Convex and quasiconvex functions on trees

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The first part is based on:

RBB, Kalita, Nath, Sarma;
Convex and quasiconvex functions on trees and their applications, Linear Algebra Appl., 2017

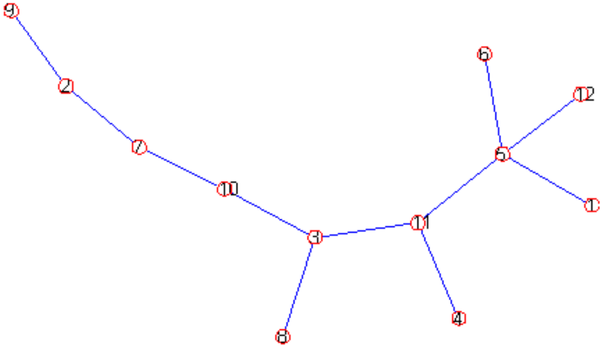
Matrices associated with a graph

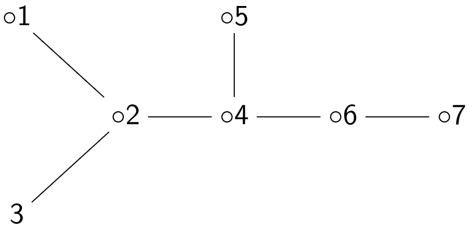
Adjacency matrix A

Laplacian matrix
$L = diag(d_1, \ldots, d_n) - A$

Distance matrix D

Trees





The distance matrix of the tree is given by

$$D = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix}$$

How do we define the “center” of a graph?

Classical notions:

Center

Median

Centroid

Corresponding functions on the vertex set:

Center —– Eccentricity

Median —– Transmission index

Centroid —– Maximum branch weight (or weight)

Eccentricity

Eccentricity of a vertex i is the maximum of $d(i,j)$ over j.

Center of a tree

The center of a graph is the subgraph induced by the vertices of minimum eccentricity.

Theorem (Jordan, 1869) The center of a tree consists of either a single vertex or a pair of adjacent vertices.

The center is located by a simple recursive procedure.

The eccentricity increases monotonically along any path starting at a center.

A proof using the pigeonhole principle

Graham, Entringer and Szekely,
American Math. Monthly, 1994

Median of a tree

Transmission index:

$$Tr(i) = \sum_j d(i,j)$$

The median of a graph is a vertex v with minimum transmission index.

In any graph there is either one median or two medians which are adjacent.

The transmission index is monotonically increasing along any path starting at a median.

Weight of a vertex:

The weight of v is the number of vertices in the largest component of $T - v$.

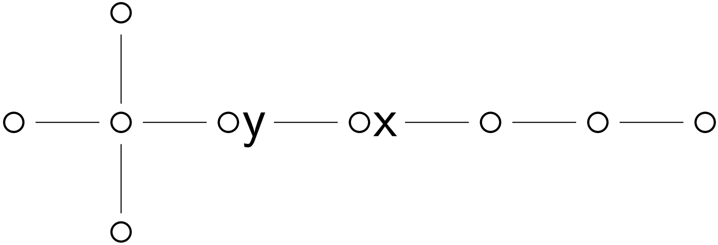
The vertex v is a centroid of the tree T if it has the least weight.

In any graph there is either one centroid or two centroids that are adjacent.

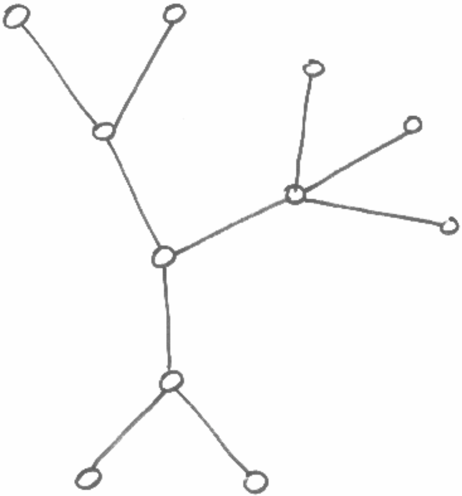
The weight is monotonically increasing along any path starting at a centroid.

