On the multiplicity of A_α-eigenvalues for $\mathbb{T}$-gain graphs

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Outline

- **Introduction & Preliminaries**
- **Literature survey**
- **Our contribution**
- **References**

Introduction & Preliminaries

Spectral graph theory

Spectral graph theory

Dealing with **Graphs** and **Matrices**

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Matrices associated with graphs:

Adjacency Matrix

Incidence Matrix

Laplacian Matrix

Distance Matrix, and many more.

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A generalization of adjacency matrix

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A generalization of adjacency matrix

Main aim:

How is **the multiplicity of an eigenvalue** related to **n** and Δ for such matrices?

What is A_α-matrix of a $\mathbb{T}$-gain graph ?

What is A_α-matrix of a $\mathbb{T}$-gain graph ?

Why is this a generalization of adjacency matrix ?

Recall some basic spectral graph theoretic terminology

- Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. If two vertices v_i and v_j are connected by an edge, then we write $v_i \sim v_j$ and the edge between them is $e_{i,j}$

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- ***Adjacency matrix*** $A(G)$ of G is an $(n \times n)$ symmetric matrix whose $(i, j)th$ entry is defined as follows:

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- The ***nullity*** of G, denoted by $\eta(G)$, is the nullity of $A(G)$ which is the multiplicity of zero eigenvalue of $A(G)$.

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$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G), \quad \alpha \in [0, 1].$$

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- Let $m_\alpha(G,\lambda)$ denotes the multiplicity of λ as an eigenvalue of $A_\alpha(G)$, for $\alpha \in [0,1)$.

  Particularly, $m_0(G,0) = \eta(G)$.

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- A ***gain function*** on G is a mapping $\varphi : \overrightarrow{E(G)} \to \mathbb{T}$ such that $\varphi(\overrightarrow{e_{s,t}}) = \varphi(\overrightarrow{e_{t,s}})^{-1}$, for every $e_{s,t} \in E(G)$.

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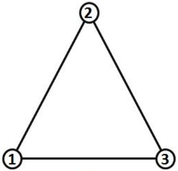
- A ***complex unit gain graph*** (or $\mathbb{T}$-gain graph) on an underlying graph G is a graph (G, φ) together with a gain function φ. It is denoted by Φ. That is $\Phi = (G, \varphi)$.

Nathan Reff, ***Linear Algebra Appl. 2012.***

Example: $\mathbb{T}$-gain graphs



G

Underlying graph

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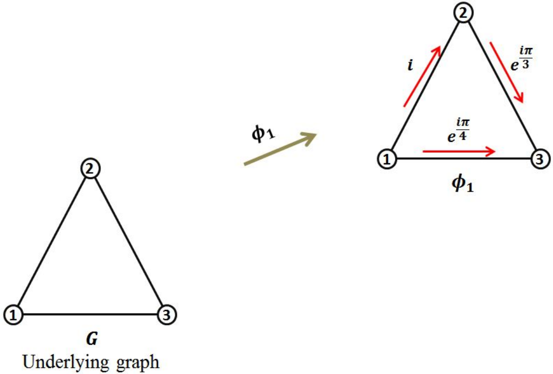

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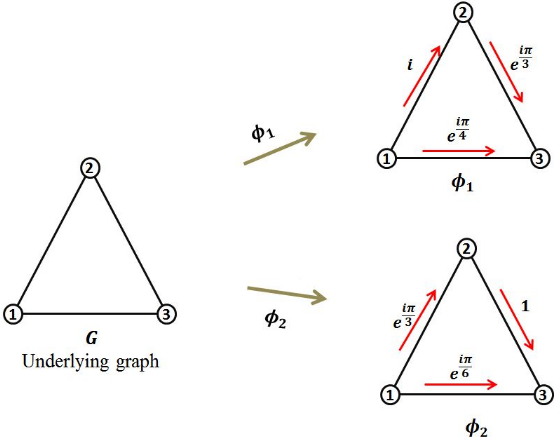

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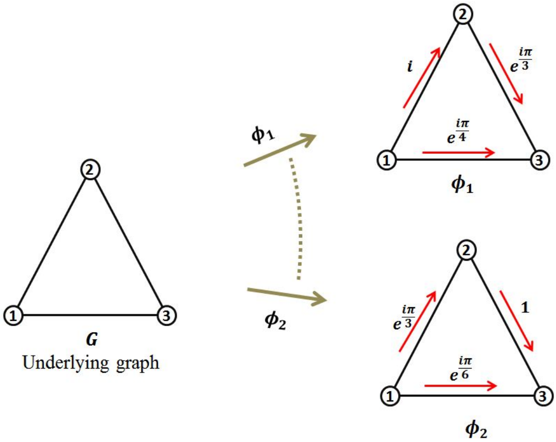

$\mathbb{T}$-gain adjacency matrix

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- The ***adjacency matrix*** of a $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ is a Hermitian matrix, denoted by $A(\Phi)$ and its (s,t)*th* entry is defined as follows:

$$A(\Phi)_{st} = \begin{cases} \varphi(\overrightarrow{e_{s,t}}) & \text{if } v_s \sim v_t, \\ 0 & \text{otherwise.} \end{cases}$$

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- We can observed that adjacency matrix of undirected graph, adjacency matrix of signed graph and Hermitian adjacency matrix of digraph can be considered as $A(\Phi)$, where the gains φ are from the set $\{1\}$, $\{1, -1\}$ and $\{1, \pm i\}$, respectively.

