# Spectra of some partitioned matrices

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## Thesis Title

Spectra of graphs constructed by various new graph operations

## New graph operations in my thesis

- $(H_1, H_2)$-merged subdivision graph of a graph
- $\mathcal{M}$-join of graphs
- $M$-generalized corona of graphs constrained by vertex subsets
- $(M, \mathcal{M})$-corona-join of graphs constrained by vertex subsets

## Notations

- $J_{n\times m}$ - The $n \times m$ matrix in which all the entries are 1
- $\sigma(M)$ - The spectrum of a matrix $M$
- $\mathcal{R}_{n\times m}(s) := \{[m_{ij}] \in M_{n\times m}(\mathbb{C}) | \sum_{j=1}^{m} m_{ij} = s \text{ for } i = 1, 2, \ldots, n\}$
- $\mathcal{C}_{n\times m}(c) := \{[m_{ij}] \in M_{n\times m}(\mathbb{C}) | \sum_{i=1}^{n} m_{ij} = c \text{ for } j = 1, 2, \ldots, m\}$
- $\mathcal{RC}_{n\times m}(s, c) := \mathcal{R}_{n\times m}(s) \cap \mathcal{C}_{n\times m}(c)$.
- $A \cup B$, $A \cap B$, $A + B$ denote the union, intersection, sum of sets (multi-sets) $A$ and $B$
- $kA$ Sum of a multi-set $A$ with itself $k$ times
- $A \subseteq B$ $A$ is a subset (multi-subset) of $B$
- $A \setminus B$ The difference of a set (multi-set) $A$ from $B$
- $|A|$ Cardinality of the set (multi-set) $A$

# Related results in literature

The following result was proved by Goddard in 1995.

## Proposition 1.1.

*([1]) Let $A \in M_{n\times n}(\mathbb{C})$ and $B \in M_{m\times m}(\mathbb{C})$. If there exists a matrix $P \in M_{n\times m}(\mathbb{C})$ such that $\operatorname{rank}(P) = r$ and $AP = PB$, then $A$ and $B$ have at least $r$ common eigenvalues. Moreover, if $m \geq n$ and $r = n$, then $\sigma(B) \supseteq \sigma(A)$; if $m \leq n$ and $r = m$, then $\sigma(B) \subseteq \sigma(A)$.*

Haynsworth proved the following result in 1960.

## Theorem 1.1.

*([3, Theorem 2]) Let*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} \end{bmatrix}, \tag{1.1}$$

*where $A_{ij} \in M_{n_i\times n_j}(\mathbb{C})$ and $B_{ij} \in M_{m_i\times m_j}(\mathbb{C})$ for $i, j = 1, 2, \ldots, k$. Let $X_j \in M_{n_j\times m_j}(\mathbb{C})$ such that $\operatorname{rank}(X_j) = r$ for $j = 1, 2, \ldots, k$. If $A_{ij}X_j = X_iB_{ij}$ for $i, j = 1, 2, \ldots, k$, then $A$ and $B$ have at least $kr$ common eigenvalues. Moreover, if $r = m_i$ for $i = 1, 2, \ldots, k$, then $\sigma(B) \subseteq \sigma(A)$.*

Throughout this presentation, unless we mentioned otherwise, we assume the following.

(1) $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}$, where $A_{ij} \in M_{n_i \times n_j}(\mathbb{C})$ for $i, j = 1, 2, \ldots, k$;

(2) $\beta = \{s_1, s_2, \ldots, s_t\} \subseteq \{1, 2, \ldots, k\}$ and $s_1 < s_2 < \cdots < s_t$;

(3) $B = \begin{bmatrix} B_{s_1 s_1} & B_{s_1 s_2} & \cdots & B_{s_1 s_t} \\ B_{s_2 s_1} & B_{s_2 s_2} & \cdots & B_{s_2 s_t} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s_t s_1} & B_{s_t s_2} & \cdots & B_{s_t s_t} \end{bmatrix}$, where $B_{ij} \in M_{m_i \times m_j}(\mathbb{C})$ for $i, j \in \beta$.

