On equitable partition of matrices and its applications

Dr. Fouzul Atik

Department of Mathematics
SRM University-AP, Amaravati

- Consider an $n \times p$ matrix M whose rows and columns are indexed by the elements of $X = \{1, 2, \cdots, n\}$ and $Y = \{1, 2, \cdots, p\}$ respectively. Let α be a subset of X and β be a subset of Y.

- Consider an $n \times p$ matrix M whose rows and columns are indexed by the elements of $X = \{1, 2, \cdots, n\}$ and $Y = \{1, 2, \cdots, p\}$ respectively. Let α be a subset of X and β be a subset of Y.
- The submatrix of M, whose rows are indexed by elements of α and columns are indexed by elements of β, is denoted by $M[\alpha : \beta]$.

- Consider an $n \times p$ matrix M whose rows and columns are indexed by the elements of $X = \{1, 2, \cdots, n\}$ and $Y = \{1, 2, \cdots, p\}$ respectively. Let α be a subset of X and β be a subset of Y.
- The submatrix of M, whose rows are indexed by elements of α and columns are indexed by elements of β, is denoted by $M[\alpha : \beta]$.
- We denote the same matrix by $M[\alpha]$, $M[\alpha :]$ and $M[: \beta]$ according as $\alpha = \beta$, $\beta = Y$ and $\alpha = X$ respectively.

- Consider an $n \times p$ matrix M whose rows and columns are indexed by the elements of $X = \{1, 2, \cdots, n\}$ and $Y = \{1, 2, \cdots, p\}$ respectively. Let α be a subset of X and β be a subset of Y.
- The submatrix of M, whose rows are indexed by elements of α and columns are indexed by elements of β, is denoted by $M[\alpha : \beta]$.
- We denote the same matrix by $M[\alpha]$, $M[\alpha :]$ and $M[: \beta]$ according as $\alpha = \beta$, $\beta = Y$ and $\alpha = X$ respectively.
- By P^T we denote the transpose of the matrix P and by X^c we denote the complement of the set X.

- Consider an $n \times p$ matrix M whose rows and columns are indexed by the elements of $X = \{1, 2, \cdots, n\}$ and $Y = \{1, 2, \cdots, p\}$ respectively. Let α be a subset of X and β be a subset of Y.
- The submatrix of M, whose rows are indexed by elements of α and columns are indexed by elements of β, is denoted by $M[\alpha : \beta]$.
- We denote the same matrix by $M[\alpha]$, $M[\alpha :]$ and $M[: \beta]$ according as $\alpha = \beta$, $\beta = Y$ and $\alpha = X$ respectively.
- By P^T we denote the transpose of the matrix P and by X^c we denote the complement of the set X.
- The spectrum of the matrix A is denoted by $Spec(A)$. $J_{m \times n}$ is the all ones matrix of order $m \times n$.

- We consider a square matrix A whose rows and columns are indexed by elements of $X = \{1, 2, \cdots, n\}$. Let $\pi = \{X_1, X_2, \cdots, X_m\}$ be a partition of X.

- We consider a square matrix A whose rows and columns are indexed by elements of $X = \{1, 2, \cdots, n\}$. Let $\pi = \{X_1, X_2, \cdots, X_m\}$ be a partition of X.
- The *characteristic matrix* $C = (c_{ij})$ of π is an $n \times m$ order matrix such that $c_{ij} = 1$ if $i \in X_j$ and 0 otherwise.

Equitable partition and quotient matrix

- We consider a square matrix A whose rows and columns are indexed by elements of $X = \{1, 2, \cdots, n\}$. Let $\pi = \{X_1, X_2, \cdots, X_m\}$ be a partition of X.
- The *characteristic matrix* $C = (c_{ij})$ of π is an $n \times m$ order matrix such that $c_{ij} = 1$ if $i \in X_j$ and 0 otherwise.
- We partition the matrix A according to π as $\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$, where $A_{ij} = A[X_i : X_j]$ and $i, j = 1, 2, \cdots, m$.