Example: $\mathbb{T}$-gain adjacency matrix

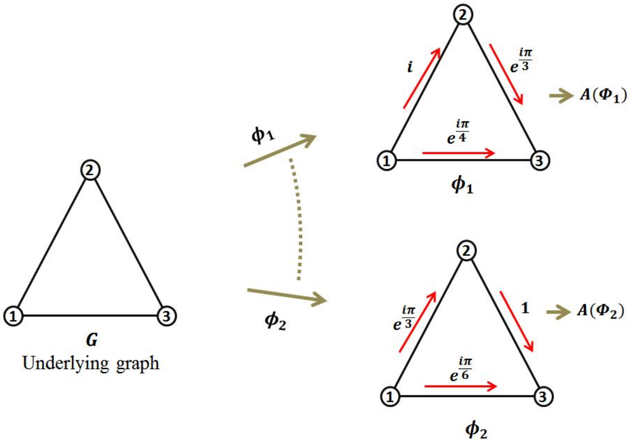

Figure: Graph G

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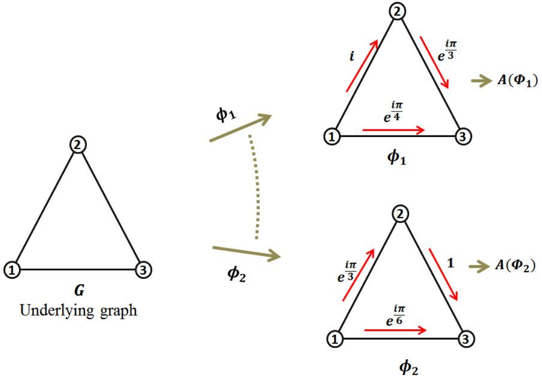

Where, $A(\Phi_1) = \begin{pmatrix} 0 & i & e^{\frac{i\pi}{4}} \\ -i & 0 & e^{\frac{i\pi}{3}} \\ e^{-\frac{i\pi}{4}} & e^{-\frac{i\pi}{3}} & 0 \end{pmatrix}$ $A(\Phi_2) = \begin{pmatrix} 0 & e^{\frac{i\pi}{3}} & e^{\frac{i\pi}{6}} \\ e^{-\frac{i\pi}{3}} & 0 & 1 \\ e^{-\frac{i\pi}{6}} & 1 & 0 \end{pmatrix}$

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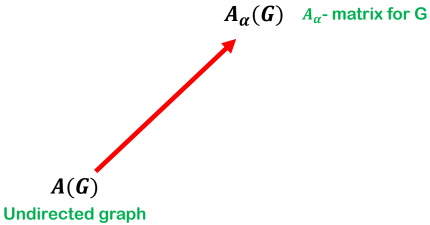
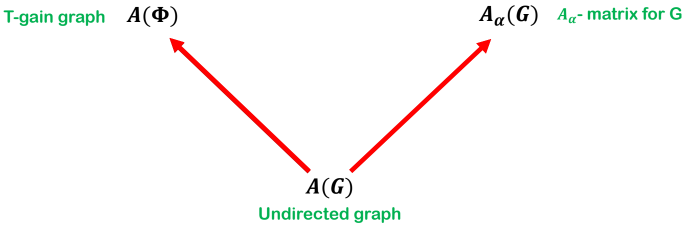
$A(G)$

Undirected graph


$A_\alpha(G)$
A_α- matrix for G
$A(G)$
Undirected graph


T-gain graph
$A(\Phi)$
$A_\alpha(G)$
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$A(G)$
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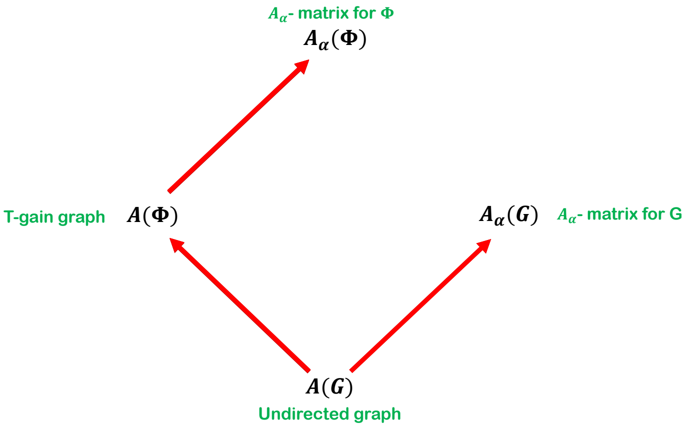
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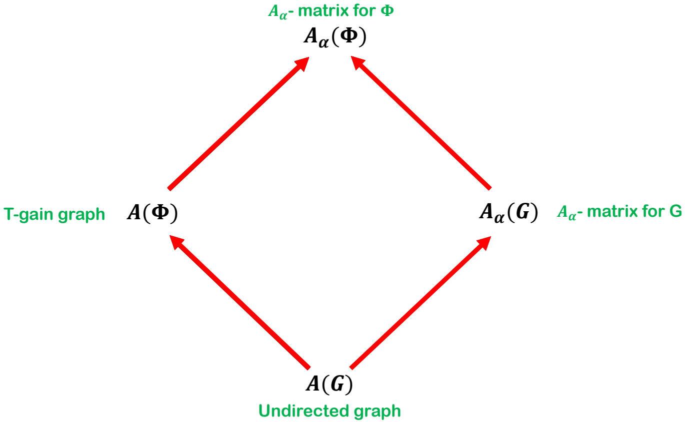