## Theorem 1.2.

*Let* $X_j \in M_{n_j \times m_j}(\mathbb{C})$ *for* $j \in \beta$, *and let* $r = \sum_{j=s_1}^{s_t} rank(X_j)$. *If*

$$A_{ij}X_j = \begin{cases} X_i B_{ij} & \text{for } i, j \in \beta; \\ \mathbf{0} & \text{for } i \in \beta^c; j \in \beta, \end{cases} \tag{1.2}$$

*then* $A$ *and* $B$ *have at least* $r$ *common eigenvalues. Moreover, if* $rank(X_i) = m_i$ *(provided* $m_i \leq n_i$*) for* $i \in \beta$, *then* $\sigma(B) \subseteq \sigma(A)$.

## proof outline

Take

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1t} \\ P_{21} & P_{22} & \dots & P_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k1} & P_{k2} & \dots & P_{kt} \end{bmatrix},$$

where $P_{ij} = \begin{cases} X_j & \text{if } i = s_j; \\ \mathbf{0} & \text{otherwise}, \end{cases}$

for $i = 1, 2, \dots, k$; $j = 1, 2, \dots, t$. in Proposition 1.1.

## Theorem 1.3.

*If there exists a sequence* $S = (X_{s_1}, X_{s_2}, \ldots, X_{s_t})$ *of non-zero vectors* $X_j \in \mathbb{C}^{n_j}$ *such that*

$$A_{ij}X_j = \begin{cases} a_{ij}X_i & \text{for } i, j \in \beta; \\ \mathbf{0} & \text{for } i \in \beta^c; j \in \beta, \end{cases} \tag{1.3}$$

*with* $a_{ij} \in \mathbb{C}$ *for* $i, j \in \beta$*, then* $\sigma(A) \supseteq \sigma(E_S)$*, where*

$$E_S = \begin{bmatrix} a_{s_1 s_1} & a_{s_1 s_2} & \cdots & a_{s_1 s_t} \\ a_{s_2 s_1} & a_{s_2 s_2} & \cdots & a_{s_2 s_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s_t s_1} & a_{s_t s_2} & \cdots & a_{s_t s_t} \end{bmatrix}.$$

## Remark 1.1.

(1) Each $X_j$ is an eigenvector of $A_{jj}$ corresponding to the eigenvalue $a_{jj}$ for $j \in \beta$.

(2) The matrix $E_S$ mentioned in Theorem 1.3 depends on the sequence $S$. In this case, we say that $E_S$ is the matrix corresponding to the sequence $S$.

Next we study under which constraints the sum of the spectra of the matrices corresponding to some sequences is contained in the spectrum of $A$.

## Proposition 1.2.

*Let $s_i, t_j \in \{1, 2, \ldots, k\}$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, p$. Let $X_h^{(q)} \in \mathbb{C}^{n_h}$ for $h = 1, 2, \ldots, k$; $q = 1, 2$, and let $S_1 = (X_{s_1}^{(1)}, X_{s_2}^{(1)}, \ldots, X_{s_r}^{(1)})$ and $S_2 = (X_{t_1}^{(2)}, X_{t_2}^{(2)}, \ldots, X_{t_p}^{(2)})$ be sequences of no-zero vectors, which satisfy (1.3) with $a_{ij}^{(1)}$, $a_{ij}^{(2)} \in \mathbb{C}$, respectively. If $X_{s_i}^{(1)}$ and $X_{t_j}^{(2)}$ are linearly independent whenever $s_i = t_j$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, p$, then*

$$\sigma(A) \supseteq \sigma(E_{S_1}) + \sigma(E_{S_2}).$$

*Moreover, if $S_1$, $S_2$, $\ldots$, $S_q$ are the sequences of non-zero vectors such that each pair $S_i$, $S_j$ for $i, j = 1, 2, \ldots, q$ satisfies the above constraints, then*

$$\sigma(A) \supseteq \sum_{i=1}^{q} \sigma(E_{S_i}).$$

## Proof outline

Let $P_{S_1} = \begin{bmatrix} Y_{i1}^T & Y_{i2}^T & \cdots & Y_{ik}^T \end{bmatrix}^T$, where

