# Graph Theory

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**Definition**

A graph $G = (V, E)$ consists of a finite set $V$ of vertices and a set $E$ of 2-elements subsets of $V$, whose members are called edges. The number of edges in $V$, denoted by $|V|$, is the order of the graph $G$ and the number of edges in $E$ is the size of the graph $G$.

Let $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ be the vertex and edge set of $G$, respectively.

- If $e = v_i v_j \in E(G)$, then we say the vertices $v_i$ and $v_j$ are adjacent, and the vertices $v_i$ and $v_j$ are incident with the edge $e$.

## Example

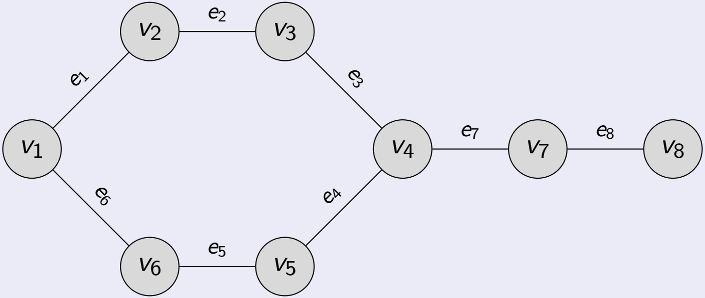



Figure: Example of Graph

In this graph: the vertices $v_1$ and $v_2$ are adjacenct, but the vertices $v_5$ and $v_8$ are not adjacent. The edge $e_1$ is incident to the vertices $v_1$ and $v_2$, but it is not incidnet with the vertex $v_3$.

The *degree* of a vertex $v$ of $G$, denoted by $\deg(v)$, is the number of vertices adjacent to it. Equivalently, $\deg(v)$ is the number of edges incident with the vertex $v$ in $G$.

**Theorem**

*For any graph $G$, the sum of the degrees of the vertices of $G$ equals to twice the number of edges in $G$. That is, if the graph $G$ has n vertices and m edges, then*

$$\sum_{i=1}^{n} \deg(v_i) = 2m.$$

**Proof.**

Proof by double counting. □

A vertex $v$ of $G$ is said to be an *even vertex* if $\deg(v)$ is even. A vertex $v$ of $G$ is said to be an *odd vertex* if $\deg(v)$ is odd.

**Corollary**

*Every graph contains an even number of odd vertices.*

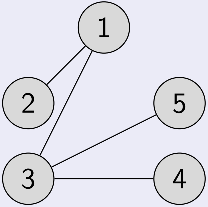
- A graph $G$ is said to be *complete graph* if every two vertices are adjacent in $G$. The complete graph on $n$ vertices is denoted by $K_n$.
- A graph $G$ is said to be *null graph* is no two vertices are adjacent in $G$, that is there is no edge in $G$.

## Definition

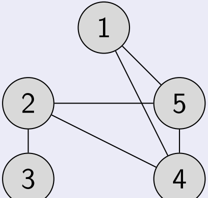
The complement of a graph $G$ on $n$ vertices, denoted by $G^c$, is defined as follows:

$$V(G^c) = V(G), E(G^c) = E(K_n) \setminus E(G)$$

## Example




$G$



## Definition

Let $G_1$ and $G_2$ be two graphs. A function $\phi : V(G_1) \to V(G_2)$ is said to be an *isomorphism* if:

- $\phi$ is a bijection,
- two vertices $v_i$ and $v_j$ in $G_1$ are adjacent if and only if $\phi(v_i)$ and $\phi(v_j)$ are adjacent in $G_2$.

Two graphs $G_1$ and $G_2$ are *isomorphic*, if there exists an isomorphism between them.

## Example

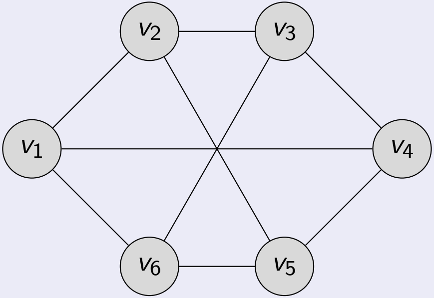



Figure: Isomorphic Graphs

The two graphs above are isomorphic.

## Remark

1. "is isomorphic to" is an equivalence relation.
2. If $G_1$ and $G_2$ are isomorphic, then $|V(G_1)| = |V(G_2)|$. The converse need not be true. Construct an example.