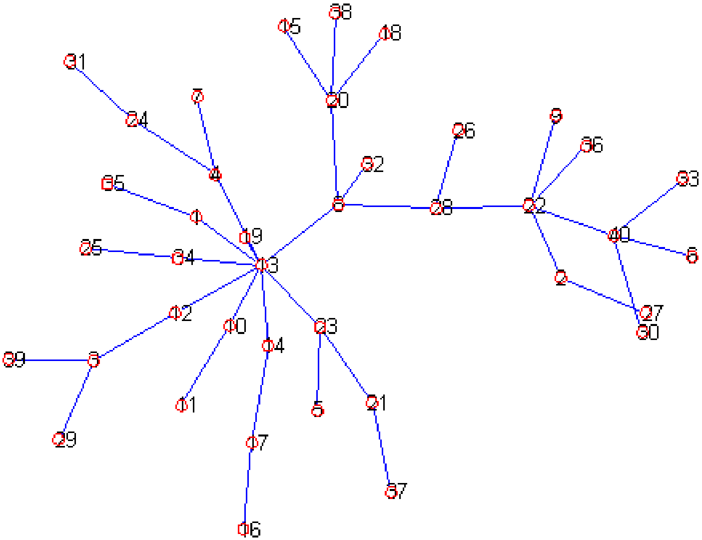
Theorem In any tree the median and the centroid coincide.

Consider the tree:



In this tree x is the center and y is the centroid (and the median).





Perron center of a tree

Let u be the Perron vector of the distance matrix D of a tree. A vertex i is a Perron center if u_i is the least coordinate of u.

Theorem In any tree there is either one Perron center or two Perron centers which are adjacent. Moreover the coordinates of u increase along any path starting at a Perron center.

Theorem In any tree there is either one Perron center or two Perron centers which are adjacent. Moreover the coordinates of u increase along any path starting at a Perron center.

Observation In almost any tree the Perron center and the median coincide.

Subtree core For a vertex v, let $f(v)$ be the number of subtrees passing through v. The subtree core consists of vertices at which $f(v)$ is maximized.

Let T be a tree with $V(T) = \{1, \ldots, n\}, n \geq 3$, and let $f : V(T) \longrightarrow (0, \infty)$.

A vertex $i \in V(T)$ will be called an f-center if

$$f(i) = \min_j f(j).$$

Some desirable properties of f:

(i) There is a unique f-center or exactly two f-centers which are adjacent.
(ii) Along any path starting at a center, and not containing another center, the values of f monotonically increase.
Clearly (ii) implies (i).

Convexity and quasiconvexity

Let T be a tree with $V(T) = \{1, \ldots, n\}$, $n \geq 3$, and let $f : V(T) \longrightarrow (0, \infty)$. We say that f is *convex* if for any distinct $i, j, k \in V(T)$ with $i \sim j, j \sim k$ we have $2f(j) \leq f(i) + f(k)$

strictly convex if $2f(j) < f(i) + f(k)$.

quasiconvex if for any distinct $i, j, k \in V(T)$ with $i \sim j, j \sim k$ we have $f(j) \leq \max\{f(i), f(k)\}$

strictly quasiconvex if $f(j) < \max\{f(i), f(k)\}$.

Clearly, if f is convex (strictly convex) then it is quasiconvex (strictly quasiconvex).

If f is strictly quasiconvex, then for $i \sim j \sim k$,
$f(i) < f(j) \Longrightarrow f(j) < f(k)$.

Hence, if f is strictly quasiconvex, then (ii), and hence (i), holds.

Theorem Let T be a tree with $V(T) = \{1, \ldots, n\}$, let $g : (0, \infty) \longrightarrow (0, \infty)$ be an increasing, convex function such that $g(x) > 0$ if $x > 0$. Let $x_1, \ldots, x_n$ be positive numbers and let

$$f(i) = \sum_{j=1}^{n} g(d_{ij}) x_j, i = 1, \ldots, n.$$

Then f is strictly convex.

If $\sum_{j=1}^{n} g(d_{ij})x_j$ is replaced by $\max_{j=1}^{n} g(d_{ij})x_j$ then f is strictly quasiconvex.

It follows that eccentricity is strictly quasiconvex while transmission index is strictly convex.
Hence both eccentricity and transmission index satisfy (i) and (ii).

Properties of the Perron center also follow from the Theorem.

This part is based on joint work with Ronit Neogy.

Telephone center (Mitchell) The switchboard number of a vertex x in a tree T, denoted $sb(x)$, is defined to be the maximum number of distinct paths having x as an interior vertex. The telephone center of T is the set of all vertices of T with the largest switchboard number.

Security center (Slater)

If $x, y \in V(G)$, then $V(x, y)$ denotes the set of vertices in G that are closer to x than to y. For a vertex x in a tree T, the security number of x, denoted $sec(x)$ is the smallest value of $|V(x, v)| - |V(v, x)|$ over all $v \in V(T) - \{x\}$. The security center of T consists of all vertices of T with the largest security number.

Accretion center (Slater)
An ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of the n vertices of T is called a sequential labeling provided the subgraph $T[\{x_1, x_2, \ldots, x_j\}]$ induced by $(x_1, x_2, \ldots, x_j)$ is connected, for all $j \in \{1, \ldots, n\}$. The sequential number of a vertex x, denoted $seq(x)$, is the number of sequential labelings of T with x as first entry.