$$Y_{ij} = \begin{cases} X_i^{(1)} & \text{if } j = s_i; \\ \mathbf{0} & \text{otherwise} \end{cases},$$

for $j = 1, 2, \ldots, k$; $i = 1, 2, \ldots, r$.
Let $P_{S_2} = \begin{bmatrix} Z_{i1}^T & Z_{i2}^T & \cdots & Z_{ik}^T \end{bmatrix}^T$, where

$$Z_{ij} = \begin{cases} X_i^{(2)} & \text{if } j = t_i; \\ \mathbf{0} & \text{otherwise} \end{cases},$$

for $j = 1, 2, \ldots, k$; $i = 1, 2, \ldots, p$.
Then $rank(Q) = r + p$, where $Q = \begin{bmatrix} P_{S_1} & P_{S_2} \end{bmatrix}$ and

$$AQ = Q \begin{bmatrix} E_{S_1} & \mathbf{0} \\ \mathbf{0} & E_{S_2} \end{bmatrix}.$$

## Corollary 1.1.

*Let $A_{ij}$ be a square matrix of order $n$ for $i, j \in \beta$. Let $X^{(1)}, X^{(2)}, \ldots, X^{(r)}$ be linearly independent eigenvectors of $A_{ij}$ corresponding to the eigenvalues $a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(r)}$, respectively for $i, j \in \beta$. Then we have the following.*

(1) *If $A_{ij}X^{(h)} = \mathbf{0}$ for $i \in \beta^c$; $j \in \beta$, then*

$$\sigma(A) \supseteq \sum_{h=1}^{r} \sigma(E_h),$$

*where*

$$E_h = \begin{bmatrix} a_{s_1 s_1}^{(h)} & a_{s_1 s_2}^{(h)} & \cdots & a_{s_1 s_t}^{(h)} \\ a_{s_2 s_1}^{(h)} & a_{s_2 s_2}^{(h)} & \cdots & a_{s_2 s_t}^{(h)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s_t s_1}^{(h)} & a_{s_t s_2}^{(h)} & \cdots & a_{s_t s_t}^{(h)} \end{bmatrix}$$

*for $h = 1, 2, \ldots, r$.*

(2) *If $A_{ij} = c_{ij} J_{n \times n}$, where $c_{ij} \in \mathbb{C}$ for $i \in \beta^c$; $j \in \beta$ and $X^{(h)}$ is orthogonal to $J_{n \times 1}$ for $h = 1, 2, \ldots, r$, then $\sigma(A) \supseteq \sum_{h=1}^{r} \sigma(E_h)$, where $E_h$ is as mentioned in part (1).*

## Proof.

1. For $h = 1, 2, \ldots, r$, let $S_h = (X_{s_1}^{(h)}, X_{s_2}^{(h)}, \ldots, X_{s_t}^{(h)})$, where $X_{s_i}^{(h)} = X^{(h)}$ for $i = 1, 2, \ldots, t$. Since each pair $S_i$, $S_j$ satisfies the constraints of Proposition 1.2 for $i, j = 1, 2, \ldots, r$; $i \neq j$, the result follows. Here we denote $E_{S_h}$ by $E_h$.
2. Since $X^{(h)}$ is orthogonal to $J_{n\times 1}$ for each $h = 1, 2, \ldots, r$, $A_{ij}X^{(h)} = \mathbf{0}$ for $i \in \beta^c$; $j \in \beta$. So, the result follows by using part (1) of this corollary.

□

**Example 1** Consider the matrix $M = \begin{bmatrix} 1 & 4 & 2 & 2 \\ -4 & 3 & 2 & 2 \\ 0 & 5 & -2 & 4 \\ -6 & 3 & 4 & -2 \end{bmatrix}$.

$M = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where

$A_{11} = \begin{bmatrix} 1 & 4 \\ -4 & 3 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$,

$A_{21} = \begin{bmatrix} 0 & 5 \\ -6 & 3 \end{bmatrix}$ and $A_{22} = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$.

