Inverses of Non-bipartite Unicyclic Graphs

A Talk in
E-Seminar@IITKGP
by
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What is an Adjacency Matrix of a graph?

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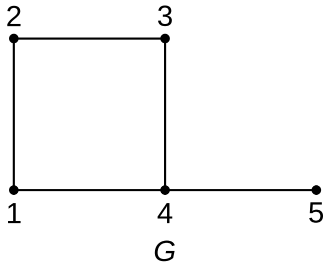
Definition

- The adjacency matrix $A(G) = [a_{ij}]$ of a graph G on vertices $1, \ldots, n$ is the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

Example



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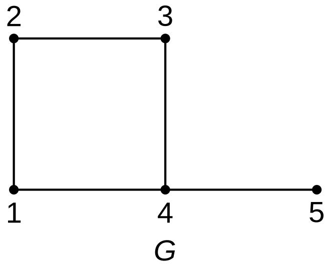

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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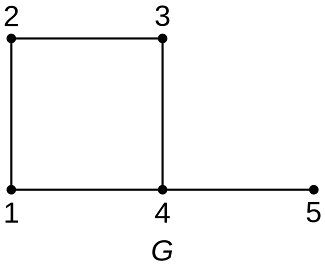

Singular

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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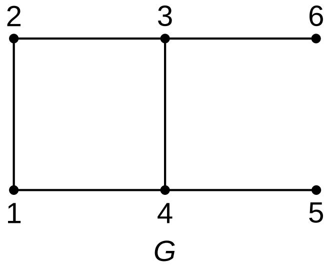

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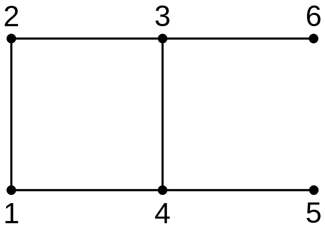

G

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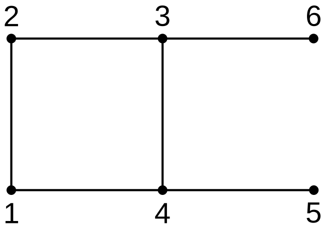

G

Non-singular

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

What is a Singular Graph?

Definition

- A graph G is called singular if $A(G)$ is *singular*, otherwise it is called nonsingular.

What is a perfect Matching?

- A spanning subgraph H of a graph G is called a spanning linear subgraph if each component of H is either an edge or a cycle.
- A perfect matching is a spanning linear subgraph whose components are edges only.

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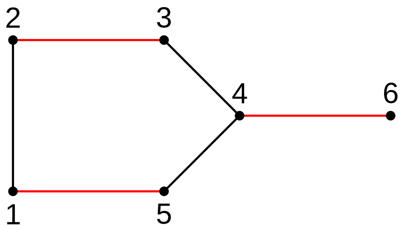

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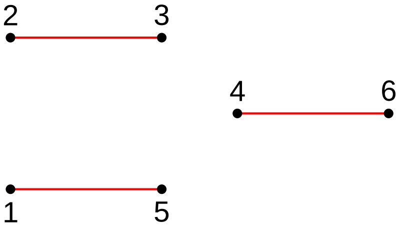

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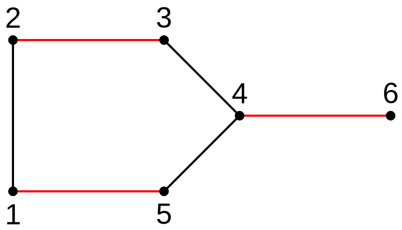

A Formula for the Determinant

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Theorem (Harary, Sachs)

If G is a graph of order n, then

$$\det(A(G)) = \sum_H (-1)^{n-P_H-C_H} 2^{C_H},$$

where the summation is taken over all spanning linear subgraphs of G, P_H and C_H are the number of components in H which are edges and cycles, respectively.

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- If a graph has a unique perfect matching, then it is non-singular.

Invertible Graph

Definition

[4, Harary and Minc] A nonsingular graph G is said to be invertible if $B = A(G)^{-1}$ is a matrix with entries 0 or 1.

