\sqrt{3}\mathtt{i}}{2}$

$$\mathcal{B}_A(G_1) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \mathcal{B}_A(G_2) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{B}_A(G_3) = \left\{ \begin{pmatrix} \sqrt[3]{6} \\ \sqrt[3]{6^2} \\ 3 \end{pmatrix}, \begin{pmatrix} \sqrt[3]{6}\omega \\ \sqrt[3]{6^2}\omega \\ 3 \end{pmatrix}, \begin{pmatrix} \sqrt[3]{6}\omega^2 \\ -\sqrt[3]{6^2}\omega^2 \\ 3 \end{pmatrix} \right\}$$

Lacunae in associating a multidigraph by the adjacency matrix

1. The adjacency spectrum may contain complex entries.
2. The matrix fails to possess a complete set of linearly independent eigenvectors.
3. It is difficult to determine the change in orientation of any directed edge either from its eigenvalues or from its eigenvectors.

Criteria for a new associated matrix of a multidigraph

1. The matrix should be well defined, i.e. for each matrix there should be a unique multidigraph (at least upto isomorphism of graphs) and vice versa.
2. It should be a generalization of the adjacency matrix of an undirected graph.
3. An undirected edge should be treated equivalent to two oppositely oriented directed edges.
4. The matrix should be Hermitian.

New associated matrix of a multidigraph: Complex adjacency matrix

f_{ij}: number of forward edges from i to j,
i.e., number of directed edges from vertex i to vertex j

b_{ij}: number of backward edges from i to j,
i.e., number of directed edges from vertex j to vertex i

Definition

The complex adjacency matrix $A_{\mathbb{C}}(G)$ of a multidigraph G is a square $n \times n$ matrix whose (i, j)-entry is given by

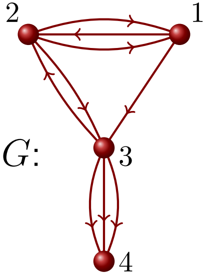
$$a_{ij} = \left(\frac{f_{ij} + b_{ij}}{2}\right) + \left(\frac{f_{ij} - b_{ij}}{2}\right) \mathtt{i}.$$

$$\mathbb{W} = \left\{ \frac{a}{2} + \frac{b}{2}\mathtt{i} : a, b \in \mathbb{Z}, a \geq |b| \geq 0 \text{ and } 2|(a-b) \right\}, \quad \mathbb{W}_+ = \mathbb{W} \backslash \{0\}$$

Example:

$$A_{\mathbb{C}}(G) = \begin{bmatrix} 0 & \frac{3}{2} - \frac{\mathtt{i}}{2} & \frac{1}{2} + \frac{\mathtt{i}}{2} & 0 \\ \frac{3}{2} + \frac{\mathtt{i}}{2} & 0 & 1 & 0 \\ \frac{1}{2} - \frac{\mathtt{i}}{2} & 1 & 0 & \frac{3}{2} + \frac{3\mathtt{i}}{2} \\ 0 & 0 & \frac{3}{2} - \frac{3\mathtt{i}}{2} & 0 \end{bmatrix}$$

$$\sigma_{A_{\mathbb{C}}}(G) = (-2.51, \ -1.38, \ 1.19, \ 2.70)$$



$$\mathcal{B}_A(G) = \left\{ \begin{pmatrix} 1 \\ -071 - 1.04\mathtt{i} \\ 0.29 + 2.13\mathtt{i} \\ -1.44 - 1.09\mathtt{i} \end{pmatrix}, \begin{pmatrix} 1 \\ -0.91 - 0.12\mathtt{i} \\ -0.22 - 0.33\mathtt{i} \\ 0.60 + 0.11\mathtt{i} \end{pmatrix}, \begin{pmatrix} 1 \\ 0.82 + 0.43\mathtt{i} \\ -0.51 + 0.02\mathtt{i} \\ -0.61 + 0.67\mathtt{i} \end{pmatrix}, \begin{pmatrix} 1 \\ 1.27 + 0.01\mathtt{i} \\ 1.55 - 0.46\mathtt{i} \\ 0.60 - 1.11\mathtt{i} \end{pmatrix} \right\}$$

Relationship between $A(G)$ and $A_{\mathbb{C}}(G)$

Let G be a multidigraph.