Here $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A_{22}$ corresponding to the eigenvalue $-6$, which is orthogonal to $J_{2\times 1}$.

So, by using Corollary 1.1 (2), $-6$ is an eigenvalue of $M$.

## Corollary 1.2.

*([3, Corollary 2]) If $A_{ij}$ for $i, j = 1, 2, \ldots, k$ are real symmetric matrices of order $n$ such that they commutes with each other, then*

$$\sigma(A) = \sum_{h=1}^{n} \sigma(E_h),$$

*where*

$$E_h = \begin{bmatrix} a_{11}^{(h)} & a_{12}^{(h)} & \cdots & a_{1k}^{(h)} \\ a_{21}^{(h)} & a_{22}^{(h)} & \cdots & a_{2k}^{(h)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}^{(h)} & a_{k2}^{(h)} & \cdots & a_{kk}^{(h)} \end{bmatrix},$$

*with $a_{ij}^{(h)}$ is an eigenvalue of $A_{ij}$ corresponding to the same eigenvector $X$ for each $i, j = 1, 2, \ldots, k$; $h = 1, 2, \ldots, n$.*

# Partitioned matrices with generalized stochastic matrices as its blocks

- A matrix $M$ is said to be **generalized stochastic**, if $M \in \mathcal{R}_{n\times m}(r)$ for some $r \in \mathbb{C}$.
- The matrix $A$ is said to be **block-stochastic matrix**, if each $A_{ij} \in \mathcal{R}_{n_i\times n_j}(a_{ij})$ for $i, j = 1, 2, \ldots, k$. We denote the matrix $\delta_A := \left[a_{ij}\right]$ for $i, j = 1, 2, \ldots, k$

## Corollary 1.3.

*If $A_{ij} \in \mathcal{R}_{n_i\times n_j}(a_{ij})$ for $i, j \in \beta$ and $A_{ij} = \mathcal{R}_{n_i\times n_j}(0)$ for $i \in \beta^c$ and $j \in \beta$, then $\sigma(A) \supseteq \sigma(\delta_{[A,\beta]})$, where*

$$\delta_{[A,\beta]} = \begin{bmatrix} a_{s_1s_1} & a_{s_1s_2} & \cdots & a_{s_1s_t} \\ a_{s_2s_1} & a_{s_2s_2} & \cdots & a_{s_2s_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s_ts_1} & a_{s_ts_2} & \cdots & a_{s_ts_t} \end{bmatrix}.$$

## Proof.

Let $X_i = J_{n_i\times 1}$ for $i \in \beta$ and let $S = (X_{s_1}, X_{s_2}, \ldots, X_{s_t})$. Then $S$, $A$ and $a_{ij}$ satisfies (1.3). So the result follows from Theorem 1.3. $\square$

**Example 2**

Consider the matrix $M = \begin{bmatrix} 1 & 4 & 1 & -1 \\ -4 & 3 & -2 & 2 \\ 0 & 5 & -2 & 5 \\ -6 & 3 & 4 & -1 \end{bmatrix}$.

We can partition $M$ as $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$,

where $A_{11} = \begin{bmatrix} 1 & 4 \\ -4 & 3 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$,

$A_{21} = \begin{bmatrix} 0 & 5 \\ -6 & 3 \end{bmatrix}$ and $A_{22} = \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}$.

Notice that $A_{12} \in \mathcal{R}_{2\times 2}(0)$ and $A_{22} = \mathcal{R}_{2\times 2}(3)$.

So, taking $\beta = \{2\}$ in Corollary 1.3, we can obtain that 3 is an eigenvalue of $M$.

The following result, which was proved by Haynsworth [2] in 1959.