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Theorem (Harary and Minc)

A connected graph G is invertible if and only if $G = P_2$

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- Godsil posed the problem of characterizing the bipartite graphs with a unique perfect matching that possess an inverse.

Key Work

- Barik, Neumann and Pati described the inverse of the adjacency matrix of a bipartite graph with a unique perfect matching.
- Akbari and Kirkland characterized the bipartite, unicyclic graphs with a unique perfect matching that possess an inverse.
- Tifenbach and Kirkland identified those that are self-inverse.
- Panda supplied a characterization of bipartite, unicyclic graphs with a unique perfect matching that possess bicyclic inverses.

The inverse Graph

- Let G be a nonsingular graph and $A(G)^{-1} = (\alpha_{ij})$.
- By G^+, we denote the graph on the same vertex set as that of G constructed as follows: two distinct vertices i and j are adjacent in G^+ if and only if $\alpha_{ij} \neq 0$.
- We call G^+ the inverse graph of G.

Terminology

- By $\mathcal{P}_{ij}$, we denote the set of all i-j paths P in G such that $G - V(P)$ has a spanning linear subgraph.
- In a non-bipartite unicyclic graph with a unique perfect matching

$$|\mathcal{P}_{i,j}| = 0 \text{ or } 1 \text{ for any } i, j.$$

- A path $P = [u_1, u_2, \ldots, u_k]$ is called *mm*-alternating if the the edges on P are alternately matching and non-matching edges, with $[u_1, u_2]$ and $[u_{k-1}, u_k]$ as matching edges.

Formula for the Inverse

Formula for the Inverse

Lemma (Kalita and Sarma)

Let U be a non-bipartite unicyclic graph of order n with a unique perfect matching. Then U is invertible and for $i \neq j$ the ij-th entry α_{ij} of the adjacency matrix of U^+ is given by

$$\alpha_{ij} = \begin{cases} (-1)^{\frac{n}{2}+|C_H|+|E(H)|+|E(P)|}\, 2^{|C_H|} & \text{if } |\mathcal{P}_{ij}| = 1 \\ 0 & \text{if } \mathcal{P}_{ij} = \emptyset, \end{cases}$$

where $P \in \mathcal{P}_{ij}$, H is the spanning elementary subgraph of $U - V(P)$, $|C_H|$ is the number of cycle components of H.
Furthermore, $\alpha_{ii} = 0$ or ± 2.

Adjacency in the Inverse Graph

Adjacency in the Inverse Graph

Theorem (Kalita and Sarma)

Let U be a non-bipartite unicyclic graph with a unique perfect matching, and let i, j be two vertices of U. Then $i \sim j$ in U^+ if and only if U has an mm-alternating path between i and j

What is a Mixed Graph?

Definition

- A mixed graph is a graph with two different types of edges, say red and blue.

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- A mixed graph is a graph with two different types of edges, say red and blue.
- The adjacency matrix $A(G) = [a_{ij}]$ of a mixed graph G is the matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ and } [i,j] \text{ is a red edge} \\ -1 & \text{if } i \sim j \text{ and } [i,j] \text{ is a blue edge} \\ 0 & \text{else.} \end{cases}$$

When is the Inverse of a Graph a Mixed Graph?

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- What is a Peg?

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Let U be a unicyclic graph. A matching edge that is incident with exactly one vertex of the cycle is called a **peg**.

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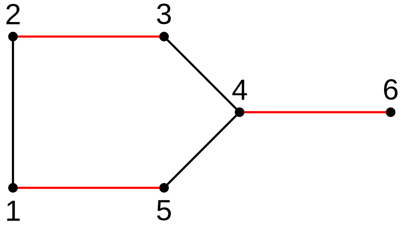
Example

Red edges are the matching edges. The edge $[4, 6]$ is a peg.



When is the Inverse of a graph a Mixed Graph?

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Theorem (Kalita and Sarma)

Let U be a non-bipartite, unicyclic graph with a unique perfect matching. Then U^+ is a mixed graph if and only if U has at least three pegs.

