Signed Distance in Signed Graphs

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Introduction

A signed graph $\Sigma = (G, \sigma)$ is an underlying graph $G = (V, E)$ with a signature function $\sigma : E \to \{1, -1\}$. A signed graph $\Sigma = (G, \sigma)$ is said to be *balanced* if the product $\sigma(C) = \prod_{e \in E(C)} \sigma(e) = 1$ for all cycles C in Σ. It is called *antibalanced* if $-\Sigma$ is balanced; equivalently, every even cycle is positive and every odd cycle is negative.

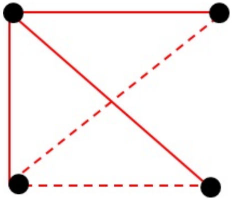

Figure: Example of a signed graph

Introduction

Switching is an important operation in signed graphs which will be used very often in our discussion. Given a signed graph $\Sigma = (G, \sigma)$, we can construct another signed graph $\Sigma^{\zeta} = (G, \sigma^{\zeta})$ where $\zeta : V \to \{1, -1\}$ and $\sigma^{\zeta}(uv) = \zeta(u)\sigma(uv)\zeta(v)$. We say then that Σ is switched to Σ^{ζ}.

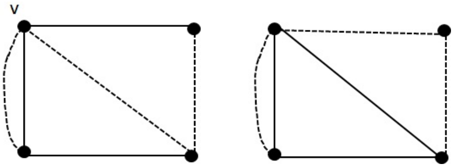

Figure: Switched with respect to v

Introduction

In this talk we introduce two types of signed distance for signed graphs and corresponding distance matrices, and we characterize balanced signed graphs using these matrices.

Several characterizations of balance are already available in the literature and we list some that are used in our discussion as follows.

Introduction

Harary's bipartition theorem

A signed graph Σ is balanced if and only if there is a bipartition of its vertex set, $V = V_1 \cup V_2$, such that every positive edge is induced by V_1 or V_2 while every negative edge has one endpoint in V_1 and one in V_2.[a]

[a] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. 2 (1953–1954) 143–146.

A bipartition of V as in Theorem 5 is called a *Harary bipartition* of Σ.

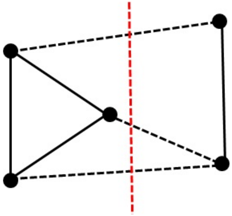

Figure: Example

Introduction

Harary's path criterion

A signed graph is balanced if and only if, for any two vertices u and v, every uv-path has the same sign. [b]

[b] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. 2 (1953–1954) 143–146.

Restatement of Acharya

Let Σ be a signed graph with underlying graph G and let w be a positive weight function on the edges. Then (Σ, w) is cospectral with (G, w) if and only if Σ is balanced.[c]

[c] B. D. Acharya, Spectral criterion for cycle balance in networks, J. Graph Theory, 4(1980) 1–11.

Introduction

Corollary :Acharya's spectral criterion

A signed graph $\Sigma = (G, \sigma)$ is balanced if and only if the spectra of the adjacency matrices of Σ and G coincide.[d]

[d] B. D. Acharya, Spectral criterion for cycle balance in networks, J. Graph Theory, 4(1980) 1–11.

Stanić's spectral criterion

A signed graph $\Sigma = (G, \sigma)$ is balanced if and only if the largest eigenvalues of the adjacency matrices of Σ and G coincide.[e]

[e] Z. Stanić, Integral regular net-balanced signed graphs with vertex degree at most four. Ars Math. Contemp. 17 (2019) 103–114.

Introduction

Corollary

A signed graph with positively weighted edges, $(\Sigma, w) = (G, \sigma, w)$, is balanced if and only if the largest eigenvalues of the adjacency matrices $A(\Sigma, w)$ and $A(G, w)$ coincide. [f]

[f] Z. Stanić, Integral regular net-balanced signed graphs with vertex degree at most four. Ars Math. Contemp. 17 (2019) 103–114.

Switching criterion

A signed graph $\Sigma = (G, \sigma)$ is balanced if and only if it can be switched to an all positive signed graph.[g]

[g] T. Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1982) 47–74. Erratum, Discrete Appl. Math. 5 (1983) 248.

Main Results

All the underlying graphs in our consideration are simple and connected, unless otherwise stated. Given a signed graph $\Sigma = (G, \sigma)$, the sign of a path P in Σ is defined as $\sigma(P) = \prod_{e \in E(P)} \sigma(e)$. We denote a shortest path between two given vertices u and v by $P_{(u,v)}$ and the collection of all shortest paths $P_{(u,v)}$ by $\mathcal{P}_{(u,v)}$; and $d(u, v)$ denotes the usual distance between u and v.