## Corollary 1.4.

*([2, Theorem 2]) If A is block-stochastic with* $A_{ij} = \left[a_{hq}^{(ij)}\right]$, $h = 1, 2, \ldots, n_i$; $q = 1, 2, \ldots, n_j$; $i, j = 1, 2, \ldots, k$, *then*

$$\sigma(A) = \sigma(\delta_A) + \sigma(C),$$

*where*

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \cdots & C_{kk} \end{bmatrix} \tag{1.4}$$

*with* $C_{ij} = [a_{hq}^{(ij)} - a_{h1}^{(ij)}]$ *for* $h = 2, 3, \ldots, n_i$; $q = 2, 3, \ldots, n_j$; $i, j = 1, 2, \ldots, k$. *If either* $n_i$ *or* $n_j$ *is 1, then the block* $C_{ij}$ *is omitted, so* $i$ *and* $j$ *do not necessarily take all values of* $1, 2, \ldots, k$.

## Corollary 1.5.

*Let $A_{ii}$ be a block-stochastic matrix for $i = 1, 2, \ldots, k$ and let*

$$A_{ij} = \begin{bmatrix} \rho_{11}^{(ij)} J_{n_{i1} \times n_{j1}} & \rho_{12}^{(ij)} J_{n_{i1} \times n_{j2}} & \cdots & \rho_{1p_j}^{(ij)} J_{n_{i1} \times n_{jp_j}} \\ \rho_{21}^{(ij)} J_{n_{i2} \times n_{j1}} & \rho_{22}^{(ij)} J_{n_{i2} \times n_{j2}} & \cdots & \rho_{2p_j}^{(ij)} J_{n_{i2} \times n_{jp_j}} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p_i 1}^{(ij)} J_{n_{ip_i} \times n_{j1}} & \rho_{p_i 2}^{(ij)} J_{n_{ip_i} \times n_{j2}} & \cdots & \rho_{p_i p_j}^{(ij)} J_{n_{ip_i} \times n_{jp_j}} \end{bmatrix},$$

*where $\rho_{hq}^{(ij)} \in \mathbb{C}$ for $h = 1, 2, \ldots, p_i$; $q = 1, 2, \ldots, p_j$; $i, j = 1, 2, \ldots, k$; $i \neq j$. Then*

$$\sigma(A) = \sigma(\delta_A) + \sum_{i=1}^{k} \left[\sigma(A_{ii}) \setminus \sigma(\delta_{A_{ii}})\right].$$

## Proof.

Since $A_{ii}$ is block stochastic, by Corollary 1.4,

$$\sigma(A_{ii}) = \sigma(\delta_{A_{ii}}) + \sigma(C^{(i)}), \tag{1.5}$$

where $C^{(i)}$ can be obtained from (1.4) for $i = 1, 2, \ldots, k$.
From (1.5), we can obtain that

$$\sigma(C^{(i)}) = \sigma(A_{ii}) \setminus \sigma(\delta_{A_{ii}}). \tag{1.6}$$

Since $A$ is block stochastic, again by using Corollary 1.4, we obtain that

$$\sigma(A) = \sigma(\delta_A) + \sigma(C), \tag{1.7}$$

where $C$ is as given in (1.4) with

$$C_{ij} = \begin{cases} C^{(i)} & \text{for } i = j; \\ \mathbf{0} & \text{for } i \neq j, \end{cases}$$

for $i, j = 1, 2, \ldots, k$. So,

$$\sigma(C) = \sum^{k} \sigma(C^{(i)}) = \sum^{k} \left[\sigma(A_{ii}) \setminus \sigma(\delta_{A_{ii}})\right].$$

## Corollary 1.6.

*Let* $A_i \in \mathcal{R}_{n_i \times n_i}(a_i)$ *for* $i = 1, 2, \dots, k$. *Let*

$$A = \begin{bmatrix} A_1 & \rho_{12} J_{n_1 \times n_2} & \dots & \rho_{1k} J_{n_1 \times n_k} \\ \rho_{21} J_{n_2 \times n_1} & A_2 & \dots & \rho_{2k} J_{n_2 \times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k1} J_{n_k \times n_1} & \rho_{k2} J_{n_k \times n_2} & \dots & A_k \end{bmatrix},$$

*where* $\rho_{ij} \in \mathbb{C}$ *for* $i, j = 1, 2, \dots, k$; $i \neq j$. *Then*

$$\sigma(A) = \sigma(\delta_A) + \sum_{i=1}^{k} \left[\sigma(A_i) \setminus \{a_i\}\right],$$