Adjacency matrix of the Inverse

Theorem (Kalita and Sarma)

Let U be a non-bipartite unicyclic graph with a unique perfect matching such that U^+ is a mixed graph. Suppose that $A(U^+) = (\alpha_{ij})$. Then

$$\alpha_{ij} = \begin{cases} (-1)^{\frac{||P||-1}{2}} & \text{if } U \text{ contains an } i-j \text{ mm-alternating path } P, \\ 0 & \text{otherwise .} \end{cases}$$

Here $||P||$ denotes the number of edges in P.

Bipartiteness of the Inverse Graph

Lemma (Kalita and Sarma)

Let U be a non-bipartite unicyclic graph with a unique perfect matching. If U has r pegs, then U^+ contains a cycle of length r for $r \geq 3$. In particular, if $r = 1$, then U^+ contains a triangle.

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Let U be a non-bipartite unicyclic graph with a unique perfect matching. If U has r pegs, then U^+ contains a cycle of length r for $r \geq 3$. In particular, if $r = 1$, then U^+ contains a triangle.

Theorem (Kalita and Sarma)

Let U be a non-bipartite unicyclic graph with a unique perfect matching. Then the inverse graph U^+ is always non-bipartite.

What is a quasi-bipartite Graph?

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Definition

A mixed graph G is called quasi-bipartite if there exists a partition $V(G) = V_1 \cup V_2$ such that every edge between V_1 and V_2 is red and every edge within V_1 and V_2 is blue.

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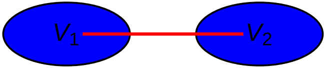

Is the Inverse graph quasi-bipartite?

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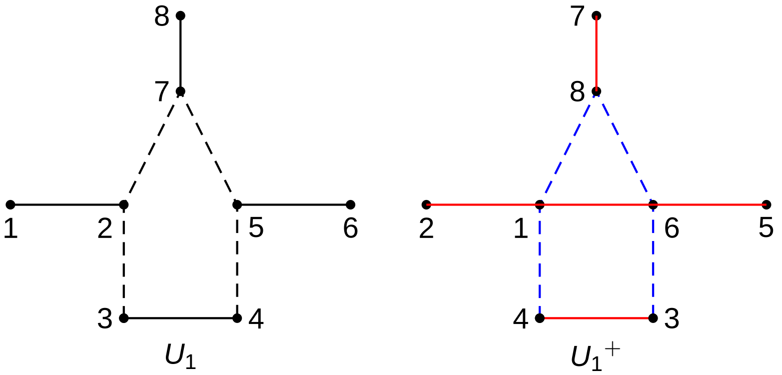


Inverse graph is not Quasi-bipartite

Quasi-Bipartiteness?

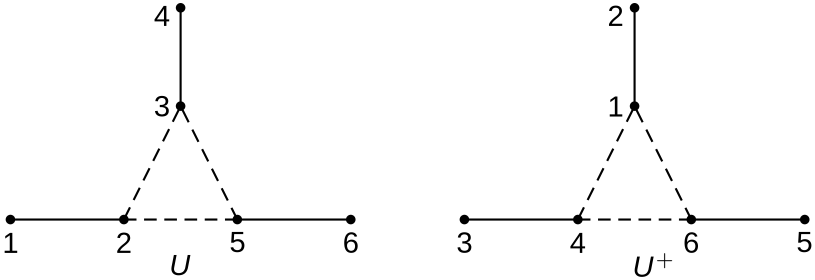
Quasi-Bipartiteness?

Theorem (Kalita and Sarma)

Let U be a non-bipartite unicyclic graph with a unique perfect matching such that U^+ is a mixed graph. Then U^+ is quasi-bipartite if and only if the number of matching edges in the cycle is even.

