INDIAN INSTITUTE OF TECHNOLOGY HYDERABAD

MA4050 - Combinatorics and Graph Theory/Combinatorics
Problem Sheet 1 Autumn 2024

Problem 1. Do there exist graphs with the following degree sequences:

(a) $2, 3, 4, 4, 5$.

(b) $0, 1, 2, 3, 4, 5, 6$.

(c) $1, 1, 1, 1, 1, 1$

(d) $1, 2, 3, 4, 5, 6$

Problem 2. Prove that if a graph G has at least two vertices, then G contains two vertices of the same degree.

Problem 3. Prove that if there is a walk in the graph G between the vertices u and v, then there is a path between the vertices u and v in G. Also, show that every circuit contains a cycle.

Problem 4. Let G be a graph and G^c be the complement of G. Show that either G or G^c is connected.

Problem 5. Let G be a graph and G^c be the complement of G. Show that either G or G^c is connected.

Problem 6. Compute the eigenvalues of the follwoing graphs:

(a) Complete graph K_n,

(b) Complete bipartite graph $K_{m,n}$,

(c) The cycle graph C_n.

Problem 7. Compute the automorpshism group of the following graphs:

(a) Complete graph K_n,

(b) Complete bipartite graph $K_{m,n}$,

(c) The cycle graph C_n,

(d) The path graph P_n.

Problem 8. Let G be a tree and v be a vertex in G. Then all the connected components of the graph $G - v$, the graph obtained from G by deleting v, are trees. These connected components are called the branches at the vertex v. Show that every tree contains a vertex such that every branch at this vertex contains at most half the vertices of the tree.

Problem 9. Give an example of a connected graph containing more cut-edges (bridges) than the cut-vertices

Problem 10. Let G be a connected graph on n vertices other than the complete graph. If e is a bridge of G, then show that e is incident with a cut-vertex in G.

Problem 11. Let G be a connected graph containing only even vertices. Prove that G cannot contain cut-edges.

Problem 12. If a tree G contains a vertex of degree d, then G contains at least d pendent vertices.

Problem 13. Let G be a tree. Show that any two maximum-paths in G must have a common vertex.

Problem 14. Let $n \geq 2$. Show that there is a tree with degree sequence $(d_1, d_2, \ldots, d_n)$ if and only if $d_i > 0$ for all i and $\sum_{i=1}^{n} d_i = 2(n-1)$.

Problem 15. Let G be a weighted graph. Consider the following algorithm to construct a minimum spanning tree: Choose v_1 in G. Choose an edge incident with v_1 with minimum weight. After picking $S = \{v_1, \ldots, v_k\}$, choose an edge with one endpoint in S and another in S^c and with the smallest weight among all such edges. Let v_{k+1} be the endpoint of this edge not in S, and add this vertex and the associated edge to T. Continue until all vertices of G are in T. Show that any tree obtained by the above algorithm is a minimum spanning tree.

Problem 16. Compute the number of spanning trees in the following graphs:

(a) Complete graph on n vertices minus one edge.

(b) The complete bipartite graph $K_{p,q}$.