- $A(G) = \text{real}(A_{\mathbb{C}}(G)) + \text{imag}(A_{\mathbb{C}}(G))$

- If $A(G) = [a_{ij}]$ and $A_{\mathbb{C}}(G) = [c_{ij}]$, then

$$c_{ij} = \left(\frac{a_{ij} + a_{ji}}{2}\right) + \left(\frac{a_{ij} - a_{ji}}{2}\right)\mathtt{i}.$$

Some interlacing results

Theorem (Cauchy's interlacing theorem)

Let $B \in \mathcal{M}_n$ *be Hermitian, let* $y \in \mathbb{C}^n$ *and* $a \in \mathbb{R}$ *be given, and let* $A = \begin{bmatrix} B & y \\ y^* & a \end{bmatrix} \in \mathcal{M}_{n+1}$. *Then* $\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \ldots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A)$.

Theorem

Let G *be a multidigraph on* $n+1$ *vertices and* H *be obtained by deleting a vertex* v *from* G *along with the directed edges associated to (incident to or incident from)* v *in* G. *If* $\{\lambda_i(G)\}_{i=1}^{n+1}$ *and* $\{\lambda_i(H)\}_{i=1}^{n}$ *are the sets of eigenvalues of* $A_{\mathbb{C}}(G)$ *and* $A_{\mathbb{C}}(H)$ *written in nondecreasing order, respectively, then*

$$\lambda_1(G) \leq \lambda_1(H) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G) \leq \lambda_n(H) \leq \lambda_{n+1}(G).$$

Some interlacing results

Theorem

Let G be a multidigraph on vertices $1, \ldots, n$ and H be a multidigraph produced from G by deleting a directed edge e from G. If $\{\lambda_i(G)\}_{i=1}^n$ and $\{\lambda_i(H)\}_{i=1}^n$ are the sets of eigenvalues of $A_{\mathbb{C}}(G)$ and $A_{\mathbb{C}}(H)$ written in nondecreasing order, respectively, then

$$\lambda_1(G) \leq \lambda_2(H),$$

$$\lambda_{i-1}(H) \leq \lambda_i(G) \leq \lambda_{i+1}(H), \text{ for } i = 2, \ldots, n-1,$$

$$\lambda_{n-1}(H) \leq \lambda_n(G).$$

Multidigraphs which satisfy SO-property

- The adjacency spectrum of an undirected graph is symmetric about origin if and only if the graph is bipartite.

Theorem

The complex adjacency spectrum of a bipartite multidigraph is symmetric about origin.

A multidigraph is said to satisfy SO-property if its complex adjacency spectrum is symmetric about origin.

Multidigraphs which satisfy SO-property

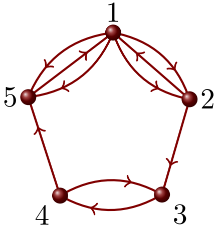
Theorem

The complex adjacency spectrum of a bipartite multidigraph is symmetric about origin.

Converse of the above statement is NOT true.

$$\sigma_{A_{\mathbb{C}}}(G) = \left(0, \pm\sqrt{\frac{7 \pm 3\sqrt{2}}{2}}\right)$$

G:



A multidigraph is said to satisfy SO-property if its complex adjacency spectrum is symmetric about origin.

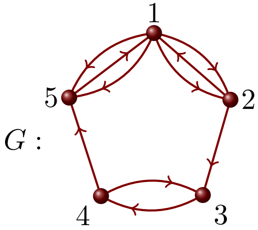
Multidigraphs which satisfy SO-property

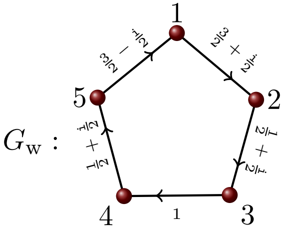
Which non-bipartite multidigraphs satisfy the SO-property?