*where*

$$\delta_A = \begin{bmatrix} a_1 & \rho_{12} n_2 & \dots & \rho_{1k} n_k \\ \rho_{21} n_1 & a_2 & \dots & \rho_{2k} n_k \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k1} n_1 & \rho_{k2} n_2 & \dots & a_k \end{bmatrix}.$$

## Corollary 1.7.

*Let* $M \in \mathcal{RC}_{n\times m}(p_1,p_2)$ *and* $B_{ij} = b_{ij}I_n + b'_{ij}J_n + b''_{ij}MM^T$, $P_{is} = p_{is}J_{n\times m} + p'_{is}M$, $Q_{hj} = q_{hj}J_{m\times n} + q'_{hj}M^T$ *and* $C_{hs} = c_{hs}I_m + c'_{hs}J_m + c''_{hs}M^TM$, *where* $b_{ij}, b'_{ij}, b''_{ij}, p_{is}, p'_{is}, q_{hj}, q'_{hj}, c_{hs}, c'_{hs}, c''_{hs} \in \mathbb{R}$ *for* $i,j = 1,2,\ldots,k_1$; $h,s = 1,2,\ldots,k_2$. *Let*

$$A = \begin{bmatrix} B & P \\ Q & C \end{bmatrix}, \tag{1.8}$$

*where*

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k_1} \\ B_{21} & B_{22} & \cdots & B_{2k_1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k_11} & B_{k_12} & \cdots & B_{k_1k_1} \end{bmatrix}, \qquad P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1k_2} \\ P_{21} & P_{22} & \cdots & P_{2k_2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k_11} & P_{k_12} & \cdots & P_{k_1k_2} \end{bmatrix},$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1k_1} \\ Q_{21} & Q_{22} & \cdots & Q_{2k_1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k_21} & Q_{k_22} & \cdots & Q_{k_2k_1} \end{bmatrix}, \qquad C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k_2} \\ C_{21} & C_{22} & \cdots & C_{2k_2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k_21} & C_{k_22} & \cdots & C_{k_2k_2} \end{bmatrix}.$$

Let $r = \begin{cases} rank(M)+1 & \text{if } p_1 = 0 \text{ or } p_2 = 0; \\ rank(M) & \text{otherwise.} \end{cases}$

Then

$$\sigma(A) = \sigma(\delta_A) + (n-r)\sigma(B') + (m-r)\sigma(C') + \sum_{0 \neq \lambda_t \in \sigma(MM^T) \setminus \{p_1 p_2\}} \sigma(E_{\lambda_t}), \tag{1.9}$$

$$\begin{aligned} B' &= [b_{ij}] \text{ for } i,j = 1,2,\ldots,k_1; \\ C' &= [c_{ij}] \text{ for } i,j = 1,2,\ldots,k_2; \end{aligned}$$

and

$$E_{\lambda_t} = \begin{bmatrix} E_{1t} & \lambda_t E_{2t} \\ E_{3t} & E_{4t} \end{bmatrix},$$

with

$$E_{1t} = [b_{ij} + \lambda_t b''_{ij}], \quad E_{2t} = [p'_{is}], \quad E_{3t} = [q'_{hj}], \quad E_{4t} = [c_{hs} + \lambda_t c''_{hs}],$$

for all $t$ such that $0 \neq \lambda_t \in \sigma(MM^T) \setminus \{p_1 p_2\}$; $i,j = 1,2,\ldots,k_1$; $h,s = 1,2,\ldots,k_2$.