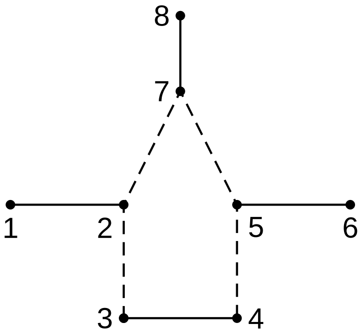
Structure of Inverse Graph

Structure of Inverse Graph

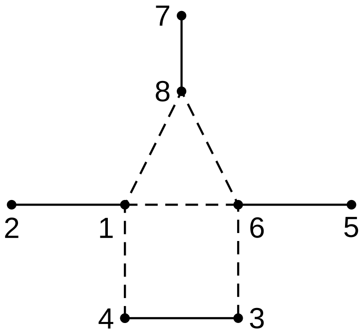


Inverse graph is Unicyclic

Structure of Inverse Graph



U_1



U_1^+

Inverse graph is bicyclic

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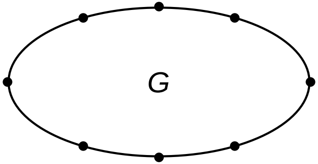

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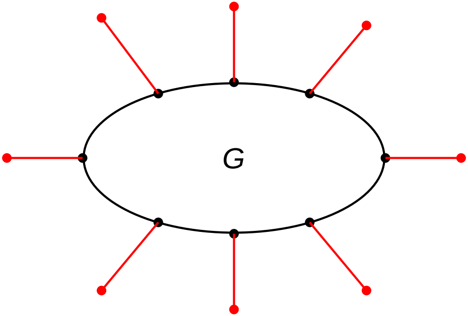


Unicyclic inverse

Unicyclic inverse

Theorem

Let U be a non-bipartite unicyclic graph with a unique perfect matching such that U^+ is a mixed graph. Then the following are equivalent:

(i) U^+ is unicyclic,

(ii) U is a simple corona,

(iii) $U \cong U^+$.

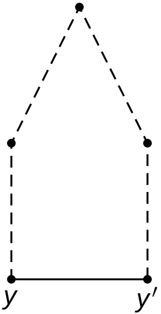
Construction of Type-A graphs

Construction of Type-A graphs

Let $\mathcal{U}_1$ be the class of non-bipartite unicyclic graphs constructed in the following steps:

1. Take an odd cycle $\Gamma = [v_1, v_2, \ldots, v_n, v_1]$, where $n \geq 5$.
2. Take $n-2$ simple corona trees $T_1, T_2, \ldots, T_{n-2}$. Choose a quasi-pendant vertex u_i in T_i for $i = 1, \ldots, n-2$. Attach T_i at the vertex v_i of Γ by identifying the vertex u_i for $i = 1, \ldots, n-2$.

Example of Type-A



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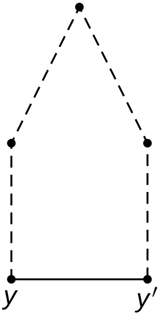
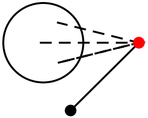


Corona Tree

T_1



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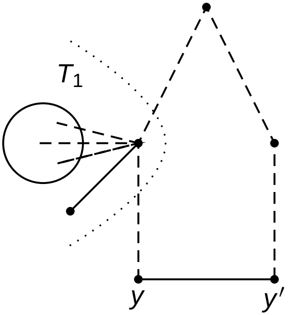

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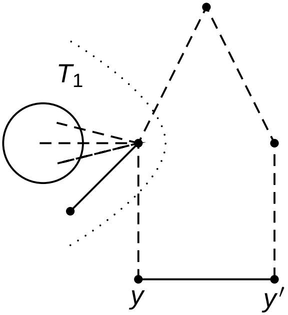

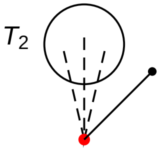

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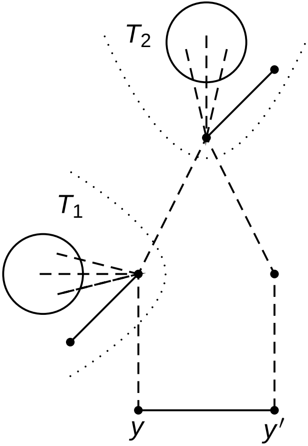