Multidigraphs which satisfy SO-property

Theorem

Let $G = C_n(w)$ be an odd cycle multidigraph on n vertices, where $w = (w_i)_{i=1}^n \in \mathbb{W}_+^n$. Then the weight of G is purely imaginary if and only if G satisfies SO-property.





weight of G is $\frac{5\mathrm{i}}{4}$

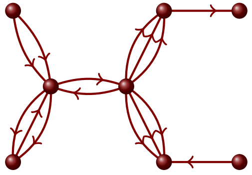
Theorem

A multidigraph satisfies SO-property if weights of all its odd cycle sub-multidigraphs are purely imaginary.

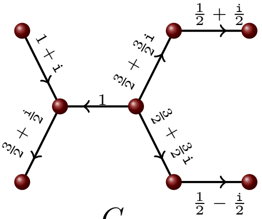
Spectral properties of a multi-directed tree

Definition

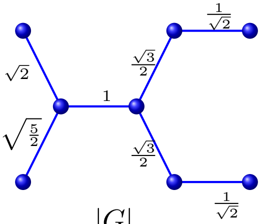
Let G be a multidigraph on vertices $1, 2, \ldots, n$ and G_{w} be its associated weighted digraph. Then the modular graph *of G, denoted by $|G|$, is the weighted graph which is obtained from G_{w} by replacing each of its directed edge by an edge of weight equal to the modulus of the corresponding weight of the directed edge in G_{w}. That is, if $i \xrightarrow{w} j$ in G_{w} (or, that is, in G) for some $w \in \mathbb{W}$, then the vertices i, j are adjacent in $|G|$ and the weight of the edge $\{i, j\}$ is $|w|$.*



G



G_{w}



$|G|$

Theorem

Let T be a multi-directed tree on n vertices and $|T|$ be its modular tree. Let $A_{\mathbb{C}}(T)$ and $A(|T|)$ be the complex adjacency matrix and the adjacency matrix of T and $|T|$, respectively. Then both T and $|T|$ share same $A_{\mathbb{C}}$-spectrum, that is

$$\sigma_{A_{\mathbb{C}}}(T) = \sigma_A(|T|).$$

Furthermore, if x and y are eigenvectors of $A_{\mathbb{C}}(T)$ and $A(|T|)$, respectively, corresponding to an eigenvalue λ, then $|x| = |y|$.

Rough sketch of the proof:

$A_{\mathbb{C}}(T) = (c_{ij})_{n\times n}$ and $A(|T|) = [a_{ij}]_{n\times n}$

$y = (y_i)_{i=1}^n$, $A(|T|)y = \lambda y$

$$\sum_{k=1}^{n} a_{ij} y_j = \lambda y_i \quad \text{for } i = 1, \ldots, n$$
$$\sum_{k=1}^{n} c_{ij} y_j \mathrm{e}^{\mathtt{i}\operatorname{Arg}(\overline{c}_{ij})} = \lambda y_i \quad \text{for } i = 1, \ldots, n$$

Now choose a vector $x = (x_i)_{i=1}^n$ such that $|x| = |y|$ and for $i \xrightarrow{w} j$ in T

$$x_j = \begin{cases} |y_j|\mathrm{e}^{\mathtt{i}\theta}, & \text{if } y_i y_j \geq 0 \\ |y_j|\mathrm{e}^{\mathtt{i}(\theta+\pi)}, & \text{otherwise} \end{cases}$$

where $\theta = \operatorname{Arg}(x_i) + \operatorname{Arg}(\overline{w})$ and $\operatorname{Arg}(x_1) = 0$.

➪ Application to any Hermitian matrix:

Theorem

Let A be a Hermitian matrix of order $n \times n$ with all its diagonal entries zero such that its associated graph is a tree. Then A and $|A|$ have the same set of eigenvalues. More generally, if D is a real diagonal matrix of order $n \times n$, then $D + A$ and $D + |A|$ also have the same set of eigenvalues.

$$P = \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2} & 3 & 1-3\mathtt{i} & 0 & \mathtt{i} \\ 0 & 1+3\mathtt{i} & -1 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & -\mathtt{i} & 0 & 0 & 1.8 \end{bmatrix}, \ |P| = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 3 & \sqrt{10} & 0 & 1 \\ 0 & \sqrt{10} & -1 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1.8 \end{bmatrix}$$

$$\sigma(P) = (-3.0229, -0.2406, 0.1424, 1.6478, 5.2734) = \sigma(|P|)$$

Spectral Properties of Simple Digraphs

Complex adjacency matrix of a digraph

(Digraphs which contains at most one directed edge between any two pairs of vertices)