**Example 3**

Consider the matrix

$$A = \left[\begin{array}{ccc|cccc} 3 & -2 & 3 & 1 & -2 & -1 & 2 \\ -2 & 11 & -5 & -3 & 6 & 3 & -6 \\ 3 & -5 & 6 & 2 & -4 & -2 & 4 \\ \hline 2 & 2 & 2 & 2 & -3 & -2 & 1 \\ 2 & 2 & 2 & -3 & 5 & 1 & -5 \\ 2 & 2 & 2 & -2 & 1 & 2 & -3 \\ 2 & 2 & 2 & 1 & -5 & -3 & 5 \end{array}\right].$$

Taking $M = \begin{bmatrix} 1 & -2 & -1 & 2 \\ -3 & 6 & 3 & -6 \\ 2 & -4 & -2 & 4 \end{bmatrix}$, $A$ can be viewed as

$$\begin{bmatrix} B_{11} & P_{11} \\ Q_{11} & C_{11} \end{bmatrix},$$

where $B_{11} = I_3 + J_2 + \frac{1}{10}MM^T$, $P_{11} = M$, $Q_{11} = 2J_{4\times 3}$, $C_{11} = 2I_4 - J_4 + \frac{1}{14}M^TM$. Taking $p_1 = 0$ and $rank(M) = 1$, $r = 2$ in Corollary 1.7, we get

$$\sigma(A) = \sigma(\delta_A) + \sigma(E_{\lambda_t}) + \sigma(B') + 2\sigma(C'),$$

where $\delta_A = \begin{bmatrix} 4 & 0 \\ 6 & -2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 15 & 140 \\ 0 & 12 \end{bmatrix}$, $B' = [1]$ and $C' = [2]$.
Thus $\sigma(A) = \{-2, 1, 2, 2, 4, 12, 15\}$.

**Example 4** Consider the matrix

$$A = \left[\begin{array}{ccc|cccc|cc} 3 & -2 & 3 & 1 & -2 & -1 & 2 & 0 & 0 \\ -2 & 11 & -5 & -3 & 6 & 3 & -6 & 0 & 0 \\ 3 & -5 & 6 & 2 & -4 & -2 & 4 & 0 & 0 \\ \hline 2 & 2 & 2 & 2 & -3 & -2 & 1 & -2 & -2 \\ 2 & 2 & 2 & -3 & 5 & 1 & -5 & -2 & -2 \\ 2 & 2 & 2 & -2 & 1 & 2 & -3 & -2 & -2 \\ 2 & 2 & 2 & 1 & -5 & -3 & 5 & -2 & -2 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 2 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \end{array}\right].$$

Then $A$ can be viewed as

$$A = \left[\begin{array}{c|c|c} A_{11}^{(1)} & A_{12}^{(1)} & \mathbf{0} \\ \hline A_{21}^{(1)} & A_{22}^{(1)} & -2J_{4\times 2} \\ \hline -J_{2\times 3} & \mathbf{0} & A_{11}^{(2)} \end{array}\right],$$

where $A_{11}^{(1)}$, $A_{12}^{(1)}$ $A_{21}^{(1)}$ and $A_{22}^{(1)}$ are the blocks of $A$ as mentioned above. Then by using Corollary 1.5, we have

$$\sigma(A) = \sigma(\delta_A) + [\sigma(M_{11}) \setminus \sigma(\delta_{M_{11}})] + [\sigma(M_{22}) \setminus \sigma(\delta_{M_{22}})],$$

where $\delta_A = \begin{bmatrix} 4 & 0 & 0 \\ 6 & -2 & -4 \\ -3 & 0 & 1 \end{bmatrix}$, $M_{11} = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{bmatrix}$, $M_{22} = A_{11}^{(2)}$.
Notice that $\sigma(\delta_A) = \{4, -2, 1\}$, $\sigma(M_{22}) = \{1, -3\}$, $\delta(M_{22}) = \begin{bmatrix} 1 \end{bmatrix}$ and by using Example 3, $\sigma(M_{11}) \setminus \sigma(\delta_{M_{11}}) = \{1, 2, 2, 12, 15\}$.
Thus we have $\sigma(A) = \{4, -2, 1, 1, 2, 2, 12, 15, -3\}$.