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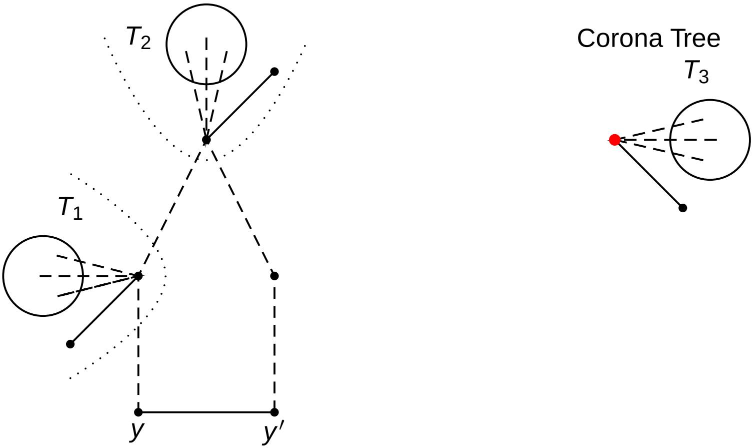

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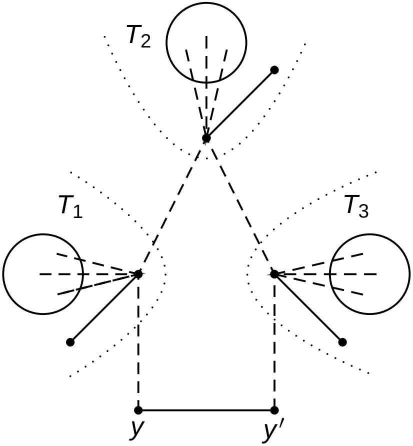


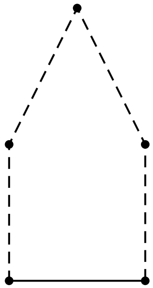
Construction of Type-B graphs

Construction of Type-B graphs

Let $\mathcal{U}_2$ be the class of non-bipartite unicyclic graphs constructed in the following steps:

1. Take an odd cycle $\Gamma = [v_1, v_2, \ldots, v_n, v_1]$, where $n \geq 3$.
2. Take $n-1$ simple corona trees $T_1, T_2, \ldots, T_{n-1}$. Choose a quasi-pendant vertex u_i in T_i for $i = 1, \ldots, n-1$. Attach T_i at the vertex v_i of Γ by identifying the vertex u_i for $i = 1, \ldots, n-1$.
3. Take two corona trees T and T', an edge $P_2 = [x, x']$.
4. Pick quasi-pendant vertices w and w' of T and T', respectively.
5. Join T and T' by adding the edges $[w, x]$ and $[x', w']$ to obtain the new tree T_0.
6. Finally, attach T_0 at the vertex v_n of Γ, by identifying a quasi-pendant vertex of T_0.

Example of Type-B



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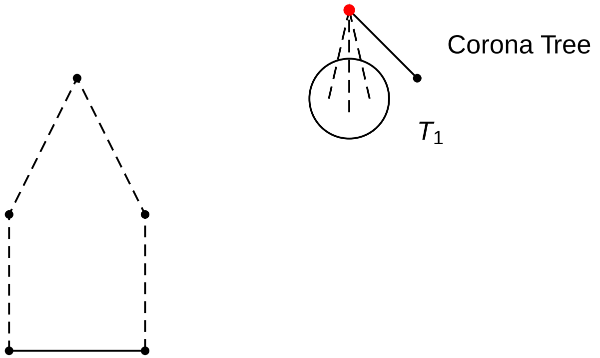

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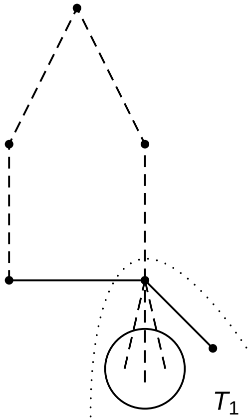

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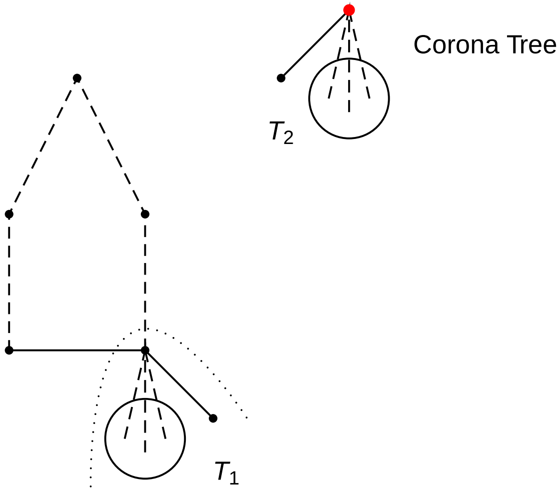