Definition

Let $D = (V, E)$ be a digraph with $V = \{1, 2, \ldots, n\}$. Then the complex adjacency matrix of D, denoted by $A_{\mathbb{C}}(D) = [a_{ij}]$, is a square $n \times n$ matrix whose rows and columns are indexed by V and whose ijth entry is given by

$$a_{ij} = \begin{cases} \frac{1}{2} + \frac{\mathtt{i}}{2} & \text{if } i \to j, \\ \frac{1}{2} - \frac{\mathtt{i}}{2} & \text{if } i \leftarrow j, \\ 0 & \text{otherwise.} \end{cases}$$

Properties of complex adjacency matrix of a digraph

- The sum of all the 2×2 principal minors of $A_{\mathbb{C}}(D)$ equals $-|E|$.

  Reason:

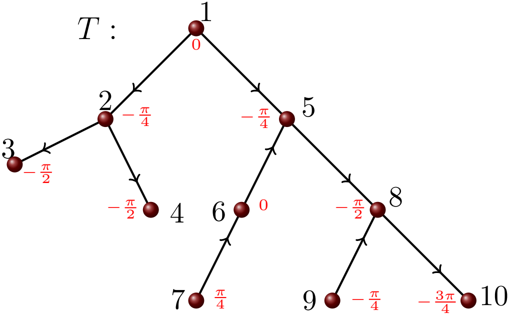
$$\frac{1}{2}\begin{bmatrix} 0 & 1+\mathtt{i} \\ 1-\mathtt{i} & 0 \end{bmatrix} \text{ or } \frac{1}{2}\begin{bmatrix} 0 & 1-\mathtt{i} \\ 1+\mathtt{i} & 0 \end{bmatrix}$$

  The determinant of each one is -1.

- If p and q are the number of proper and improper 3-cycles of D, then the sum of all the 3×3 principal minors of $A_{\mathbb{C}}(D)$ equals to $\frac{1}{2}(p-q)$.

Complex Adjacency Spectra of a Directed Tree

All directed trees having same base structure share same complex adjacency spectrum.



$$|v| = \begin{pmatrix} 1 \\ 0.815 \\ 0.378 \\ 0.378 \\ 1.338 \\ 0.792 \\ 0.367 \\ 1.091 \\ 0.506 \\ 0.506 \end{pmatrix}$$

$\sigma_{A_{\mathbb{C}}}(T) = (\pm 3.046, \pm 2.334, \pm 1.679, \pm 0.669, 0, 0)$ and v is the eigenvector corresponding to the eigenvalue 3.046.

Complex Adjacency Spectra of Cycle Digraphs

Remark

The complex adjacency spectrum of a proper cycle $\overrightarrow{C}_n$ is given by

$$\sigma_{A_{\mathbb{C}}}(\overrightarrow{C}_n) = \left(\sqrt{2}\cos\left(\frac{2\pi k}{n} + \frac{\pi}{4}\right)\right)_{k=1}^{n}$$

and the eigenvectors of $A_{\mathbb{C}}(\overrightarrow{C}_n)$ are

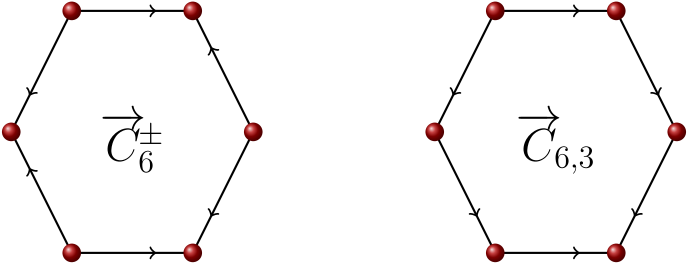
$$x_\omega = (1, \omega, \omega^2, \ldots, \omega^{n-1})^T \text{ where } \omega^n = 1.$$

How does the change in orientation of some of the arcs of a cycle digraph affect its complex adjacency spectrum?