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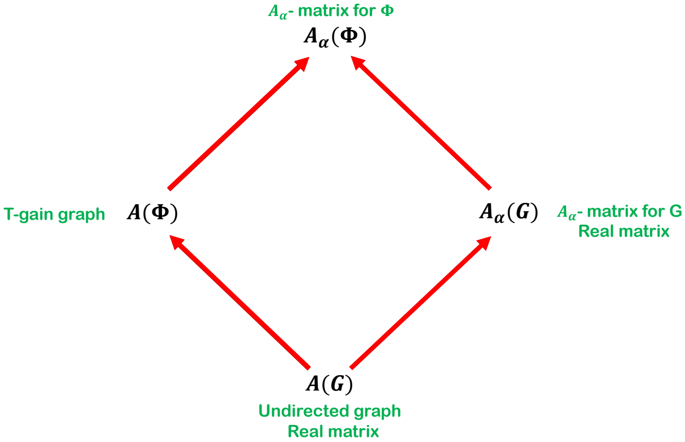

A_α- matrix for Φ
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T-gain graph
$A(\Phi)$
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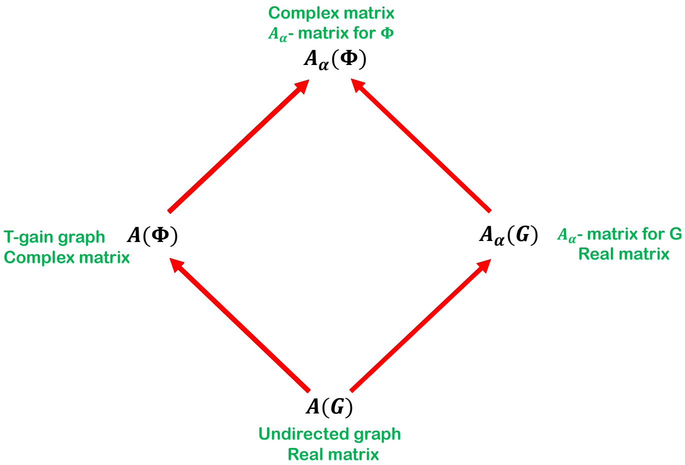
A_α- matrix for G
Real matrix
$A(G)$
Undirected graph
Real matrix


Complex matrix
A_α- matrix for Φ
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T-gain graph
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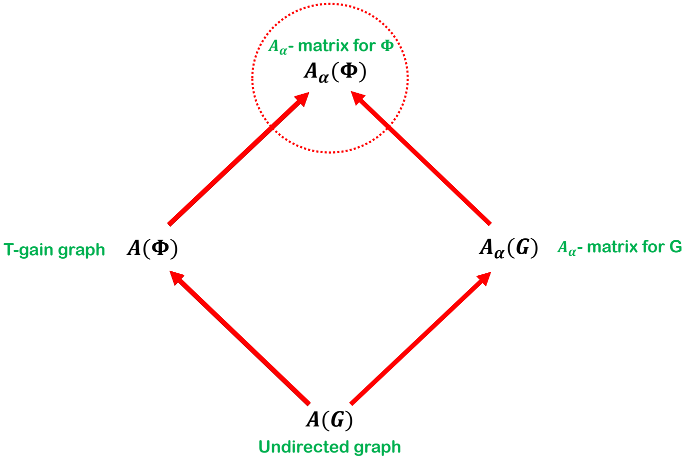
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Literature survey

How is $m_\alpha(\Phi, \lambda)$ related to n and Δ ?

(1) For undirected Tree T, $\eta(T) \leq n - 2\lceil \frac{n-2}{\Delta} \rceil$, with characterization of equality.
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(4) For connected graph G, $\eta(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}$, with characterization of equality.

- Zhi Wen Wang, Ji Ming Guo, ***Linear Algebra Appl. 2019***
- Wanting Sun, Shuchao Li, ***Linear Algebra Appl. 2019***
- Bo Chenga, Muhuo Liub, Bolian Liud, ***Linear Algebra Appl. 2019***
- Long Wang, Xianya Geng, ***Journal of Graph Theory 2020***

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One immediate result.

(5) Let $m_\alpha(G,\lambda)$ be the multiplicity of λ as an eigenvalue of $A_\alpha(G)$. Then

$$m_\alpha(G,\lambda) \leq \frac{(\Delta-2)n+2}{\Delta-1}, \quad \text{with characterization of equality.}$$

Long Wanga, Xianwen Fanga, Xianya Genga, Fenglei Tianb,
Linear Algebra Appl. 2019

Remark: $m_0(G,0) = \eta(G)$, above result is a generalization.

- Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph.
- $A(\Phi)$ and $D(\Phi)$ are the ***adjacency matrix*** and ***degree matrix*** of Φ, respectively.
- Then $A_\alpha(\Phi) := \alpha D(\Phi) + (1-\alpha)A(\Phi)$, for $\alpha \in [0, 1]$.
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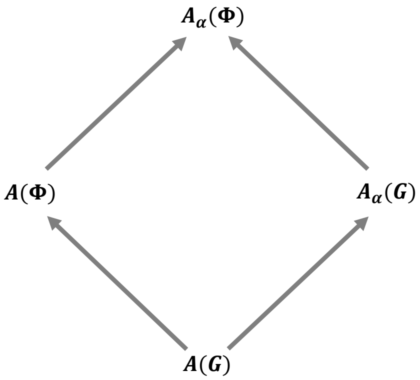
(6) For $\mathbb{T}$-gain graph Φ, $\eta(\Phi) \leq \frac{(\Delta-1)n}{\Delta}$, with characterization of equality.
Yong Lu, Jingwen Wu, ***Linear Algebra Appl. 2020***



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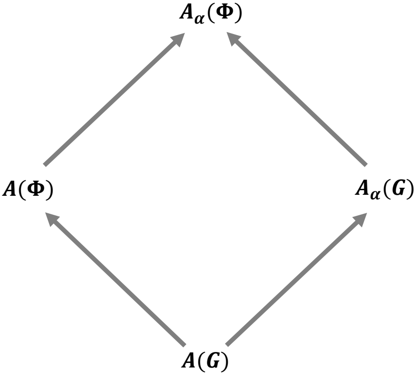

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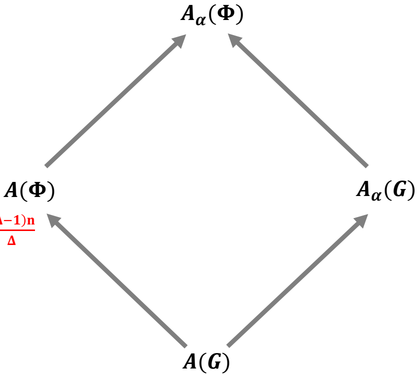

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Our result (2021): $m_\alpha(\Phi, \lambda) \leq \frac{(\Delta-2)n+2}{\Delta-1}$

$A_\alpha(\Phi)$

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Our Contribution

Theorem (A. Samanta, M. Rajesh Kannan, 2021)

Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph of n vertices with maximum vertex degree $\Delta \geq 2$. If $m_\alpha(\Phi, \lambda)$ is the multiplicity of λ as an A_α-eigenvalue of Φ, where $\alpha \in [0, 1)$, then

$$m_\alpha(\Phi, \lambda) \leq \frac{(\Delta - 2)n + 2}{(\Delta - 1)}. \tag{1}$$

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$$m_\alpha(\Phi, \lambda) \leq \frac{(\Delta - 2)n + 2}{(\Delta - 1)}. \quad (1)$$

Charecterization (A. Samanta, M. Rajesh Kannan, 2021)

Equality occurs in (1) if and only if one of the following holds:

(i) $\Phi \sim (K_{\frac{n}{2},\frac{n}{2}}, 1)$ and $\lambda = \frac{\alpha n}{2}$.

(ii) $\Phi = (C_n, \varphi)$ with $\varphi(C_n) = 1$ and $\lambda \in \left\{2\alpha + 2(1 - \alpha)\cos\left(\frac{2\pi j}{n}\right) : j = 0, 1, \ldots, \lceil \frac{n}{2} \rceil - 1\right\}$.

(iii) $\Phi = (C_n, \varphi)$ with $\varphi(C_n) = -1$ and $\lambda \in \left\{2\alpha + 2(1 - \alpha)\cos\left(\frac{(2j+1)\pi}{n}\right) : j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\right\}$.

(iv) $\Phi = (K_n, \varphi)$ with $\mu \in \sigma(\Phi)$ has multiplicity $(n - 1)$ and $\lambda = \alpha(n - 1) + (1 - \alpha)\mu$.

The following are the main significance of the above Theorem.

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- The above Theorem extend the Result (5) for $\mathbb{T}$-gain graphs.
- Particular case of the above Theorem simplify the proof of the Result (5).

Sketch of the Proof

Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph with vertices $V(\Phi) = \{v_1, v_2, \cdots, v_n\}$ and maximum vertex degree $\Delta \geq 2$.

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- Key ideas: **Zero forcing number** $Z(G)$

Zero forcing set

The notion of a zero-forcing set of a simple graph G was introduced in
AIM Minimum Rank-Special Graphs Work Group, ***Linear Algebra Appl. 2008***.

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- The ***derived coloring*** of a given coloring of G is the resulting coloring after applying the color-change rule such that no more changes are possible.