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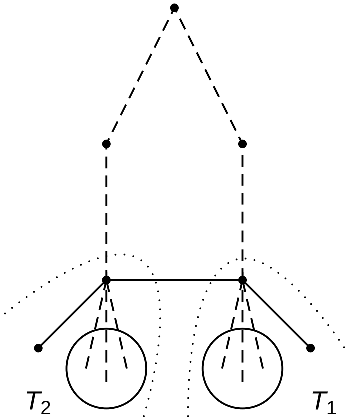

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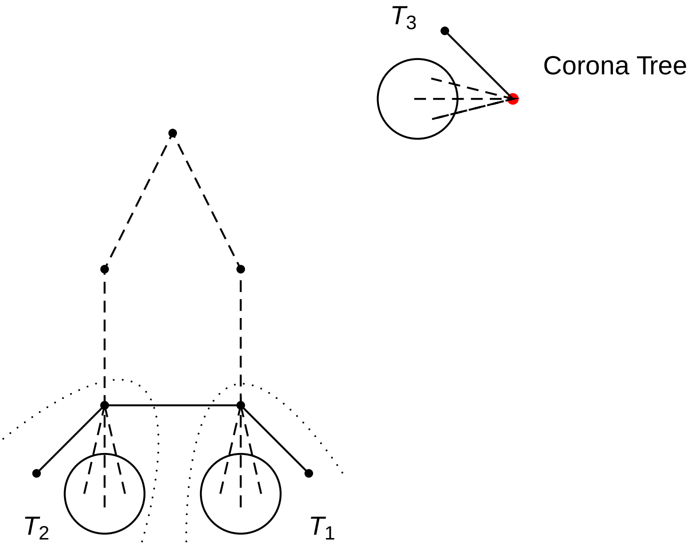

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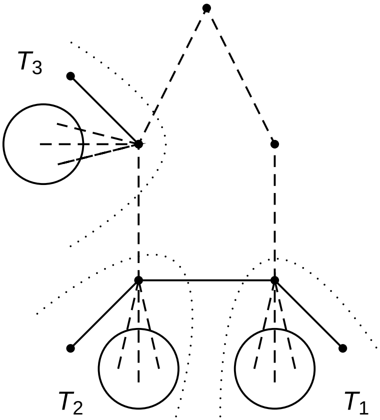

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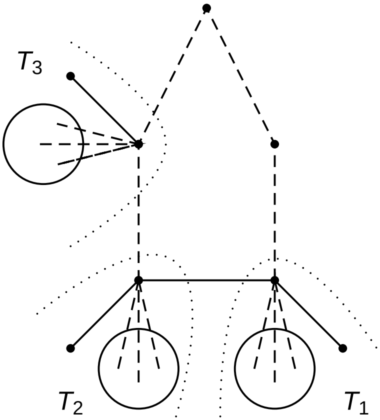
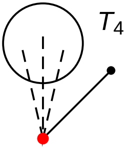
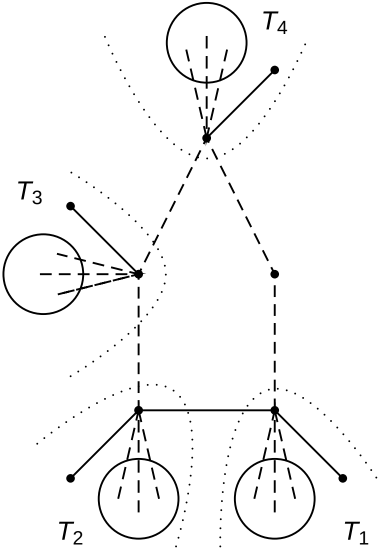