- We consider a square matrix A whose rows and columns are indexed by elements of $X = \{1, 2, \cdots, n\}$. Let $\pi = \{X_1, X_2, \cdots, X_m\}$ be a partition of X.
- The *characteristic matrix* $C = (c_{ij})$ of π is an $n \times m$ order matrix such that $c_{ij} = 1$ if $i \in X_j$ and 0 otherwise.
- We partition the matrix A according to π as $\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$, where $A_{ij} = A[X_i : X_j]$ and $i, j = 1, 2, \cdots, m$.
- If q_{ij} denotes the average row sum of A_{ij} then the matrix $Q = (q_{ij})$ is called a *quotient matrix* of A. If the row sum of each block A_{ij} is a constant then the partition π is called *equitable*.

- We consider a square matrix A whose rows and columns are indexed by elements of $X = \{1, 2, \cdots, n\}$. Let $\pi = \{X_1, X_2, \cdots, X_m\}$ be a partition of X.
- The *characteristic matrix* $C = (c_{ij})$ of π is an $n \times m$ order matrix such that $c_{ij} = 1$ if $i \in X_j$ and 0 otherwise.
- We partition the matrix A according to π as $\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$, where $A_{ij} = A[X_i : X_j]$ and $i, j = 1, 2, \cdots, m$.
- If q_{ij} denotes the average row sum of A_{ij} then the matrix $Q = (q_{ij})$ is called a *quotient matrix* of A. If the row sum of each block A_{ij} is a constant then the partition π is called *equitable*.
-

$$A = \begin{bmatrix} 2 & -2 & 1 & 1 & 0 & 1 \\ -1 & 2 & 1 & 1 & 1 & 2 \\ -1 & 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & -1 & 1 & -1 & 1 \\ 2 & 0 & 0 & 0 & 2 & -1 \\ 2 & -2 & 2 & -2 & 0 & 3 \end{bmatrix}$$

- Let $X = \{1, 2, \cdots, 6\}$ and $\pi = \{X_1, X_2, X_3\}$ be a partition of X, where $X_1 = \{1\}, X_2 = \{2, 3\}$ and $X_3 = \{4, 5, 6\}$.
- We consider the following matrix A whose rows and columns are indexed by elements of X.

$$A = \left[\begin{array}{c|cc|ccc} 2 & -2 & 1 & 1 & 0 & 1 \\ \hline -1 & 2 & 1 & 1 & 1 & 2 \\ -1 & 0 & 3 & 1 & 2 & 1 \\ \hline 2 & 1 & -1 & 1 & -1 & 1 \\ 2 & 0 & 0 & 0 & 2 & -1 \\ 2 & -2 & 2 & -2 & 0 & 3 \end{array}\right].$$

- Here the matrix A is partitioned according to π. Then the quotient matrix is given by

$$Q = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix}$$

- Here the partition π is equitable partition for the matrix A.

The following are well known results on an equitable partition of a matrix.

Theorem 1 (Brouwer and Haemers [4])

Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition. Then the spectrum of A contains the spectrum of Q.

The following are well known results on an equitable partition of a matrix.

Theorem 1 (Brouwer and Haemers [4])

Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition. Then the spectrum of A contains the spectrum of Q.

Theorem 2 (Atik and Panigrahi, 2018)

The spectral radius of a nonnegative square matrix A is the same as the spectral radius of a quotient matrix corresponding to an equitable partition.

The following are well known results on an equitable partition of a matrix.

Theorem 1 (Brouwer and Haemers [4])

Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition. Then the spectrum of A contains the spectrum of Q.

Theorem 2 (Atik and Panigrahi, 2018)

The spectral radius of a nonnegative square matrix A is the same as the spectral radius of a quotient matrix corresponding to an equitable partition.

Stochastic matrix:
A square matrix whose entries are nonnegative and for which each row sum equals to one is known as a *stochastic matrix*. Therefore stochastic matrices can be considered to have an equitable partition with one partition set and by previous theorem 1 is the spectral radius of a stochastic matrix.