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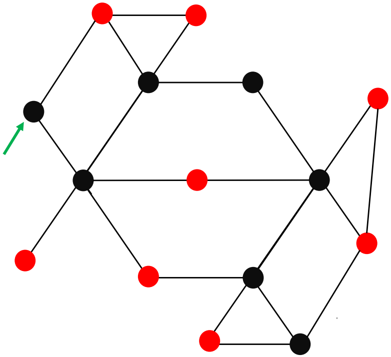
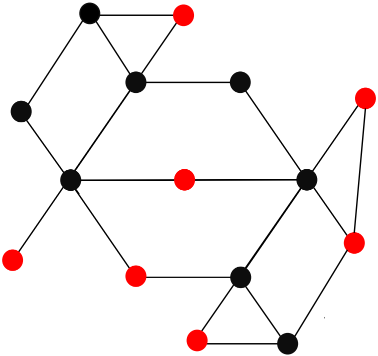
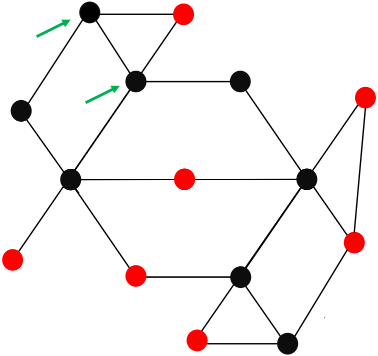
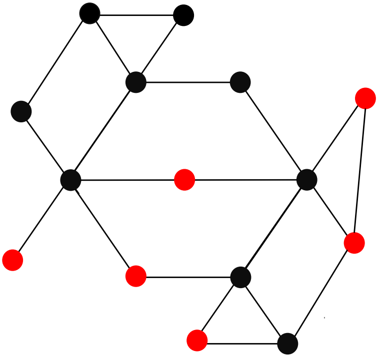
- The ***derived coloring*** of a given coloring of G is the resulting coloring after applying the color-change rule such that no more changes are possible.
- A subset Z of the vertex set of G is called a ***zero forcing set*** of G, if initially the vertices of Z are all colored black and the remaining vertices are colored red, the derived coloring of G are all black.

Counter example of zero forcing set

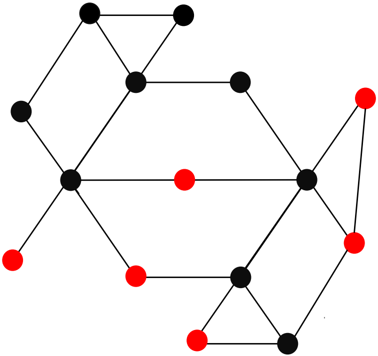




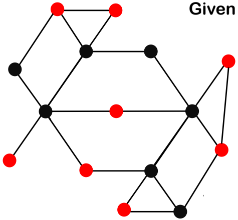


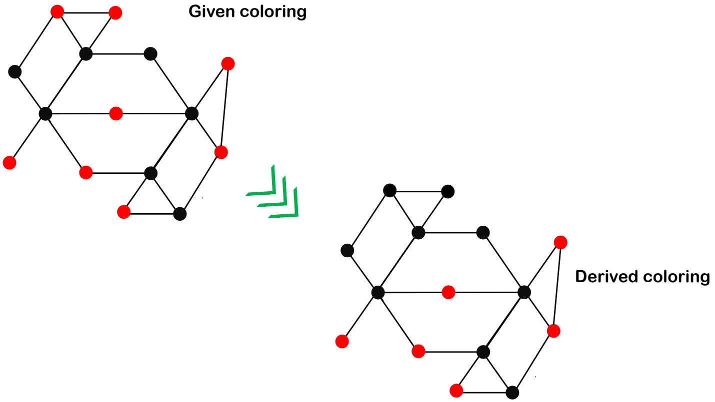


Derived coloring



Given coloring




Given coloring
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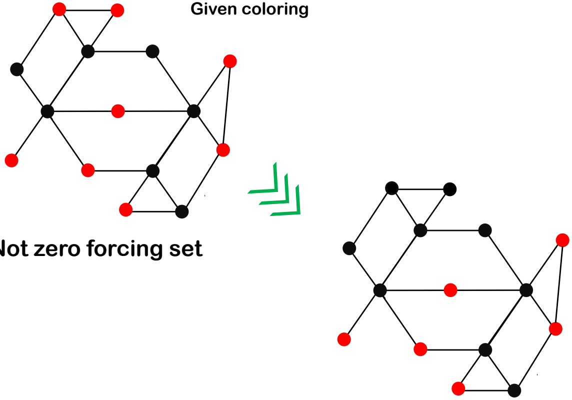
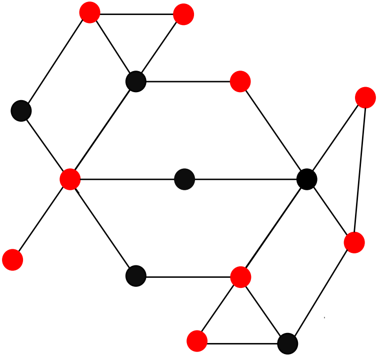
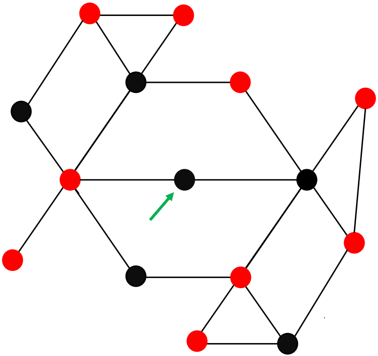