Corona Tree

Example of Type-B



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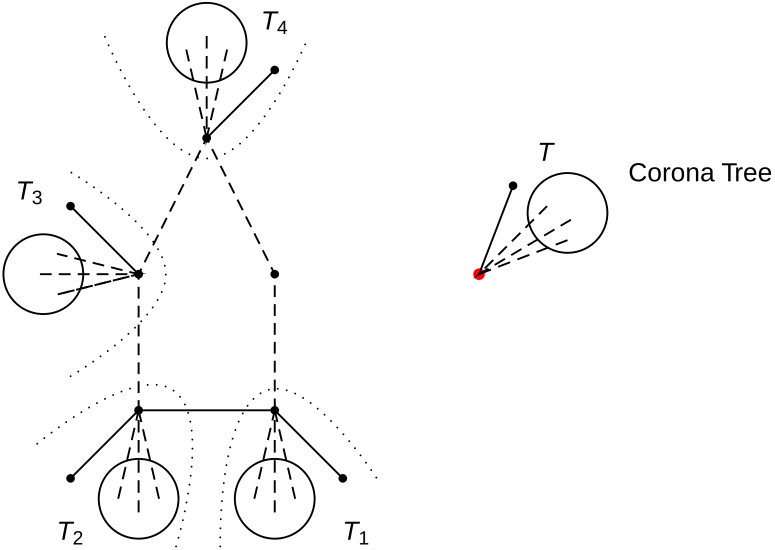

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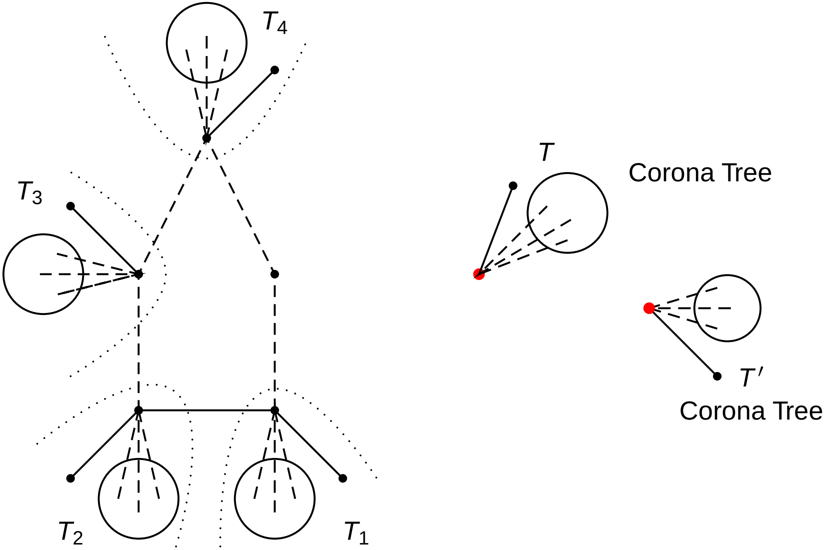

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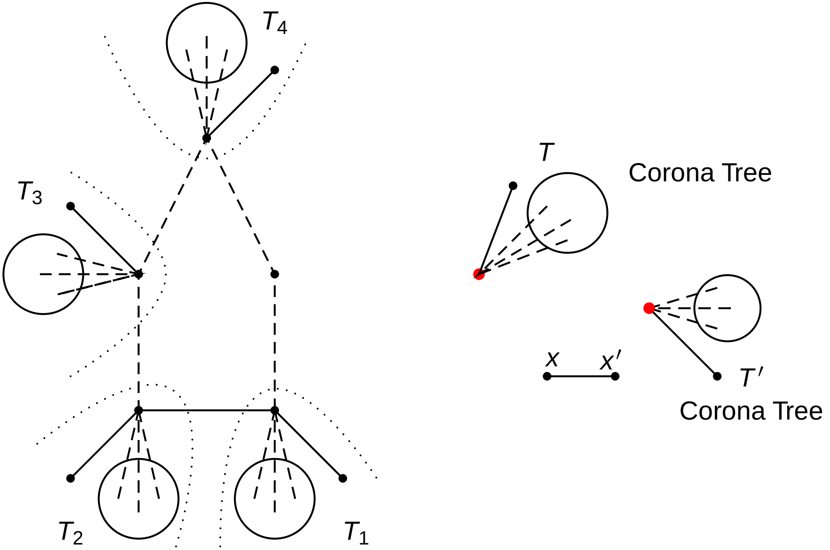