The accretion center of T is the set of all vertices of T with the largest sequential number.

Weight balance center (Reid and DePalma) The weight balance of a vertex x in a tree T is defined to be the integer $\min\{|n_1 - n_2|\}$ where the minimum is taken over all subtrees T_1 and T_2 of T such that $V(T) = V(T_1) \cup V(T_2)$, $V(T_1) \cap V(T_2) = \{x\}$, $|V(T_1)| = n_1$ and $|V(T_2)| = n_2$.

The weight balance center of T is the set of all vertices of T with smallest weight balance.

Processing center(Gerstel and Zaks) A processing sequence for a tree T of order n is a permutation $x_1, x_2, \ldots, x_n$ of its vertices so that x_1 is a leaf of T, and for each i, $2 \leq i \leq n$, x_i is a leaf of the subtree $T - \{x_1, x_2, \ldots, x_{i-1}\}$.

The processing number of a vertex x, denoted $proc(x)$, is the index of the earliest possible position for x over all processing sequences.

The processing center of a tree is the set of vertices with the largest processing number.

Telephons center —— switchboard number

Security center —— security number

Accretion center —— sequential number

Weight balance center —— weight balance

Processing center —— processing number

The basic reference for these definitions is the survey paper: K. B. Reid, Centrality Measures in Trees, Interdisciplinary Mathematical Sciences, Advances in Interdisciplinary Applied Discrete Mathematics, pp. 167-197 (2010)

Centers based on the Laplacian matrix

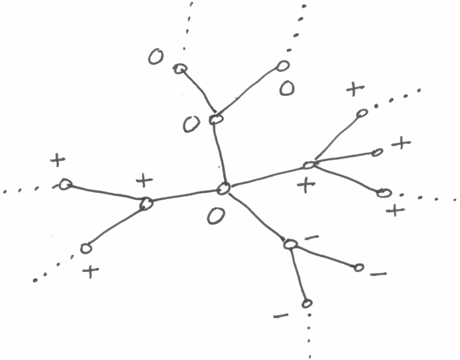
This part is based on:
(RBB, Chai Wah Wu) Control localization in dynamical systems connected via a weighted tree, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2016.

The work is motivated by the problem of localization of control in dynamical systems coupled via a weighted tree, when only a single system receives control.

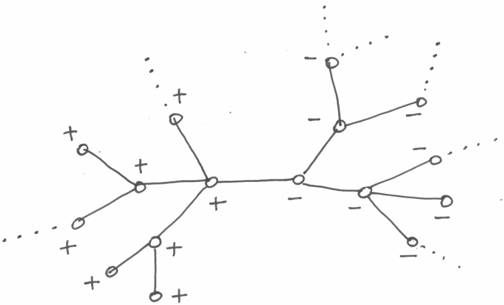
Since L is singular, $\lambda_1(L) = 0$. The eigenvalue $\lambda_2(L)$ is known as the algebraic connectivity of the graph and it is positive if and only if the graph is connected. This terminology is due to Fiedler who proved some fundamental properties of an eigenvector corresponding to the algebraic connectivity.

Trees are classified into two types according to whether the Fiedler vector has a zero coordinate or otherwise.

Type I tree



Type II tree



The Fiedler vector exhibits monotonicity properties.

Theorem Let T be a tree with Fiedler vector y. Then $|y|$ is convex. It is strictly convex for a Type II tree.

The corresponding center (which may be called Fiedler center) is known as a characteristic vertex.

Let T be a tree with vertex set $\{1, \ldots, n\}$, and let L be the Laplacian matrix of T. For $i = 1, \ldots, n$. We set

Let L_i to be the matrix obtained by deleting row and column i from L.

Let $\alpha_i =$ Perron root of L_i^{-1}.

Let $\tilde{L}_i(c)$ to be the matrix obtained by adding $c > 0$ to the i-th diagonal element in L. Let

$\beta_i(c) =$ Perron root of $\tilde{L}_i(c)^{-1}$.

It can be seen that

$$\lim_{c \to \infty} \beta_i(c) = \alpha_i.$$

Theorem $\alpha_i, i = 1, \ldots, n$ is strictly convex. Furthermore, the corresponding center coincides with the characterisic vertex.

Theorem $\beta_i(c), i = 1, \ldots, n$ is strictly convex.

We conjecture that $\beta_i(c)$ is also minimized at a characteristic vertex.

Since

$$\lim_{c \to \infty} \beta_i(c) = \alpha_i.$$

the conjecture is true for large c.

Thank You!