## Definition

Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$. The degree sequence of the graph $G$ is defined as:

$$(\deg(v_1), \deg(v_2), \ldots, \deg(v_n)).$$

## Theorem

*If $G_1$ and $G_2$ are isomorphic graphs, then the degrees of the vertices of $G_1$ are exactly the degrees of the vertices of $G_2$. That is, the degree sequence of $G_1$ and $G_2$ are the same.*

## Remark

The converse of the Theorem 3.4 need not be true. Construct a counter example.

## Definition

Let $G$ be a graph. A graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

## Definition

Let $G$ be a graph, and let $u, v \in V(G)$.

- A $(u, v)$-walk is an alternating sequence of vertices and edges of $G$ beginning at $u$ and ending at $v$.
- A $(u, v)$-trail is a $(u, v)$-walk which does not repeat any edge.
- A $(u, v)$-path is a $(u, v)$-walk which does not repeat any vertex.

### Definition

Let $G$ be a graph, and $u, v \in V(G)$.

- The vertices $u$ and $v$ are said to be connected in $G$ if there exists a $(u, v)$-path in $G$.
- The graph $G$ is said to be connected, if any two vertices of $G$ are connected. $G$ is disconnected if it is not connected.
- A connected subgraph $H$ of a graph $G$ is called a component of $G$ if $H$ is not contained in any connected subgraph of $G$ having more edges or vertices than $H$.

### Definition

Let $G$ be a graph, and $u, v \in V(G)$.

- A $(u, v)$-trial with $u = v$ with at least three edges is called a circuit.
- $(u, v)$-path with $u = v$ is called a cycle.

## Definition

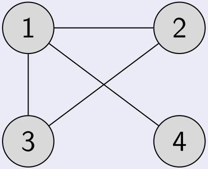
Let $v$ be a vertex of $G$, and $e$ be an edge of $G$.

- The subgraph obtained by deleting the edge from $G$, denoted by $G - e$, is the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.
- The subgraph $G \setminus v$ is defined as follows: $V(G \setminus v) = V(G) \setminus \{v\}$, and the edge set of $G \setminus v$ consists of all edges of $G$ except those incident with $v$.

Let $G$ be a graph with $V(G) = \{1, 2, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. The adjacency matrix of $G$, denoted by $A(G) = (a_{ij})$, is the $n \times n$ matrix whose rows and the columns are indexed by $V(G)$ defined as follow:

$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{if otherwise} \end{cases}$$

## Example




Graph $G$

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $G$ be a connected graph with vertices $\{v_1, v_2, \ldots, v_n\}$ and let A be the adjacency matrix of G. Then,

1. $A$ is symmetric.
2. Sum of the $2 \times 2$ principal minors of $A$ equals to $-|E(G)|$.
3. Sum of the $3 \times 3$ principal minors of $A$ equals to twice the number of triangles in the graph.

**Theorem**

*The $(i,j)^{th}$ entry of the matrix $A^k$ equals the number of walks of length $k$ from the vertex $i$ to the vertex $j$.*

**Corollary**

*The number of edges in $G$ is $\frac{\lambda_1^2+\cdots+\lambda_n^2}{2}$.*

## Definition

Let $G$ be a connected graph. For $u, v \in V(G)$, the distance between the vertices $u$ and $v$, denoted by $d(u, v)$, is defined as the length of a shortest path between the vertices $u$ and $v$. The diameter of the graph $G$, denoted by $\operatorname{diam}(G)$, is defined as follows:

$$\operatorname{diam}(G) = \max_{u,v \in V(G)} d(u, v).$$

## Example

1. If $G$ is the complete graph on $n$ vertices, then $\operatorname{diam}(G) = 1$.
2. If $G = C_n$, then $\operatorname{diam}(G) = \lfloor \frac{n}{2} \rfloor$.
3. If $G = P_n$, then $\operatorname{diam}(G) = n - 1$.

## Theorem

*If $v_i$ and $v_j$ are vertices of $G$ with $d(v_i, v_j) = m$, then the matrices $I, A, \ldots, A^m$ are linearly independent.*

## Corollary

*Let $G$ be a connected graph with $k$ distinct eigenvalues and let $d$ be the diameter of $G$. Then $k > d$.*

Let $G$ be a graph with adjacency matrix $A$. Often we refer to the eigenvalues of $A$ as the eigenvalues of $G$.