Let A be a square matrix having an equitable partition. Then the theorem below finds some matrices whose eigenvalues are the eigenvalues of A other than the eigenvalues of the quotient matrix Q.

Theorem 3

Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition $\pi = \{X_1, X_2, \cdots, X_k\}$. Also let C be the characteristic matrix of π and α be an index set which contains exactly one element from each X_i, $i = 1, 2, \cdots, k$. Then the spectrum of A is equal to the union of spectrum of Q and spectrum of Q^*, where $Q^* = A[\alpha^c] - C[\alpha^c :]A[\alpha : \alpha^c]$.

We consider the partition $\pi = \{X_1, X_2\}$ for the matrix S, where $X_1 = \{1, 2\}$ and $X_2 = \{3, 4, 5\}$:

$$S = \left[\begin{array}{cc|ccc} 0.21 & 0.32 & 0.12 & 0 & 0.35 \\ 0.23 & 0.3 & 0.17 & 0.2 & 0.1 \\ \hline 0.15 & 0.2 & 0.21 & 0.2 & 0.24 \\ 0.3 & 0.05 & 0.3 & 0.35 & 0 \\ 0.17 & 0.18 & 0.15 & 0.2 & 0.3 \end{array}\right]$$

Then π is also an equitable partition for the matrix S. For this π corresponding quotient matrix is

$$Q = \begin{bmatrix} 0.53 & 0.47 \\ 0.35 & 0.65 \end{bmatrix}$$

and we have $|X_1||X_2| = 6$ choice of α as in Theorem 3. For each α corresponding Q^* are as follows:

$$\alpha = \{1,3\},\ Q^*_\alpha = \begin{bmatrix} -0.02 & 0.2 & -0.25 \\ -0.15 & 0.15 & -0.24 \\ -0.02 & 0. & 0.06 \end{bmatrix},\ \alpha = \{1,4\},\ Q^*_\alpha = \begin{bmatrix} -0.02 & 0.05 & -0.25 \\ 0.15 & -0.09 & 0.24 \\ 0.13 & -0.15 & 0.3 \end{bmatrix},$$

$$\alpha = \{1,5\},\ Q^*_\alpha = \begin{bmatrix} -0.02 & 0.05 & 0.2 \\ 0.02 & 0.06 & 0. \\ -0.13 & 0.15 & 0.15 \end{bmatrix},\ \alpha = \{2,3\},\ Q^*_\alpha = \begin{bmatrix} -0.02 & -0.2 & 0.25 \\ 0.15 & 0.15 & -0.24 \\ 0.02 & 0. & 0.06 \end{bmatrix}$$

$$\alpha = \{2,4\},\ Q^*_\alpha = \begin{bmatrix} -0.02 & -0.05 & 0.25 \\ -0.15 & -0.09 & 0.24 \\ -0.13 & -0.15 & 0.3 \end{bmatrix},\ \alpha = \{2,5\},\ Q^*_\alpha = \begin{bmatrix} -0.02 & -0.05 & -0.2 \\ -0.02 & 0.06 & 0. \\ 0.13 & 0.15 & 0.15 \end{bmatrix}$$

One more observation

- We consider the following matrix

$$A = \left[\begin{array}{c|cc|ccc} 2 & -2 & -2 & 1 & 1 & 1 \\ \hline -1 & 2 & -1 & 1.5 & 1 & 2 \\ -1 & -1 & 2 & 1.5 & 2 & 1 \\ \hline 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 0 & 1 \end{array}\right].$$

- We consider the following matrix

$$A = \left[\begin{array}{c|cc|ccc} 2 & -2 & -2 & 1 & 1 & 1 \\ \hline -1 & 2 & -1 & 1.5 & 1 & 2 \\ -1 & -1 & 2 & 1.5 & 2 & 1 \\ \hline 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 0 & 1 \end{array}\right].$$