First we define auxiliary signs:

- $\sigma_{\max}(u, v) = -1$ if all shortest uv-paths are negative, and $+1$ otherwise.
- $\sigma_{\min}(u, v) = +1$ if all shortest uv-paths are positive, and -1 otherwise.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

Then we define signed distances:

- $d_{\max}(u,v) = \sigma_{\max}(u,v)d(u,v) = \max\{\sigma(P_{(u,v)}) : P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u,v)$.
- $d_{\min}(u,v) = \sigma_{\min}(u,v)d(u,v) = \min\{\sigma(P_{(u,v)}) : P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u,v)$.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

Finally, we define the signed distance matrices:

- $D^{\max}(\Sigma) = (d_{\max}(u, v))_{n\times n}$.
- $D^{\min}(\Sigma) = (d_{\min}(u, v))_{n\times n}$.

Compatibility

Two vertices u and v in a signed graph Σ are said to be *distance-compatible* (briefly, *compatible*) if $d_{min}(u, v) = d_{max}(u, v)$. And Σ is said to be (distance-)compatible if every two vertices are compatible.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

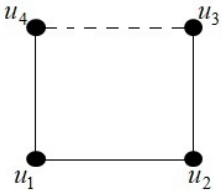

Figure: 1.Unbalanced C_4

The two signed distance matrices of the signed graph in Figure 1 are

$$D^{\max} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \end{pmatrix}, \qquad D^{\min} = \begin{pmatrix} 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \\ -2 & 1 & 0 & -1 \\ 1 & -2 & -1 & 0 \end{pmatrix}.$$

[1]Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

The following are some immediate observations:

- If there is a unique shortest uv-path $P_{(u,v)}$ in G, then $\sigma_{\max}(u,v) = \sigma_{\min}(u,v) = \sigma(P_{(u,v)})$. Thus, for an edge $e = uv$, $\sigma_{\max}(u,v) = \sigma_{\min}(u,v) = \sigma(e)$.
- For an underlying graph in which any two vertices are joined by a unique shortest path (this is called a geodetic graph), we have $\sigma_{\max}(u,v) = \sigma_{\min}(u,v) = \sigma(P_{(u,v)})$ and consequently $D^{\max}(\Sigma) = D^{\min}(\Sigma)$.

Some examples are signed graphs with underlying graph that is K_n or a tree.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

We construct two complete signed graphs from the distance matrices $D^{\max}$ and $D^{\min}$ as follows.

The associated signed complete graph $K^{D^{\max}}(\Sigma)$ with respect to $D^{\max}(\Sigma)$ is obtained by joining the non-adjacent vertices of Σ with edges having signs

$$\sigma(uv) = \sigma_{\max}(uv)$$

The associated signed complete graph $K^{D^{\min}}(\Sigma)$ with respect to $D^{\min}(\Sigma)$ is obtained by joining the non-adjacent vertices of Σ with edges having signs

$$\sigma(uv) = \sigma_{\min}(uv)$$

[1]Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

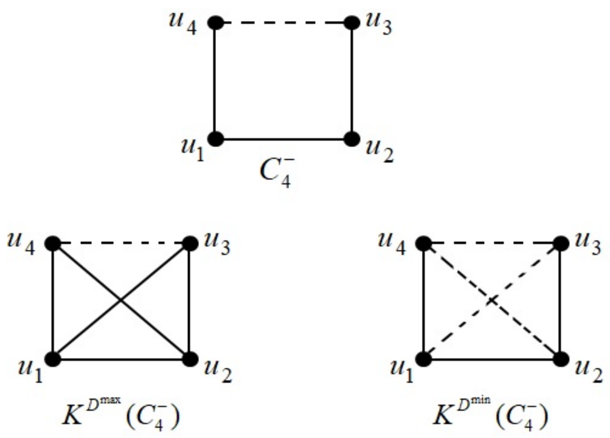

Figure: 2.Two associated signed complete graphs of C_4^-

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

All connected signed graphs fall into three classes:

(I) Σ is balanced or antibalanced or geodetic, which makes $D^{\max} = D^{\min}$

(II) Σ is unbalanced and neither antibalanced nor geodetic, but still $D^{\max} = D^{\min}$.