## Theorem

1. *For any positive integer $n$, the eigenvalues of $K_n$ are $n-1$ with multiplicity 1 and $-1$ with multiplicity $n-1$.*
2. *For any positive integers $p$ and $q$, the eigenvalues of $K_{p,q}$ are $\sqrt{pq}$, $-\sqrt{pq}$ and 0 with multiplicity $p+q-2$.*
3. *For any positive integer $n$, the eigenvalues of $C_n$ are given by the set $\{2\cos\frac{2k\pi}{n} : 1 \leq k \leq n\}$.*

## Definition

Two graphs $G_1$ and $G_2$ are cospectral if the spectrum of $A(G_1)$ and $A(G_2)$ are the same.

## Theorem

*Let $G_1$ and $G_2$ be two cospectral graphs. Then*

- *the number of vertices in $G_1$ and $G_2$ are the same,*
- *the number edges in $G_1$ and $G_2$ are the same.*

## Theorem

*If $G_1$ and $G_2$ are isomorphic, then they are cospectral.*

### Remark

If $G_1$ and $G_2$ are cospectral, then they need not be isomorphic.

An isomorphism from a graph $G$ to itself is called an automorphism on $G$. Consider the set

$$\text{Aut}(G) = \{\varphi : V(G) \to V(G) | \ \varphi \text{ is an automorphism.}\}$$

(a) As the identity map is an automorphism $\text{Aut}(G) \neq \ \phi$.

(b) It is easy to see that $\phi_1, \phi_2 \in \text{Aut}(G) \Rightarrow \phi_1 \circ \phi_2 \in \text{Aut}(X)$

(c) It is also easy to see that $\phi \in \text{Aut}(G) \Rightarrow \phi^{-1} \in \text{Aut}(G)$

**Theorem**

*The set of all automorphisms of a graph $G$, $\text{Aut}(G)$, forms a subgroup of $(\text{Sym}(n), \circ)$, the group of all permutations on $n$ letters.*

$(\text{Aut}(G), \circ)$ is called the automorphism group of the graph $G$.

## Remark

(a) $\text{Aut}(G) = \text{Aut}(G^c)$.

(b) If $X \cong Y$, then $\text{Aut}(X) \cong \text{Aut}(Y)$. For, if $\psi$ is an isomorphism from $X$ to $Y$ then $\phi : \text{Aut}(X) \to \text{Aut}(Y)$ defined by $\phi(g) = \psi g \psi^{-1}$ defines an isomorphism from $\text{Aut}(X)$ to $\text{Aut}(Y)$.

(c) Converse of the above remark need not be true. For $\text{Aut}(K_n) \cong \text{Aut}(K_n^c) \cong \text{Sym}(n)$, but $K_n \ncong K_n^c$.

## Lemma

*If $v$ is a vertex of graph $G$ and $g \in \text{Aut}(G)$, then the vertices $v$ and $g(v)$ have same degree.*

## Remark

Lemma 5.3 provides a necessary condition for a vertex $v_1$ to be mapped under an automorphism to another vertex $v_2$. This is possible only if both $v_1$ and $v_2$ have the same degree.

## Example

- $\text{Aut}(K_n) = \text{Sym}(n)$.
- For $m \neq n$, $\text{Aut}(K_{m,n}) = \text{Sym}(n) \times \text{Sym}(m)$
- $\text{Aut}(C_n) = D_n$.
- $\text{Aut}(P_n) = \text{Sym}(2)$

## Lemma

*Let $u, v \in V(G)$ , $g \in \text{Aut}(G)$. Then $d(u, v) = d(g(u), g(v))$.*

## Remark

$\text{Aut}(G) = \text{Aut}(\overline{G})$ (as automorphisms preserve adjacency as well as non-adjacency.)

## Definition

- A vertex $v$ of $G$ is called a cut-vertex of $G$, if the number of components in $G \setminus v$ is more than that of $G$.
- An edge $e$ of $G$ is called a cut-edge or bridge if the number of components in $G \setminus e$ is more than that of $G$.

## Remark

If $v$ is a cut-vertex of a connected graph $G$, then $G \setminus e$ contains two or more components. However, if $e$ is a bridge of $G$, then $G \setminus e$ has exactly two components.

### Theorem

*Let $G$ be a connected graph. An edge $e$ of $G$ is a bridge if and only if $e$ does not lie on any cycle of $G$.*

### Definition

A graph $G$ is called a tree if $G$ is connected and it does not contain cycles. A graph that has no cycles is called a forest. A graph $G$ is acyclic if it does not contain cycles.