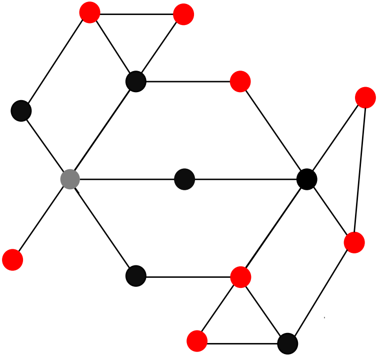
Not zero forcing set

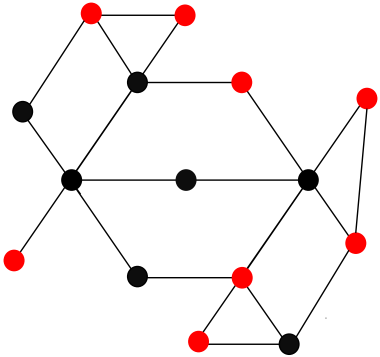
Derived coloring

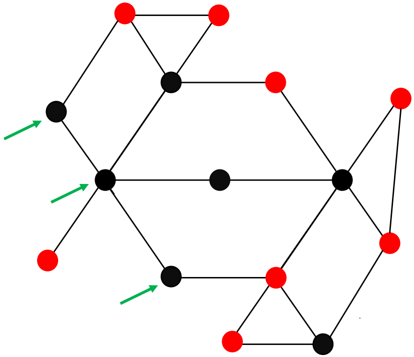
Example of Zero forcing set

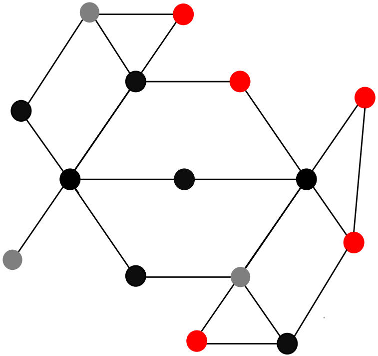


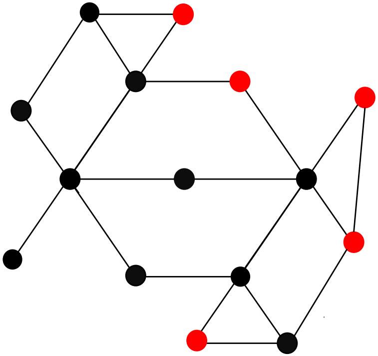


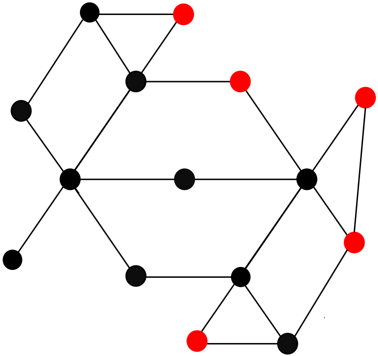


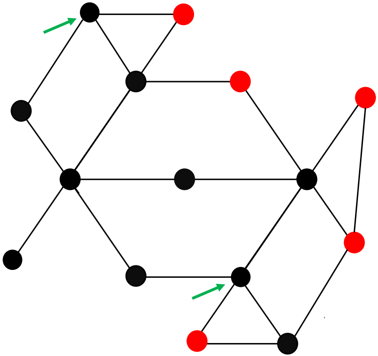


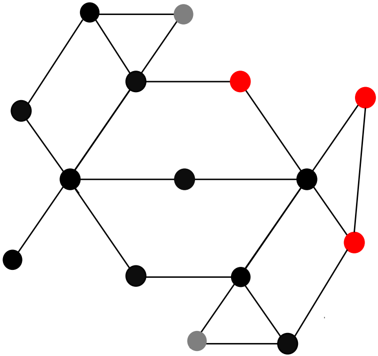


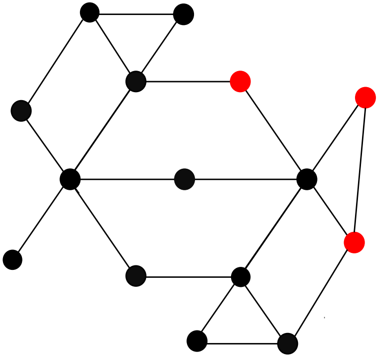


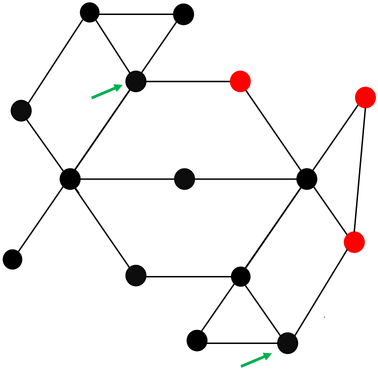


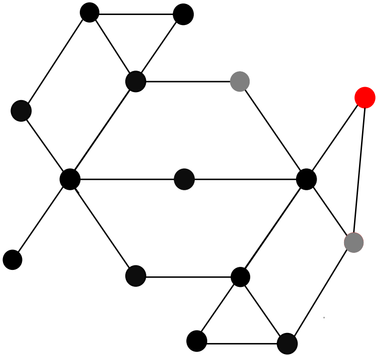


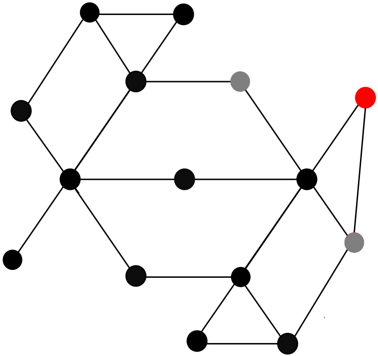


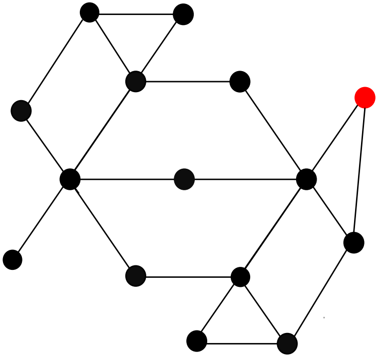


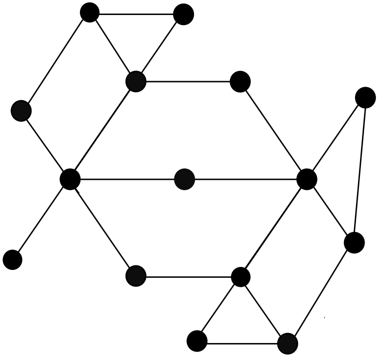


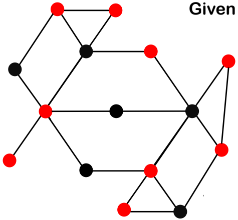




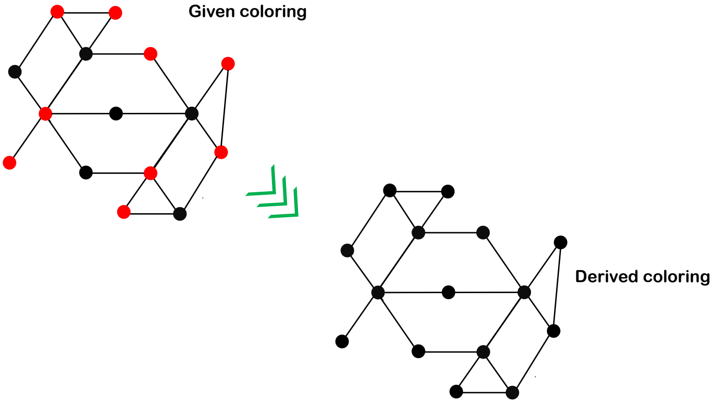


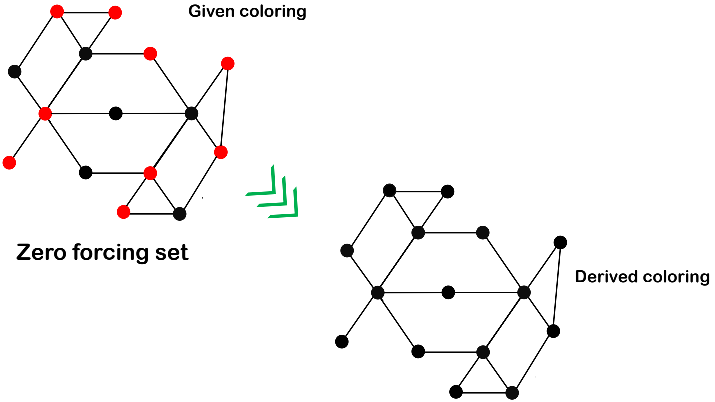






Given coloring


Given coloring
Derived coloring


Given coloring
Zero forcing set
Derived coloring

Zero forcing number: $Z(G) := \min_{Z} |Z|$ over all zero forcing set Z

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Let us present the following immediate result.

(6) For any connected G with $\Delta \geq 2$,

$$Z(G) \leq \frac{(\Delta - 2)n + 2}{(\Delta - 1)}$$

Equality occur if and only if G is either C_n, or K_n, or $K_{\frac{n}{2},\frac{n}{2}}$.

Michael Gentner at el., ***Discrete Appl. Math 2016.***

Sketch of the Proof

Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph with vertices $V(\Phi) = \{v_1, v_2, \cdots, v_n\}$ and maximum vertex degree Δ.

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Let $\eta(B)$ be the nullity of the matrix B.