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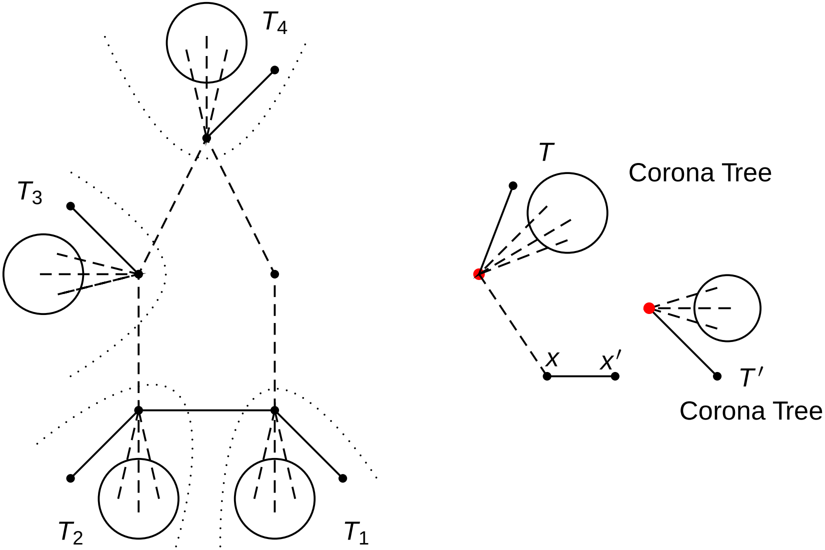
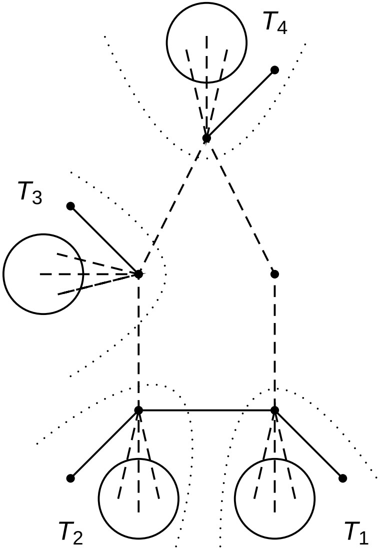
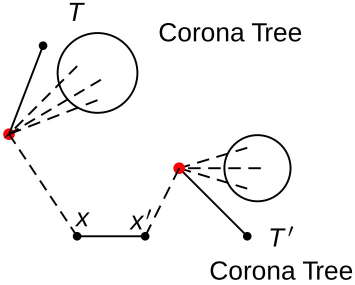
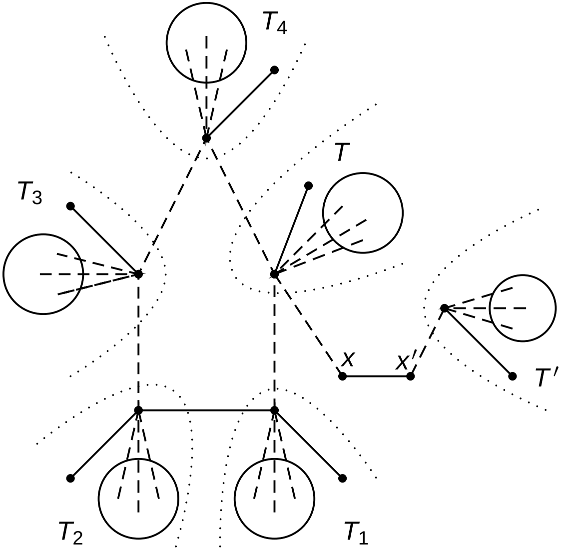


Example of Type-B





Example of Type-B



Bicyclic Inverse

Bicyclic Inverse

Theorem

Let U be a non-bipartite unicyclic graph with a unique perfect matching such that U^+ is a mixed graph. Then U^+ is bicyclic if and only if either $U \in \mathcal{U}_1$ or $U \in \mathcal{U}_2$.

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