### Theorem

*If $G$ is a tree on $n$ vertices and $m$ edges, then $m = n - 1$.*

## Theorem

*Let $G$ be a graph on $n$ vertices. Then the following are equivalent:*

(a) *$G$ is connected and acyclic.*

(b) *$G$ is connected and has $n-1$ edges.*

(c) *$G$ is connected, and every edge of $G$ is a bridge.*

(d) *If $u$ and $v$ are two distinct vertices of $G$, then there is exactly one $(u,v)$-path in $G$.*

(e) *$G$ contains no cycles, and adding an edge between any two vertices produces exactly one cycle.*

### Definition

A vertex $v$ of a graph $G$ is said to be *pendant vertex* if $\deg(v) = 1$, and is said to be a *quasi-pendant vertex* if it is adjacent to a pendant vertex.

### Theorem

*Every tree with at least two vertices has at least one pendent vertex.*

### Definition

A subgraph $H$ of a graph $G$ is said to be a spanning subgraph of $G$ if $V(G) = V(H)$. A spanning subgraph which is also a tree is called a spanning tree. In this subsection, we consider graphs with weighted edges. For weighted graph $G$, the weight of $G$, denoted by $w(G)$, is the sum of the weights of the edges of $G$.

**Problem:** Find a spanning tree $T$ of $G$ with the sum of the weights of its edges $w(T)$ is minimum.

**Krushkal's algorthm (Greedy algorithm):**

- **Step 1 :** Choose the first edge $e_1$ of $T$ from $G$ so that $w(e_1)$ is minimum among all the edges of $G$.
- **Step 2 :** Choose the second edge $e_2$ of $T$ from $G \setminus \{e_1\}$ so that $w(e_2)$ is minimum among all the edges of $G \setminus \{e_1\}$.
- **Step 3 :** Choose the third edge $e_3$ of $T$ from $G \setminus \{e_1, e_2\}$ so that $w(e_3)$ is minimum among all the edges of $G \setminus \{e_1, e_2\}$.
- **Step 4 :** Continue this process until we obtain a spanning tree $T$.

A spanning tree obtained in the above algorithm is called a minimal spanning tree (MST).

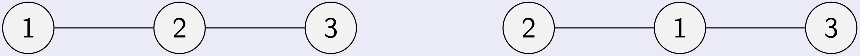
**Theorem**

*Let $G$ be a connected graph on $n$ vertices with weighted edges. If $T$ is a minimal spanning tree, then $w(T)$ is minimum among all the spanning trees of $G$.*

We want to count the number of labelled trees on $n$ vertices.

**Remark**

The following two labelled trees considered to be different.




$T_1$ $T_2$

**Definition**

A Prüfer sequence of length $(n-2)$, for $n \geq 2$, is any sequence of integers between 1 and $n$, with repetitions allowed.

**Lemma**

*There are $n^{n-2}$ Prüfer sequences of length $n-2$.*

**Theorem**

*The number of labelled trees on $n$ vertices is $n^{n-2}$.*

**Remark**

The number of non-isomorphic trees on $n$ vertices is less than $n^{n-2}$.

Let $G$ be a connected simple graph with vertex set $\{1, 2, \ldots, n\}$, and let $M = (m_{ij})$ be the $n \times n$ matrix in which $m_{ii} = \deg(v_i)$, for $1 \leq i \leq n$, and $m_{ij} = -1$ if the vertices $v_i$ and $v_j$ are adjacent, and $m_{ij} = 0$ otherwise.

**Theorem (Matrix-Tree theorem)**

*The number of spanning trees of $G$ equals to the cofactor of any element of $M$.*

## Definition

Let $G = (V, E)$ be a graph. The graph $G$ is said to be *bipartite* if the vertex set $V$ can be partitioned into two nonempty disjoint sets $A$ and $B$ such that every edge of $G$ has one end vertex in $A$ and the other end in $B$.