(III) Σ is unbalanced and neither antibalanced nor geodetic, with $D^{\max} \neq D^{\min}$.

In the first two cases, we denote the associated signed complete graph by $K^{D^{\pm}}(\Sigma)$ or simply by $K^{D^{\pm}}$.
A small example for case (II) is the triangle with one negative edge.

[1]Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

As examples for all three cases consider the signed graph $K_4 \setminus \{e\}$ with three different sign patterns as given in Figure 6. Note that this graph is not geodetic.

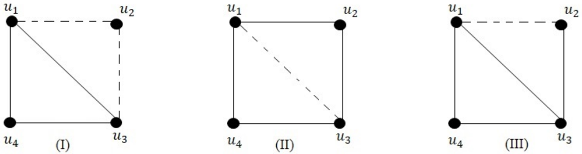

Figure: 3.Three different kinds of signature of $K_4 \setminus \{e\}$.

[1]Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

The signed graph (I) in Figure 6 is balanced and has $D^{\max} = D^{\min}$; the matrix is

$$D^{\max} = D^{\min} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -2 \\ 1 & -1 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}$$

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Main Results

The signed graph (II) in Figure 6 is neither balanced nor antibalanced, yet $D^{\max} = D^{\min}$, as follows:

$$D^{\max} = D^{\min} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}.$$

The signed graph (III) in Figure 6 is of type (III), where $D^{\max} \neq D^{\min}$. The two matrices are

$$D^{\max} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}, \qquad D^{\min} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -2 \\ 1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}.$$

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

The below figure shows another simple, 2-connected, compatible signed graph that is neither geodetic nor balanced nor antibalanced.

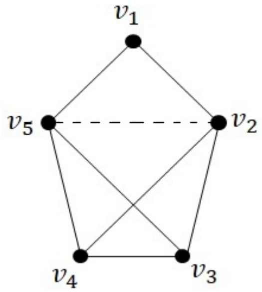

Figure: 4.A nontrivially distance-compatible signed graph of order 5.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Some new characterizations for balance

Theorem

For a signed graph Σ the following statements are equivalent:

(i) Σ is balanced.

(ii) The associated signed complete graph $K^{D^{\max}}(\Sigma)$ is balanced.

(iii) The associated signed complete graph $K^{D^{\min}}(\Sigma)$ is balanced.

(iv) $D^{\max}(\Sigma) = D^{\min}(\Sigma)$ and the associated signed complete graph $K^{D^{\pm}}(\Sigma)$ is balanced.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Some new characterizations for balance

Using Acharya's spectral criterion in Corollary 7 and Stanić's criterion in Corollary 8, the above characterization can be reformulated as

Theorem

A signed graph Σ is balanced if and only if the associated signed complete graph $K^{D^{\pm}}(\Sigma)$ has largest eigenvalue $n-1$, equivalently its spectrum is $\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Some new characterizations for balance

Lemma

Switching a signed graph Σ does not change the set of compatible pairs of vertices.

Theorem

If Σ is switched to Σ^{ζ} and if $D^{\max}(\Sigma) = D^{\min}(\Sigma) = D^{\pm}(\Sigma)$ then $D^{\max}(\Sigma^{\zeta}) = D^{\min}(\Sigma^{\zeta}) = D^{\pm}(\Sigma^{\zeta})$ and $D^{\pm}(\Sigma)$ is similar to $D^{\pm}(\Sigma^{\zeta})$.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Some new characterizations for balance

Theorem

The following properties of a signed graph Σ are equivalent.

(i) Σ is balanced.
(ii) $D^{\max}(\Sigma)$ is cospectral with $D(G)$.
(iii) $D^{\min}(\Sigma)$ is cospectral with $D(G)$.
(iv) $D^{\max}(\Sigma)$ has largest eigenvalue equal to that of $D(G)$.
(v) $D^{\min}(\Sigma)$ has largest eigenvalue equal to that of $D(G)$.

In particular, Σ is balanced if and only if $D^{\pm}(\Sigma)$ exists and is cospectral with $D(G)$.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Some new characterizations for balance

We illustrate the use of the above theorem with the example of a complete bipartite graph $K_{n,n}$.