Theorem

Let D be a cycle digraph on vertex set $\{1, 2, \ldots, n\}$ and having b number of backward directed edges. Then the $A_{\mathbb{C}}$-spectrum of D consists of

$$\sqrt{2}\cos\Big(\frac{(4k-b)\pi}{2n}+\frac{\pi}{4}\Big), \text{ for } k = 0, 1, \ldots, n-1.$$



Alternating cycle Colliding cycle

$$\sigma_{A_{\mathbb{C}}}(\overrightarrow{C}_6^{\pm}) = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$$

$$= \left(\sqrt{2}\cos\left(\frac{(4k-3)\pi}{12} + \frac{\pi}{4}\right)\right)_{k=0}^{5} = \sigma_{A_{\mathbb{C}}}(\overrightarrow{C}_{6,3})$$

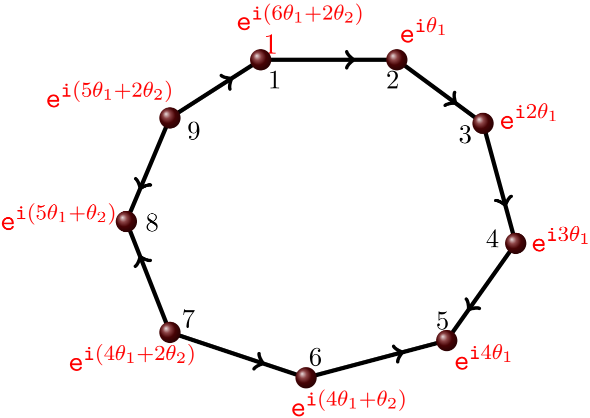
How to find the structural difference between two cycle digraphs on the same number of vertices whose $A_{\mathbb{C}}$-spectra are the same?

Theorem

Let D be a cycle digraph on vertex set $\{1, 2 \ldots, n\}$. Let f_j and b_j be the number of forward and backward directed edges in the path $[1, 2, \ldots, j]$, respectively. Then the components of an eigenvector x_k corresponding to the $A_{\mathbb{C}}$-eigenvalue λ_k (whose algebraic multiplicity is 1), $k \in \{0, 1, \ldots, n-1\}$, of $A_{\mathbb{C}}(D)$ can be chosen as the following.

$$x_k(1) = 1, \quad x_k(j) = \mathtt{e}^{\mathtt{i}(f_j\theta_1 + b_j\theta_2)}, \text{ for } 2 \leq j \leq n,$$

where $\theta_1 = (4k - b)\frac{\pi}{2n}$, $\theta_2 = \frac{\pi}{2} + \theta_1$ and b is the total number of backward directed edges in D.

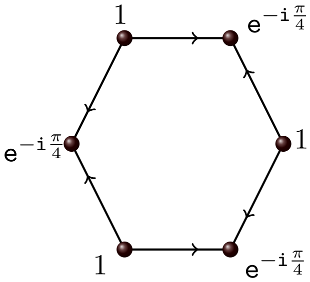


$$\theta_1 = (4k-b)\frac{\pi}{2n}, \qquad \theta_2 = \frac{\pi}{2} + \theta_1$$

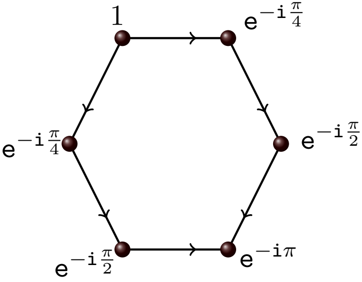
and b is the total number of backward directed edges in D.

λ_k kth eigenvalue of $A_{\mathbb{C}}(D)$

x_k eigenvector corresponding to eigenvalue λ_k;



$\overrightarrow{C}_6^{\pm}$



$\overrightarrow{C}_{6,3}$

$$\sigma_{A_{\mathbb{C}}}(\overrightarrow{C}_6^{\pm}) = \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right) = \sigma_{A_{\mathbb{C}}}(\overrightarrow{C}_{6,3})$$

The eigenvectors corresponding to the eigenvalue $\sqrt{2}$ are shown here around the vertices of the corresponding digraphs.

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Q&A

Thank You.