Define $M(\Phi) := \max\{\eta(B) : B \in \mathcal{H}(\Phi)\}$.

Particularly, $\eta(A(\Phi)) \leq M(\Phi)$.

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- Key ideas: **Zero forcing number** $Z(G)$, $\quad Z(G) \leq \frac{(\Delta-2)n+2}{(\Delta-1)}$
- Define $M(\Phi) := \max\{\eta(B) : B \in \mathcal{H}(\Phi)\}$, then $\eta(B) \leq M(\Phi)$, where B is of type Φ.

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- Then $M(\Phi) \leq Z(G)$

We use few more results. For $y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$, the *support* of y is the set of indices j such that $y_j \neq 0$, and is denoted by $supp(y)$. For real symmetric metrics, the following two results are known.

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Suppose $Z \subset V(\Phi)$. Since Z is a zero forcing set, so all the white vertices in $V(\Phi) \setminus Z$ can be colored black by color change rule. Let $v_i \in Z$ be such that it has exactly one red neighbour vertex v_t.

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Then the i-th entry $(By)_i = B_{ii}y_i + \sum\limits_{v_i \sim v_j} B_{ij}y_j = B_{it}y_t = 0$. Then $y_t = 0$. As Z is a zero forcing set, so all the components of y associated with white vertices are zero. Hence $y = 0$.

Remark: Significance of the name " Zero forcing set "

Recall the following result.

(7) Let B be any square matrix on some field with $\eta(B) > s$. Then there exists a non-zero vector $y \in Ker(B)$ vanishing at s specified positions.