Corollary

The signed graph $\Sigma = (K_{n,n}, \sigma)$ is balanced if and only if $D^{\max}$ (or $D^{\min}$) has largest eigenvalue $3n-2$; also if and only if its spectrum is $\begin{pmatrix} 3n-2 & n-2 & -2 \\ 1 & 1 & 2n-2 \end{pmatrix}$.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Examples of signed-distance spectra

For the odd unbalanced cycle C_n^- where $n = 2k+1$, we have unique shortest path between any two vertices. Thus $D^{\max}(C_n^-) = D^{\min}(C_n^-) = D^{\pm}(C_n^-)$ and we give the spectrum of an odd unbalanced cycle as follows.

Theorem

For an odd unbalanced cycle C_n^- where $n = 2k+1$, the spectrum of $D^{\pm}$ is

$$\begin{pmatrix} k(-1)^k - \dfrac{1-(-1)^k}{2} & \dfrac{k(-1)^j}{\sin((2j+1)\frac{\pi}{2n})} - \dfrac{\sin^2((2j+1)\frac{k\pi}{2n})}{\sin^2((2j+1)\frac{\pi}{2n})} \\ 1 & 2 \quad (j = 0, 1, 2, \ldots, k-1) \end{pmatrix}.$$

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Examples of signed-distance spectra

Now we consider an unbalanced signed wheel $(W_{n+1}, \sigma) = (C_n \vee K_1, \sigma)$ where n is odd and the signature σ is such that $\sigma(e) = -1$ if $e \in E(C_n)$ and 1 otherwise. Clearly, this signed graph is compatible (in fact, it is antibalanced).

Theorem

The spectrum of (W_{n+1}, σ) is

$$SpecD^{\pm}(W_{n+1}, \sigma) = \begin{pmatrix} n - 4 \pm \sqrt{n^2 - 7n + 16} & -2 - 6\cos\frac{2j\pi}{n} \\ 1 & 1 \ (j = 1, 2, \ldots, n-1) \end{pmatrix}.$$

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Compatible and incompatible signed graphs

We know three kinds of compatible signed graph: balanced, antibalanced, and geodetic, and there are others we do not know. For bipartite graphs we have better information.

Theorem

A bipartite signed graph is distance-compatible if and only if it is balanced.

Corollary

Let Σ be a bipartite signed graph. Then, Σ is balanced if and only if $D^{\max}(\Sigma) = D^{\min}(\Sigma)$.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Compatible and incompatible signed graphs

Theorem

A signed graph is compatible if and only every block of it is compatible.

The characterization problem for all compatible signed graphs is reduced by these theorems to the case of a 2-connected, non-bipartite graph that is not a cycle. That appears to be a hard problem. We know there exist examples that are not any of the three kinds stated at the beginning of this section; we conclude with such an example.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Compatible and incompatible signed graphs

Counter Example

Let P and Q be graphs of orders p and q, respectively, that are 2-connected, incomplete, and not bipartite. Consider P as an all-positive signed graph and Q as an all-negative signed graph. Let $K_{p,q}$ be the complete bipartite graph considered as an all-positive signed graph, with partition sets A and B of orders p and q, respectively. Now identify $V(P)$ with A and $V(Q)$ with B; this gives a signed graph Σ. We prove that Σ is not balanced, not antibalanced, and not geodetic, but it is distance-compatible.

[1] Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Compatible and incompatible signed graphs

Counter Example

Because P contains a positive odd cycle and Q contains a negative odd cycle, Σ is neither balanced nor antibalanced. The diameter of Σ is 2. Since P, Q are 2-connected, $p, q \geq 3$; hence any two non-adjacent vertices in P (or in Q) are joined by multiple shortest paths of length 2, all of which are positive. (A 2-path in $K_{p,q}$ or P is positive. A 2-path in Q is positive because Q is all negative.) It follows that Σ is compatible but it is not geodetic.

We note that there are many graphs suitable to be P and Q, such as the complemented line graphs of complete graphs, $\overline{L(K_n)}$ for $n \geq 5$ ($n = 5$ being the Petersen graph).

[1]Shahul Hameed K, Shijin T V, Soorya P, Germina K A and Thomas Zaslavsky, Signed Distance in Signed Graphs, Linear Alg. and its Applications 608 (2021) 236-247.

Other Papers in this concept

Other Papers in this concept

1. Shijin T V, Soorya P, Shahul Hameed K, Germina K A, On Signed Distance in Product of Signed Graphs, ArXiv.
2. Shijin T V, Germina K A, Shahul Hameed K, On the Powers of Signed Graphs, ArXiv.
3. Roshni T Roy, Germina K A, Shahul Hameed K, Thomas Zaslavsky, Signed Distance Laplacian Matrices for Signed Graphs, Communicated.

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Thank
You!!!