- Then the quotient matrices corresponding to the rows and columns are given by

$$Q = \begin{bmatrix} 2 & -4 & 3 \\ -1 & 1 & 4.5 \\ 2 & 2 & 3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 1 & 3 \\ 6 & 3 & 3 \end{bmatrix} \text{ respectively.}$$

- We consider the following matrix

$$A = \left[\begin{array}{c|cc|ccc} 2 & -2 & -2 & 1 & 1 & 1 \\ \hline -1 & 2 & -1 & 1.5 & 1 & 2 \\ -1 & -1 & 2 & 1.5 & 2 & 1 \\ \hline 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 0 & 1 \end{array}\right].$$

- Then the quotient matrices corresponding to the rows and columns are given by
$Q = \begin{bmatrix} 2 & -4 & 3 \\ -1 & 1 & 4.5 \\ 2 & 2 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 1 & 3 \\ 6 & 3 & 3 \end{bmatrix}$ respectively.

- One may expect that the matrices P and Q have different eigenvalues. But observe that P and Q have same eigenvalues. Then the question is whether this situation holds for all such P and Q or not. This is answered in the next result.

- We consider the following matrix

$$A = \left[\begin{array}{c|cc|ccc} 2 & -2 & -2 & 1 & 1 & 1 \\ \hline -1 & 2 & -1 & 1.5 & 1 & 2 \\ -1 & -1 & 2 & 1.5 & 2 & 1 \\ \hline 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 0 & 1 \end{array}\right].$$

- Then the quotient matrices corresponding to the rows and columns are given by $Q = \begin{bmatrix} 2 & -4 & 3 \\ -1 & 1 & 4.5 \\ 2 & 2 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 1 & 3 \\ 6 & 3 & 3 \end{bmatrix}$ respectively.
- One may expect that the matrices P and Q have different eigenvalues. But observe that P and Q have same eigenvalues. Then the question is whether this situation holds for all such P and Q or not. This is answered in the next result.

Theorem 4

Let Q and P be quotient matrices for rows and columns of any square matrix A corresponding to the equitable partition $\pi = \{X_1, X_2, \cdots, X_k\}$. Then P and Q have same eigenvalues.

Theorem 5 (Geršgorin[5])

Let $A = [a_{ij}] \in M_n$ and consider the n Geršgorin discs

$$\{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}, \; i = 1, 2, \ldots, n.$$

Then the eigenvalues of A are in the union of Geršgorin discs

$$G(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}.$$

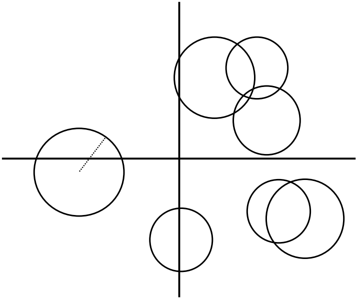


Figure: Geršgorin discs for any square matrix

For the matrix $A = [a_{ij}] \in M_n$, we hereby denote as $\mathcal{G}(A)$ the intersection of two regions as follows:

$$\mathcal{G}(A) = \left(\bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\} \right) \bigcap \left(\bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ji}|\} \right). \quad (1)$$

Theorem 6

Let Q be a quotient matrix of any square matrix A corresponding to an equitable partition $\pi = \{X_1, X_2, \cdots, X_k\}$. Also let C be the characteristic matrix of π and $\mathcal{I} = \{\alpha : \alpha$ contains exactly one element from each $X_i, \ i = 1, 2, \cdots, k\}$. Let $\mathcal{G}(A)$ be the region defined as in (1). Then the eigenvalues of A lie in

$$\left[\bigcap_{\alpha \in \mathcal{I}} \mathcal{G}(Q_\alpha) \right] \bigcup Spec\,(Q), \ \text{where } Q_\alpha = A[\alpha^c] - C[\alpha^c :]A[\alpha : \alpha^c].$$

Here we state some of the earlier results for eigenvalue localization for stochastic matrices

Theorem 7 (Cvetković et al., 2011)