AIM Minimum Rank-Special Graphs Work Group, ***Linear Algebra Appl. 2008***

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By Lemma 1, we get $y = 0$, a contradiction. Thus $M(\Phi) \leq |Z|$, and hence $M(\Phi) \leq Z(G)$.

Sketch of the Proof

- Key ideas: **Zero forcing number** $Z(G)$, $\quad Z(G) \leq \frac{(\Delta-2)n+2}{(\Delta-1)}$
- Define $M(\Phi) := \max\{\eta(B) : B \in \mathcal{H}(\Phi)\}$, then $\eta(B) \leq M(\Phi)$, where B is of type Φ.
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- For $\alpha \in [0, 1)$, $A_\alpha(\Phi) = \alpha D(\Phi) + (1-\alpha)A(\Phi)$.

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- For $\alpha \in [0, 1)$, $A_\alpha(\Phi) = \alpha D(\Phi) + (1 - \alpha)A(\Phi)$.

  Let $\lambda \in \sigma(A_\alpha(\Phi))$. Consider $B := (A_\alpha(\Phi) - \lambda I)$. Therefore $\eta(B) = m_\alpha(\Phi, \lambda)$.

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  Therefore, $\eta(B) \leq M(\Phi)$. Combining all

$$m_\alpha(\Phi, \lambda) = \eta(B) \leq M(\Phi) \leq Z(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}.$$

Characterization of equality:

If $m_\alpha(\Phi, \lambda) = \frac{(\Delta-2)n+2}{\Delta-1}$, then $Z(G) = \frac{(\Delta-2)n+2}{\Delta-1}$. By result (6), G is either $K_{\frac{n}{2},\frac{n}{2}}$ or K_n or C_n.

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- Then $\mu = 0$. Therefore $r(\Phi) = 2$
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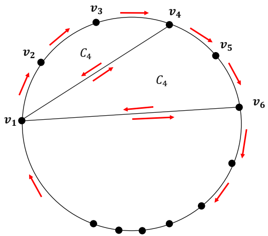
- Let us take an arbitrary cycle $C_{2k} \equiv v_1 - v_2 - \cdots - v_{2k}$.



Case 1: Suppose $\Phi = (K_{\frac{n}{2},\frac{n}{2}}, \varphi)$. Then $m_\alpha(\Phi, \lambda) = \frac{(\Delta-2)n+2}{\Delta-1} = n-2$. Therefore, there is an eigenvalue μ of $A(\Phi)$ with multiplicity $(n-2)$ such that $\lambda = \frac{\alpha n}{2} + (1-\alpha)\mu$.

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$$\begin{aligned}
\varphi(\overrightarrow{C_{2k}}) &= \varphi(\overrightarrow{e_{1,2}})\varphi(\overrightarrow{e_{2,3}})\cdots\varphi(\overrightarrow{e_{(2k-1),2k}}) \\
&= \{\varphi(\overrightarrow{e_{1,2}})\varphi(\overrightarrow{e_{2,3}})\varphi(\overrightarrow{e_{3,4}})\varphi(\overrightarrow{e_{4,1}})\} \\
&\quad \{\varphi(\overrightarrow{e_{1,4}})\varphi(\overrightarrow{e_{4,5}})\varphi(\overrightarrow{e_{5,6}})\varphi(\overrightarrow{e_{6,1}})\} \\
&\qquad \vdots \\
&\quad \{\varphi(\overrightarrow{e_{1,(2k-2)}})\varphi(\overrightarrow{e_{(2k-2),(2k-1)}})\varphi(\overrightarrow{e_{(2k-1),2k}})\varphi(\overrightarrow{e_{2k,1}})\} \\
&= 1.
\end{aligned}$$

Therefore $\Phi \sim (K_{\frac{n}{2},\frac{n}{2}}, 1)$ and $\lambda = \frac{\alpha n}{2}$. Thus statement (i) holds.

Case 2: Suppose $\Phi = (K_n, \varphi)$ and $m_\alpha(\Phi, \lambda) = \frac{(\Delta-2)n+2}{\Delta-1}$. Then $m_\alpha(\Phi, \lambda) = n - 1$. Therefore, λ is an eigenvalue of $A_\alpha(\Phi)$ with multiplicity $(n - 1)$. Therefore, $A(\Phi)$ has an eigenvalue μ with multiplicity $(n - 1)$. Statement (*ii*) holds.

Case 3: Suppose $\Phi = (C_n, \varphi)$ and $m_\alpha(\Phi, \lambda) = \frac{(\Delta-2)n+2}{\Delta-1}$. Then $m_\alpha(\Phi, \lambda) = 2$. Therefore by some known results, either statement (*iii*) or statement (*iv*) holds.

Another improved bound:

Theorem (A. Samanta, M. Rajesh Kannan, 2021)

Let Φ be any connected $\mathbb{T}$-gain graph with n vertices and the maximum vertex degree $\Delta \geq 3$. Then

$$\eta(\Phi) \leq \frac{n(\Delta - 2)}{\Delta - 1}$$

equality holds if and only if $\Phi \notin \{(K_{\frac{n}{2},\frac{n}{2}}, 1), (K_{\frac{n+1}{2},\frac{n-1}{2}}, 1)\}$.

Problem

One can consider a problem to find better bound of $\eta(G)$ in terms of n and Δ by excluding some particular graphs.

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Thank You