## Theorem

*A graph $G$ is bipartite if and only if the vertices of $G$ can be colored using 2 colors so that no two adjacent vertices receives same color.*

## Theorem

*A graph $G$ is bipartite if and only if $G$ contains no odd cycles.*

## Theorem

*Let $G$ be a simple bipartite graph on $n$ vertices. Then $G$ has at most $\frac{n^2}{4}$ edges if n edges, and at most $\frac{n^2-1}{4}$ edges if n is odd.*

## Definition

- Two edges are said to be *adjacent* if they are incident with a common vertex.
- A matching in a graph is a set of nonadjacent edges.
- Two ends of each edge of a matching $\mathcal{M}$ are said to be matched under $\mathcal{M}$. Each vertex incident with an edge of $\mathcal{M}$ is said to be covered by $\mathcal{M}$.
- A *perfect matching* is a matching which covers all the vertices of $G$.
- For a nonempty subset $S$ of vertices, define $N(S)$, the neighborhood of $S$, is the collection of neighbors of the set $S$.

$$N(S) := \{u \in V : uv \text{ is an edge for some } v \in S\}$$

## Theorem (Hall's theorem)

*Let $G = A \cup B$ be a bipartite graph. Then $G$ has a perfect matching if and only if $|A| = |B|$ and $|N(S)| \geq |S|$ for all $S \subseteq A$.*

## Corollary

*Let $k \in \mathbb{N}$. Then a $k$-regular bipartite graph $G = A \cup B$, then $G$ has the following properties:*

1. $|A| = |B|$
2. *$G$ contains a perfect matching.*

### Definition

Let $A_1, A_2, \ldots, A_n$ be finite sets. Choose $n$ distinct elements $a_1, a_2, \ldots, a_n$ so that $a_i \in A_i$ for all $i$ Such a set of objects is called a system of distinct representatives (SDR).

### Theorem

*Let $A_1, A_2, \ldots, A_n$ be a collection of finite sets. Then there exists an SDR for $A_1, A_2, \ldots, A_n$ if and only if $| \cup_{i \in I} | \geq |I|$ for all $I \subseteq \{1, \ldots, n\}$.*

A connected graph $G$ is Eulerian graph if there exists a closed trail containing every edge of $G$. Such a trial is an Eulerian trail.

**Theorem**

*Let $G$ be a graph such that for every vertex $v$ of $G$, $\deg(v) \geq 2$. Then $G$ contains a cycle.*

**Theorem (Euler, 1736)**

*A connected graph $G$ is Eulerian if and only if the degree of each vertex of $G$ is even.*

## Definition

A decomposition of a graph $G$ is a family $\mathcal{F}$ of edge-disjoint subgraphs of $G$ so that

$$\cup_{F \in \mathcal{F}} E(F) = E(G).$$

## Remark

If a graph $G$ has a cycle decomposition, then degree of every vertex of $G$ is even.

## Definition

A graph $G$ is said to be even if every vertex $v$ of $G$ has even degree.

## Theorem (Veblen's theorem(1912/13))

*A graph $G$ admits a cycle decomposition if and only $G$ is even.*

## Theorem (Graham-Pollack Theorem (1971))

*Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a decomposition of $K_n$ into complete bipartite graphs. Then $k \geq n - 1$.*

### Definition

A graph $G$ is said to be *Hamiltonian*, if there exists a cycle exists in $G$ containing all the vertices of $G$. A cylce containing all the vertices of $G$ is called a Hamiltonian cycle.

### Theorem (Ore's theorem)

*If $G$ is a simple graph with $n \geq 3$ vertices, and if $\deg(u) + \deg(v) \geq n$ for all pairs of non-adjacent vertices $u$ and $v$, then $G$ is Hamiltonian.*

### Corollary

*If $G$ is a simple graph with $n \geq 3$ vertices, and if $\deg(v) \geq \frac{n}{2}$ for each vertex $v$ of $G$, then $G$ is Hamiltonian.*

### Definition

A planer graph is a graph that can be drawn in the plane in such a way that no two edges intersect except a vertex. A planar graph already drawn in the plane so that no two edges intersect is refereed to as a plane graph.

### Theorem

*Let $G$ be a connected plane graph. Then $n - m + r = 2$.*

### Theorem

*Let $G$ be a connected planer graph with at least three vertices. Then $m \leq 3n - 6$.*

### Theorem

*The graphs $K_5$ and $K_{3,3}$ are non-planer.*

A coloring of a graph $G$ is an assignment of colors to the vertices of the graph such that no two adjacent vertices receive same color. A $k$-coloring of a graph is a coloring of $G$ using $k$-colors.

### Definition

The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number $k$ for which the graph has a $k$-coloring.

**Notation:** $\Delta(G)$ - maximum vertex degree of $G$.

### Theorem

*For any graph $G$,*

$$\chi(G) \leq 1 + \Delta(G).$$

**Theorem (Five color theorem)**

*If $G$ is a planer graph, then $\chi(G) \leq 5$.*

**Theorem (Four color theorem)**

*If $G$ is a planer graph, then $\chi(G) \leq 4$.*