Let $S = (s_{ij})$ *be a stochastic matrix, and let* s_i *be the minimal element among the off-diagonal entries of the* i*th column of* S*. Taking* $\gamma = \max_{i\in[n]}(s_{ii} - s_i)$*, for any* $\lambda \in \sigma(S) \setminus \{1\}$*, we have*

$$|\lambda - \gamma| \leq 1 - trace(S) + (n-1)\gamma.$$

Theorem 8 (Li and Li, 2014)

Let $S = (s_{ij})$ *be a stochastic matrix, and let* $S_i = \max_{j\neq i} s_{ji}$*. Taking* $\gamma' = \max_{i\in[n]}(S_i - s_{ii})$*, for any* $\lambda \in \sigma(S) \setminus \{1\}$*, we have*

$$|\lambda + \gamma'| \leq trace(S) + (n-1)\gamma' - 1.$$

Theorem 9 (Banerjee and Mehatari, 2016)

Let S *be a stochastic matrix of order* n*. Then the eigenvalues of* S *lie in the region*

$$\left[\bigcap_{i=1}^{n} G_{S(i)} \cup \{1\}\right], \text{ where } G_{S(i)} = \bigcup_{k\neq i}\{z \in \mathbb{C} : |z - s_{kk} + s_{ik}| \leq \sum_{j\neq k} |s_{kj} - s_{ij}|\}.$$

Theorem 10

Let S be a stochastic matrix of order n. Then the eigenvalues of S lie in the region

$$\left[\bigcap_{i=1}^{n} \mathcal{G}_i \cup \{1\}\right], \text{ where}$$

$$\mathcal{G}_i = \left(\bigcup_{k\neq i}\{z \in \mathbb{C} : |z - s_{kk} + s_{ik}| \leq \sum_{j\neq k} |s_{kj} - s_{ij}|\}\right) \bigcap \left(\bigcup_{k\neq i}\{z \in \mathbb{C} : |z - s_{kk} + s_{ik}| \leq \sum_{j\neq k} |s_{jk} - s_{ik}|\}\right).$$

Corollary 11

Let S be a stochastic matrix of order n and $p \in [0,1]$. Then the eigenvalues of S lie in the region

$$\left[\bigcap_{i=1}^{n} \mathcal{O}_i \cup \{1\}\right], \text{ where}$$

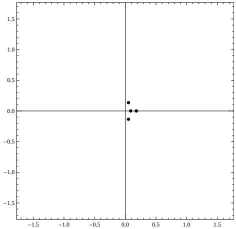
$$\mathcal{O}_i = \left(\bigcup_{k\neq i}\{z \in \mathbb{C} : |z - s_{kk} + s_{ik}| \leq (\sum_{j\neq k} |s_{kj} - s_{ij}|)^p (\sum_{j\neq k} |s_{jk} - s_{ik}|)^{1-p}\}\right).$$

In the following we give an example of a stochastic matrix for which this fact has been described graphically. We consider the following stochastic matrix:

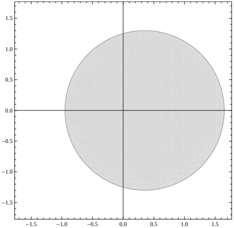
$$S = \begin{bmatrix} 0.21 & 0.32 & 0.12 & 0 & 0.35 \\ 0.23 & 0.3 & 0.17 & 0.2 & 0.1 \\ 0.15 & 0.2 & 0.21 & 0.2 & 0.24 \\ 0.3 & 0.05 & 0.3 & 0.35 & 0 \\ 0.17 & 0.18 & 0.15 & 0.2 & 0.3 \end{bmatrix}$$

The eigenvalues of S other than one are $0.18,\ 0.0878717,\ 0.0510641 + 0.134975i$ and $0.0510641 - 0.134975i$ which are plotted in figure (a).