# Eigenvectors of some partitioned matrices

Let $x$ be an eigenvalue of $E_S$ with an eigenvector $Y = \begin{bmatrix} c_{s_1} & c_{s_2} & \dots & c_{s_t} \end{bmatrix}^T$. Then we have

$$E_S Y = xY.$$

From this, we obtain

$$c_{s_1} a_{is_1} + c_{s_2} a_{is_2} + \cdots + c_{s_t} a_{is_t} = c_{s_i} x \tag{1.10}$$

for each $i \in \beta$. Let

$$Z = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_k \end{bmatrix}^T,$$

where $Z_i = \begin{cases} c_i X_i & \text{if } i \in \beta; \\ \mathbf{0} & \text{if } i \in \beta^c. \end{cases}$

Then by using (1.10) and (1.3), it can be verified that

$$AZ = xZ.$$

Therefore, $Z$ is an eigenvector of $A$ corresponding to the eigenvalue $x$.

**Construction of eigenvectors of $A$:** We proceed to construct the eigenvectors of $A$ corresponding to the eigenvalues mentioned in Theorem 1.3.
Consider the following matrix equation:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_k \end{bmatrix} = x \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_k \end{bmatrix},$$

where

$$Z_i = \begin{cases} c_i X_i & \text{if } i = s_1, s_2, \ldots, s_{t-1}; \\ X_{s_t} & \text{if } i = s_t; \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

with $c_i \in \mathbb{C}$ for $i = 1, 2, \ldots, k$. Then we have the following system of equations:

$$\left.\begin{aligned} c_{s_1}(x - a_{s_1 s_1}) - c_{s_2} a_{s_1 s_2} - \cdots - c_{s_{t-1}} a_{s_1 s_{t-1}} - a_{s_1 s_t} &= 0 \\ -c_{s_1} a_{s_2 s_1} + c_{s_2}(x - a_{s_2 s_2}) - \cdots - c_{s_{t-1}} a_{s_2 s_{t-1}} - a_{s_2 s_t} &= 0 \\ &\vdots \\ -c_{s_1} a_{s_{t-1} s_1} - c_{s_2} a_{s_{t-1} s_2} - \cdots + c_{s_{t-1}}(x - a_{s_{t-1} s_{t-1}}) - a_{s_{t-1} s_t} &= 0 \end{aligned}\right\} \quad (1.11)$$

$$-c_{s_1}a_{s_ts_1}-c_{s_2}a_{s_ts_2}-\cdots-c_{s_{t-1}}a_{s_ts_{t-1}}+(x-a_{s_ts_t}) = 0. \qquad (1.12)$$

Notice that, (1.11) can be written as

$$PC=X,$$

where

$$\begin{aligned} P &= \begin{bmatrix} x-a_{s_1s_1} & -a_{s_1s_2} & \cdots & -a_{s_1s_{t-1}} \\ -a_{s_2s_1} & x-a_{s_2s_2} & \cdots & -a_{s_2s_{t-1}} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{s_{t-1}s_1} & -a_{s_{t-1}s_2} & \cdots & x-a_{s_{t-1}s_{t-1}} \end{bmatrix}, \\ C &= \begin{bmatrix} c_{s_1} & c_{s_2} & \cdots & c_{s_{t-1}} \end{bmatrix}^T, \\ X &= \begin{bmatrix} a_{s_1s_t} & a_{s_2s_t} & \ldots & a_{s_{t-1}s_t} \end{bmatrix}^T. \end{aligned}$$

Then

$$C=P^{-1}X. \qquad (1.13)$$

Let $P_{ij}$ be the co-factor of the $(i, j)$-th entry of $P$. Then by (1.13), for each $j = s_1, s_2, \ldots, s_{t-1}$, we have,

$$c_j = \frac{1}{|P|} \sum_{i=1}^{t} a_{s_i s_t} P_{ji}. \tag{1.14}$$

Substituting the values of $c_{s_1}, c_{s_2}, \ldots, c_{s_{t-1}}$ in (1.12), we get

$$|xI_t - E_S| = 0.$$

It follows that, $\begin{bmatrix} Z_1 & Z_2 & \cdots & Z_k \end{bmatrix}^T$ is an eigenvector of $A$ corresponding to the eigenvalue $x$ of $E_S$ , where $c_{s_j}$ is given in (1.14) for $j = 1, 2, \ldots, t-1$. □

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