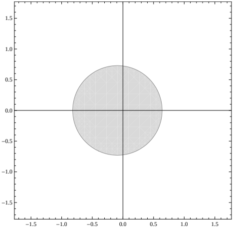
Note that the above stochastic matrix has two quotient matrix corresponding to two different equitable partitions as follows: $Q = [1]$ and $Q' = \begin{bmatrix} 0.53 & 0.47 \\ 0.35 & 0.65 \end{bmatrix}$



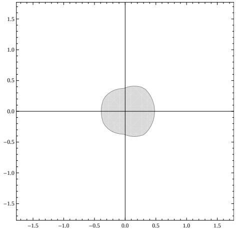
(a) Eigenvalues other than 1



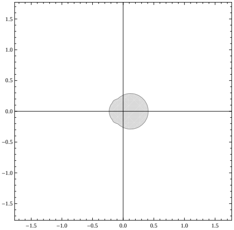
(b) Region given by Theorem 7



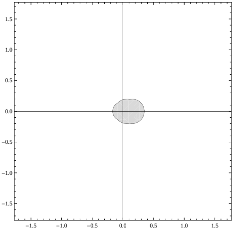
(c) Region given by Theorem 8



(d) Region given by Theorem 9



(e) Region given by Theorem 10



(f) Region given by Theorem 6

Problem to think

- We notice that in case of distance regular graph [3], the equitable partition concept has been used to find the eigenvalues of adjacency [3] and distance [2] matrices. In each case some quotient matrix corresponding to an equitable partition contain all the distinct eigenvalues of the corresponding adjacency and distance matrices.

- We notice that in case of distance regular graph [3], the equitable partition concept has been used to find the eigenvalues of adjacency [3] and distance [2] matrices. In each case some quotient matrix corresponding to an equitable partition contain all the distinct eigenvalues of the corresponding adjacency and distance matrices.
- Again consider the following matrix:

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Again $Q_1 = [3]$ and $Q_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ are quotient matrices of J corresponding to two different equitable partitions. One can observe that Q_2 contains all the distinct eigenvalues of J where as Q_1 does not.

- We notice that in case of distance regular graph [3], the equitable partition concept has been used to find the eigenvalues of adjacency [3] and distance [2] matrices. In each case some quotient matrix corresponding to an equitable partition contain all the distinct eigenvalues of the corresponding adjacency and distance matrices.
- Again consider the following matrix:

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

  Again $Q_1 = [3]$ and $Q_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ are quotient matrices of J corresponding to two different equitable partitions. One can observe that Q_2 contains all the distinct eigenvalues of J where as Q_1 does not.
- Thus it is an interesting problem to find the condition when a quotient matrix contains all the distinct eigenvalues of the original matrix. Formally, we impose the following problem:

Problem 12

Let Q be a quotient matrix of a matrix A corresponding to an equitable partition. Then what is the necessary and sufficient condition on Q to contain all the distinct eigenvalues of A ?

- On equitable partition of matrices and its applications, Linear and Multilinear Algebra, F. Atik DOI: 10.1080/03081087.2019.1572708
- F. Atik and P. Panigrahi, On the distance spectrum of distance regular graphs, Linear Algebra Appl. 478 (2015) 256-273.
- A. E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.
- A. E. Brouwer and W. H. Haemers, Spectra of graphs. Springer Science and Business Media; 2011.
- R. A. Horn and C. R. Johnson, Matrix analysis. Cambridge university press; 2013.

- A. Banerjee and R. Mehatari, An eigenvalue localization theorem for stochastic matrices and its application to randic matrices, Linear Algebra Appl. 505 (2016) 85-96.
- L. Cvetkovic, V. Kostic, J. Pena, Eigenvalue localization refinements for matrices related to positivity, SIAM Journal on Matrix Analysis and Applications, 32(3) (2011) 771-784.
- C. Li and Y. Li, A modification of eigenvalue localization for stochastic matrices, Linear Algebra Appl. 460 (2014) 231-241.
- H. Q. Lin, Y. Hong, J. F. Wang, J. L. Shu, On the distance spectrum of graphs, Linear Algebra Appl. 439 (2013), 1662-1669.
- Hui Zhou, The inverse of the distance matrix of a distance well-defined graph, Linear Algebra Appl. 517 (2017) 11-29.


Thank you!